Solution of the second examination   Math 106   Semester I   1439-1440

Exercise 1 :

a) $$\int \frac{dx}{e^{-x}\sqrt{e^{2x}-1}} \overset{t=e^x}{=} \int \frac{dt}{\sqrt{t^2-1}} = \cosh^{-1}(e^x)+c. \qquad 1+1$$

b)

$$\begin{aligned}\int \frac{\tan x}{\sqrt{4-\cos^4 x}}dx &\overset{t=\frac{1}{2}\cos^2(x)}{=} -\frac{1}{4}\int \frac{dt}{t\sqrt{1-t^2}} \qquad 2\\ &= \frac{1}{4}\mathrm{sech}^{-1}(\frac{1}{2}\cos^2(x))+c. \qquad 1\end{aligned}$$

c)

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{1}{x}-\frac{1}{\sin x} &= \lim_{x\longrightarrow 0}\frac{\sin x - x}{x\sin x} \qquad 1\\ &= \lim_{x\longrightarrow 0}\frac{\cos x - 1}{\sin x + x\cos x} \qquad 1\\ &= \lim_{x\longrightarrow 0}\frac{-\sin x}{2\cos x - x\sin x} = 0. \qquad 1\end{aligned}$$

Exercise 2 :

a)

$$\begin{aligned}\int e^{2x}\sin x dx &\overset{u=e^{2x},v'=\sin x}{=} -e^{2x}\cos x + 2\int e^{2x}\cos x dx \qquad 1\\ 1 \quad &\overset{u=e^{2x},v'=\cos x}{=} -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x dx.\end{aligned}$$

Then $\displaystyle\int e^{2x}\sin x dx = -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + c.$ 1

b)

$$\begin{aligned}\int \sec^4 x\tan^7 x dx &\overset{u=\tan x}{=} \int u^7(1+u^2)du \qquad 1\\ &= \frac{1}{8}\tan^8 x + \frac{1}{10}\tan^{10} x + c. \qquad 1\end{aligned}$$

c)

$$\int \frac{dx}{x^3\sqrt{x^2-4}} \overset{x=2\sec\theta}{=} \frac{1}{8}\int \cos^2\theta d\theta \quad 1$$
$$= \frac{1}{16}\int (1+\cos(2\theta))d\theta = \frac{1}{16}(\theta + \cos\theta\sin\theta) + c \quad 1$$
$$= \frac{1}{16}(\sec^{-1}(\frac{x}{2}) + \frac{2\sqrt{x^2-4}}{x^2}) + c \quad 1$$

Or

$$\int \frac{dx}{x^3\sqrt{x^2-4}} \overset{x^2-4=t^2}{=} \int \frac{dt}{(t^2+4)^2}$$
$$= \frac{1}{16}\tan^{-1}(\frac{t}{2}) + \frac{1}{16}\ln(\frac{t}{2}) + c$$
$$= \frac{1}{16}\tan^{-1}(\frac{\sqrt{x^2-4}}{2}) + \frac{1}{16}\ln(\frac{\sqrt{x^2-4}}{2}) + c$$

Exercise 3 :

a)

$$\int \frac{6x^2+x+8}{x^3+4x}dx = \int (\frac{2}{x} + \frac{4x+1}{x^2+4})dx \quad 1,5$$
$$= 2\ln|x| + 2\ln(x^2+4) + \frac{1}{2}\tan^{-1}(\frac{x}{2}) + c. \quad 1,5$$

b)

$$\int \frac{dx}{(x+1)^{\frac{5}{6}} - (x+1)^{\frac{1}{2}}} \overset{x+1=t^6}{=} \int \frac{6t^5}{t^5-t^3}dt \quad 1$$

(6t - 6 tanh⁻¹ t + C OK)

$$= 6\int \frac{t^2}{t^2-1}dt = 6t - 3\ln\left|\frac{1+t}{1-t}\right| + c \quad 1,5$$
$$= 6(x+1)^{\frac{1}{6}} - 3\ln\left|\frac{1+(x+1)^{\frac{1}{6}}}{1-(x+1)^{\frac{1}{6}}}\right| + c \quad 0,5$$

c) $$\int \frac{dx}{2+\cos x} \overset{t=\tan(\frac{x}{2})}{=} \int \frac{2dt}{3+t^2}dt = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}}\tan(\frac{x}{2})) + C$$

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