Question Number	Q1	Q2	Q3	Q4	Q5	Total
Mark						

Calculators are not allowed

Question 1: (9 marks (2+3+4))

(a) Without using truth tables, prove the following logical equivalence:

$$(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \equiv p \wedge q.$$

(b) Let a, b, c be integers. Prove that:
if a divides $(b+2)$ and a divides $(c-3)$, then a divides $(bc+6)$.

(c) Use mathematical induction to prove that the following statement is true for all positive integers n.

$$2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$$

Question 2: (9 marks ((3+1+1)+(3+1)))

(a) Let $P = \{(a, a\}, (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, d)\}$ be a relation on $A = \{a, b, c, d\}$

(i) Show that P is a partial ordering relation.

(ii) Is P a total ordering? (Justify your answer).

(iii) Draw the Hasse diagram of P.

(b) Let E be the relation on $\mathbb{Z}$ (integers) defined by xRy iff $3 \mid (x^2 - y^2)$.

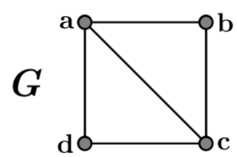
(i) Prove that E is an equivalence relation.

(ii) Show that $[2] = [4]$.

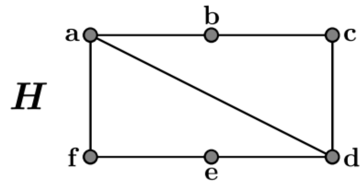
Question 3: (8 marks (2+1+(2+1)+2))

(a) Find the number of vertices n of a complete graph K_n which has 55 edges.

(b) Find the adjacency matrix of the following graph G.



(c) For the following graph H.



(i) Determine whether H is bipartite. If so give a bipartite representation.

(ii) Find a simple path from b to e of length 3.

(d) Consider two undirected graphs G_1 and G_2 defined as below:
Graph G_1: $V_1 = \{u_1, u_2, u_3, u_4\}$, $E_1 = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_1\}\}$.
Graph G_2: $V_2 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{\{v_1, v_3\}, \{v_3, v_2\}, \{v_2, v_4\}, \{v_4, v_1\}\}$.

Determine whether G_1 and G_2 are isomorphic. If they are, provide the isomorphism function $f: V_1 \to V_2$. If they are not, provide a reason.

Question 4: (6 marks (2+(1+1)+2))

(a) Let T be a tree with exactly 3 children for each internal vertex. Find the number of vertices of T such that T have 9 leaves.

(b) For the following graph I. find a spanning tree with root f.

(i) Using depth-first search.

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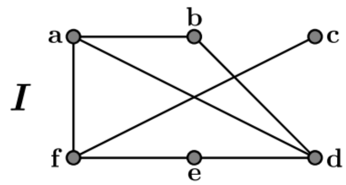

(c) Using alphabetical order, form a binary search tree for the words: banana, apple, cherry, strawberry, orange, date, elderberry, fig, grape.

Question 5: (8 marks (2+2+2+2))

(a) Prove the following Boolean identity:

$$(x+\overline{y})(\overline{x}+y)(\overline{x}+\overline{y}) = \overline{x}\,\overline{y}.$$

(b) Find the complete sum-of-products expansion (CSP) for

$$f(x,y,z) = (x+y)(\overline{y}+\overline{z}).$$

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$$g(x, y, z) = \overline{x}z + y\overline{z}.$$

(d) Use a K-map to simplify the following sum-of-products expansion (write in MSP form):

$$\overline{x}\,\overline{y}\,\overline{z} + \overline{x}y\overline{z} + x\overline{y}z + xy\overline{z} + xyz.$$

Good luck

Question Number	Q1	Q2	Q3	Q4	Q5	Total
Mark						

Calculators are not allowed

Question 1: (9 marks (2+3+4))

(a) Without using truth tables, prove the following logical equivalence:

$$(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \equiv p \wedge q.$$

$LHS \equiv [(p \vee q) \wedge (\neg p \vee q)] \wedge (p \vee \neg q)$ ~~Distribution law~~

$\equiv [(p \wedge \neg p) \vee q] \wedge (p \wedge \neg q)$ 1/4 Distribution law

$\equiv (F \vee q) \wedge (p \wedge \neg q)$ 1/2 Negation law

$\equiv q \wedge (p \vee \neg q)$ 1/2 Identity law

$\equiv (q \wedge p) \vee (q \wedge \neg q)$ 1/4 Distribution law

$\equiv (q \wedge p) \vee F$ Negation law

$\equiv q \wedge p$ identity law

$\equiv p \wedge q$ 1/2 Commutative law

(b) Let a, b, c be integers. Prove that:
if a divides $(b+2)$ and a divides $(c-3)$, then a divides $(bc+6)$.

Since $a \mid (b+2)$, there exists an integer k_1 such that

$b+2 = ak_1 \Rightarrow b = ak_1 - 2$. 1/2

Since $a \mid c-3$, there exists an integer k_2 such that

$c-3 = ak_2 \Rightarrow c = ak_2 + 3$. 1/2

Substitute these expressions into the target term $bc+6$ yields

$bc+6 = (ak_1-2)(ak_2+3)+6$ 1/2

$= a^2k_1k_2 - 2ak_2 + 3ak_1 - 6 + 6$ 1/2

$= a(ak_1k_2 - 2k_2 + 3k_1)$. 1/2

Let $m = ak_1k_2 - 2k_2 + 3k_1$. Since k_2, k_2 and a are all integers, m must also be an integer.

Therefore, we can write $bc+6 = am$. By definition, this means a divides $bc+6$. 1/2

(c) Use mathematical induction to prove that the following statement is true for all positive integers n.

$$2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n-1)}{2}$$

Let $P(n)$ be the statement : $2+7+12+\cdots+(5n-3) = \frac{n(5n-1)}{2}$

1. Basis step : We must show that the statement is true for $n=1$. LHS : The first term of the series is 2
   RHS : Substituting $n=1$ into the formula gives
   $\frac{1(5(1)-1)}{2} = \frac{4}{2} = 2$. ~~Stm~~
   Since LHS=RHS, $P(1)$ is true. 1

2. Inductive step : Assume $P(k)$ is true for an arbitrary integer k.
   That is, our inductive hypothesis is: $2+7+\cdots+(5k-3) = \frac{k(5k-1)}{2}$. 1/2
   We must show $P(k+1)$ is true. That is, we want to prove
   $2+7+\cdots+(5k-3)+5(k+1)-3 = \frac{(k+1)(5(k+1)-1)}{2}$
   $RHS = \frac{(k+1)(5k+4)}{2}$ 1/2
   $LHS = 2+7+\cdots+5(k-3)+5(k+1)-3 \overset{IH}{=} \frac{k(5k-1)}{2} + 5k+2$ 1/2
   $= \frac{5k^2-k+10k+4}{2} = \frac{5k^2+9k+4}{2}$
   $= \frac{(k+1)(5k+4)}{2} = RHS$. 1
   So $P(k+1)$ is true. By PMI $P(n)$ is true $\forall n \geq 1$ 1/2

Question 2: (9 marks ((3+1+1)+(3+1)))

(a) Let $P = \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,d)\}$ be a relation on $A = \{a,b,c,d\}$

(i) Show that P is a partial ordering relation.

- Reflexive: Since (a,a), (b,b), (c,c) and (d,d) are in P, P is reflexive 1

- Antisymmetric: The distinct pairs are (a,b), (a,c), (a,d), (b,c) and (b,d). None of their reverses (b,a), (c,a), (d,a), (c,b) or (d,b) are in P. Thus P is antisymmetric 1

- Transitive: • Since ① $(a,b)\in P$, $(b,c)\in P$ and $(a,c)\in P$
  ② $(a,b)\in P$, $(b,d)\in P$ and $(a,d)\in P$ 1
  ③ (i) $(a,c)\in P$ but c does not go to any other distinct element, there are no new pairs to check from a.
  (ii) Similarly for (b,c), (a,d) and (b,d).
  Hence P is transitive.

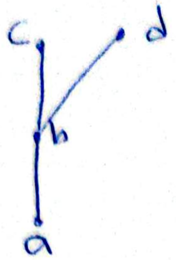
Since P is reflexive, antisymmetric and transitive, P is poset

(ii) Is P a total ordering? (Justify your answer).

No, P is not a total ordering. Since c and d are incomparable

1

(iii) Draw the Hasse diagram of P.



1

(b) Let E be the relation on $\mathbb{Z}$ (integers) defined by xEy iff $3 \mid (x^2 - y^2)$.

1/2

(i) Prove that E is an equivalence relation.

(1) Reflexive: Since $3 \mid x^2 - x^2$, xEx for all x in $\mathbb{Z}$ and E is reflexive

(2) Symmetry: Assume xEy is true. By def, this means $3 \mid x^2 - y^2$, so

$\exists k$ such that $x^2 - y^2 = 3k$

$\Rightarrow$ $y^2 - x^2 = 3(-k)$

1

$\Rightarrow$ $3 \mid y^2 - x^2$

$\Rightarrow$ yEx and E is symmetric

(3) Transitivity: Assume xEy and yEz. By def^n, this means

$3 \mid x^2 - y^2 \Rightarrow x^2 - y^2 = 3k_1$ for some $k_1 \in \mathbb{Z}$

1.5

and $3 \mid y^2 - z^2 \Rightarrow y^2 - z^2 = 3k_2$,, $k_2 \in \mathbb{Z}$

$\Rightarrow x^2 - y^2 + y^2 - z^2 = 3(k_1 + k_2) \Rightarrow x^2 - z^2 = 3(k_1 + k_2)$. Since $k_1 + k_2 \in \mathbb{Z}$,

$3 \mid x^2 - \cancel{y^2} \Rightarrow xEz$. Therefore E is transitive.

By (1), (2) and (3) E is an equivalence relation on $\mathbb{Z}$

(ii) Show that $[2] = [4]$.

Since $2E4$ $(3 \mid 2^2 - 4^2)$, $[2] = [4]$ 1

Question 3: (8 marks (2+1+(2+1)+2))

(a) Find the number of vertices n of a complete graph K_n which has 55 edges.

$$\frac{n(n-1)}{2} = 55 \quad \tfrac{1}{2} \longrightarrow 1$$

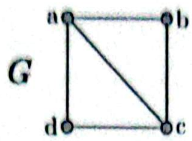
$$\Rightarrow n^2 - n - 110 = 0 \quad \tfrac{1}{2} \longrightarrow \tfrac{1}{2}$$

$$(n-11)(n+10) = 0 \quad \tfrac{1}{2} \longrightarrow \tfrac{1}{4}$$

$$\Rightarrow n = 11 \text{ or } n = -10$$

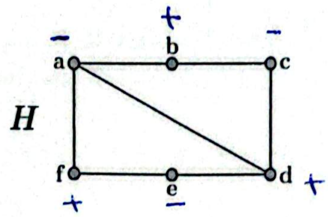
Since the number of vertices in a graph must be a positive integer, The number of vertices is 11. $\tfrac{1}{2} \longrightarrow \tfrac{1}{4}$

(b) Find the adjacency matrix of the following graph G.



$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad 1$$

(c) For the following graph H.



(i) Determine whether H is bipartite. If so give a bipartite representation.

$-\tfrac{1}{2}$

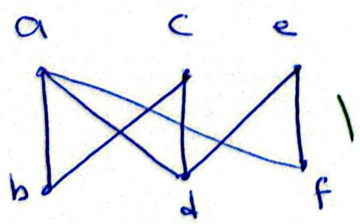
H is bipartite.

Let $V_1 = \{b, d, f\}$, $V_2 = \{a, c, e\}$

$H = V_1 \cup V_2$ $\tfrac{1}{2}$

and $V_1 \cap V_2 = \phi$ $\tfrac{1}{2}$

1



1

(ii) Find a simple path from b to e of length 3.

$b \to a \to f \to e$ 1

(d) Consider two undirected graphs G_1 and G_2 defined as below:

Graph G_1: $V_1 = \{u_1, u_2, u_3, u_4\}$, $E_1 = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_1\}\}$.

Graph G_2: $V_2 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{\{v_1, v_3\}, \{v_3, v_2\}, \{v_2, v_4\}, \{v_4, v_1\}\}$.

Determine whether G_1 and G_2 are isomorphic. If they are, provide the isomorphism function $f: V_1 \to V_2$. If they are not, provide a reason.

① Both graphs have exactly 4 vertices ($|V_1| = |V_2| = 4$)

② " " " " 4 edges ($|E_1| = |E_2| = 4$)

③ In G_1, the degree sequence is $(2,2,2,2)$

and in G_2, " " " is $(2,2,2,2)$

Let $f: V_1 \to V_2$ be defined as follows ½

½ $f(u_1) = v_1$, ½ $f(v_2) = v_3$, ½ $f(u_3) = v_2$ and $f(u_4) = v_4$.

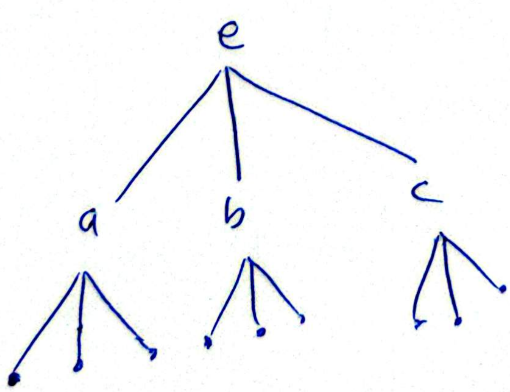
½ f is bijective. Note that f preserves adjacency and ~~for each vertices~~ ~~Edge~~ $\{x, y\}$ in an edge in E_1 iff $\{f(x), f(y)\}$ is an edge in E_2

Question 4: (6 marks (2+(1+1)+2))

(a) Let T be a tree with exactly 3 children for each internal vertex. Find the number of vertices of T such that T have 9 leaves.

The total number of vertices of T is 13

2



(b) For the following graph I, find a spanning tree with root f.

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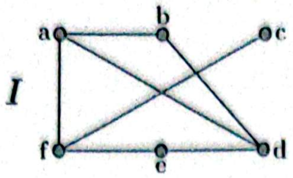
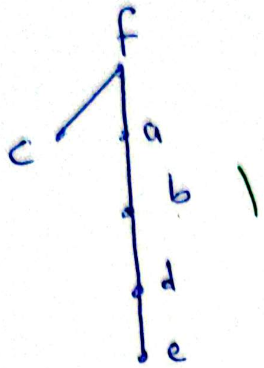
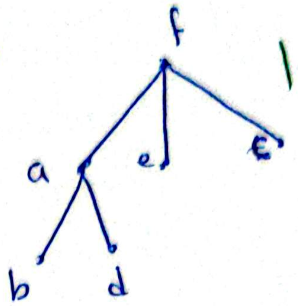


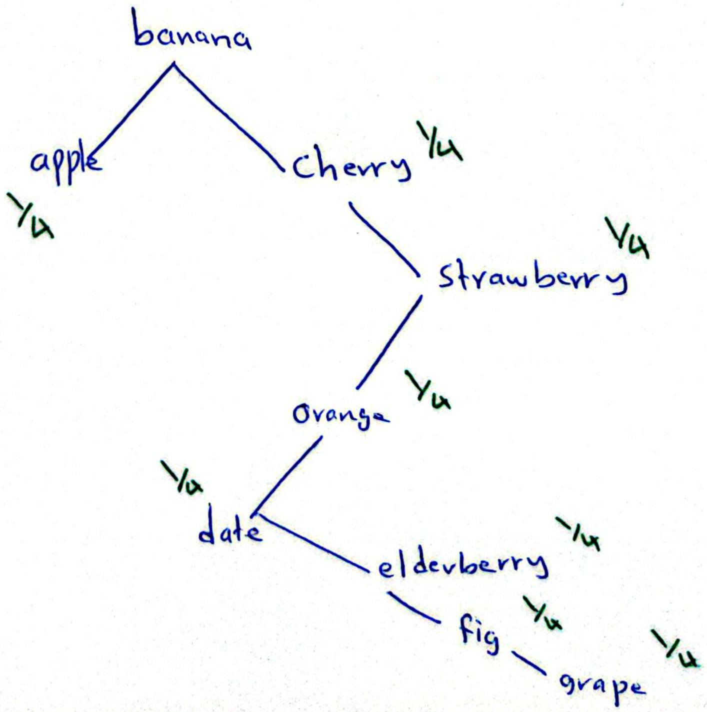
DFS



BFS



(c) Using alphabetical order, form a binary search tree for the words: banana, apple, cherry, strawberry, orange, date, elderberry, fig, grape.



Question 5: (8 marks (2+2+2+2))

(a) Prove the following Boolean identity:

$$(x+\overline{y})(\overline{x}+y)(\overline{x}+\overline{y}) = \overline{x}\,\overline{y}.$$

LHS $= (x+\bar{y}) \cdot [(\bar{x}+y)(\bar{x}+\bar{y})]$
$= (x+\bar{y}) \cdot [\bar{x} + y\bar{y}]$ Distribution law $\frac{1}{4}$
$= (x+\bar{y}) \cdot (\bar{x}+0)$ Complement law $\frac{1}{4}$
$= (x+\bar{y}) \cdot \bar{x}$ Identity law $\frac{1}{4}$
$= x \cdot \bar{x} + \bar{y} \cdot \bar{x}$ Distribution law $\frac{1}{4}$
$= 0 + \bar{y} \cdot \bar{x}$ Complement law $\frac{1}{4}$
$= \bar{y} \cdot \bar{x}$ Identity law $\frac{1}{4}$
$= \bar{x} \cdot \bar{y}$ Commutative law $\frac{1}{4}$
$=$ RHS

(b) Find the complete sum-of-products expansion (CSP) for

$$f(x,y,z) = (x+y)(\overline{y}+\overline{z}).$$

$= x\bar{y} + x\bar{z} + y\bar{y} + y\bar{z}$
$= x\bar{y} + x\bar{z} + y\bar{z}$
$= x\bar{y}(z+\bar{z}) + x\bar{z}(y+\bar{y}) + y\bar{z}(x+\bar{x})$
$= x\bar{y}z + \underline{x\bar{y}\bar{z}} + x\bar{z}y + \underline{x\bar{z}\bar{y}} + y\bar{z}x + y\bar{z}\bar{x}$
$= x\bar{y}z + x\bar{y}\bar{z} + x\bar{z}y + y\bar{z}\bar{x}$
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

(c) Find the complete product-of-sums expansion (CPS) for

$$g(x, y, z) = \overline{x}z + y\overline{z}.$$

x	y	z	$g(x,y,z)$
0	0	0	0 ✓
0	1	0	1
1	0	0	0 ✓
0	0	1	1
1	1	0	1
1	0	1	0 ✓
0	1	1	1
1	1	1	0 ✓

1/2 1/2 1/2 1/2

$g(x,y,z) = (x+y+z)(\bar{x}+y+z)(\bar{x}+y+\bar{z}) \cdot (\bar{x}+\bar{y}+\bar{z})$

$g^d = (\bar{x}+z) \cdot (y+\bar{z})$

$csp(g^d) = \bar{x}y + \bar{x}\bar{z} + zy + z\bar{z}$

$= \bar{x}y + \bar{x}\bar{z} + zy$

$= \bar{x}y(z+\bar{z}) + \bar{x}\bar{z}(y+\bar{y}) + zy(x+\bar{x})$

$= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{z}y + \bar{x}\bar{z}\bar{y} + zyx + zy\bar{x}$

$= \bar{x}yz + \bar{x}y\bar{z} + xyz + \bar{x}\bar{y}\bar{z}$

$cps(g) = (csp(g^d))^d$

$= (\bar{x}+y+z)(\bar{x}+y+\bar{z}) \cdot (x+y+z) \cdot (\bar{x}+\bar{y}+\bar{z})$

(d) Use a K-map to simplify the following sum-of-products expansion (write in MSP form):

$$\overline{x}\,\overline{y}\,\overline{z} + \overline{x}y\overline{z} + x\overline{y}z + xy\overline{z} + xyz.$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		1
$\bar{x}$		1	1	

1/2

1/2 1/2 1/2

$Msp = xz + xy + \bar{x}\bar{z}$

x	y	z
1	1	1
1	0	1

x	y	z
1	1	1
1	1	0

x	y	z
0	1	0
0	0	0

Go

King Saud University | College of Sciences | Department of Mathematics
Final Examination | Math 151 | Semester 2(1447) | Time: 3H

Calculators are not allowed

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(a) Let $P = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$ be a relation on the set $A = \{a, b, c, d\}$.

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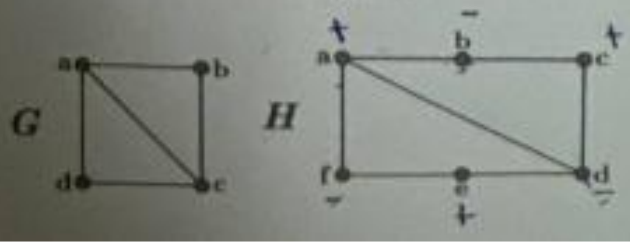

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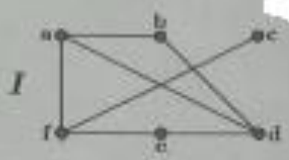

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