M 106 - INTEGRAL CALCULUS

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Solution of the first mid-term exam

Second semester 1433-1434 H

Multiple choice questions (One mark for each question)

Q.1 The average value of $f(x) = \dfrac{1}{\sqrt{x+2}}$ over $[-1,1]$ is equal to :

(a) $\sqrt{3}-1$ (b) $2\sqrt{3}-2$ (c) $\sqrt{3}+1$ (d) $2\sqrt{3}+1$

Answer :

$$f_{av} = \frac{\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{x+2}}\, dx}{1-(-1)} = \frac{1}{2}\int_{-1}^{1} (x+2)^{-\frac{1}{2}}\, dx$$

$$f_{av} = \frac{1}{2}\left[\frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{-1}^{1} = \sqrt{1+2} - \sqrt{-1+2} = \sqrt{3}-1$$

The right answer is (a)

Q.2 The approximate value of $\displaystyle\int_1^2 \frac{1}{x}\, dx$ using trapezoidal rule with $n=2$ is

(a) 0.708 (b) 2.833 (c) 1.416 (d) 0.697

Answer : $f(x) = \dfrac{1}{x}$, $[a,b]=[1,2]$, $n=2$, $\Delta x = \dfrac{2-1}{2} = \dfrac{1}{2}$

$x_0 = 1$, $x_1 = 1 + \dfrac{1}{2} = \dfrac{3}{2}$ and $x_2 = 2$.

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{2-1}{2(2)}\left[f(1) + 2f\left(\frac{3}{2}\right) + f(2)\right]$$

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{1}{4}\left[1 + \frac{4}{3} + \frac{1}{2}\right] \approx \frac{1}{4}\left(\frac{6+8+3}{6}\right) \approx \frac{17}{24} \approx 0.7083$$

The right answer is (a)

Q.3 The derivative of $\sin^{-1}(\sqrt{x}) - \frac{1}{2}\sin^{-1}(2x-1)$ is equal to

(a) $\dfrac{1}{2\sqrt{x(x-1)}}$ (b) 0 (c) $\dfrac{1}{2\sqrt{x(1-x)}} - \dfrac{1}{\sqrt{1-4x}}$ (d) 1

Answer : $\dfrac{d}{dx}\left(\sin^{-1}(\sqrt{x}) - \dfrac{1}{2}\sin^{-1}(2x-1)\right)$

$$= \frac{1}{\sqrt{1-(\sqrt{x})^2}}\ \frac{1}{2\sqrt{x}} - \frac{1}{2}\ \frac{2}{\sqrt{1-(2x-1)^2}}$$

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$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{\sqrt{1-(4x^2-4x+1)}}$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{\sqrt{1-4x^2+4x-1}}$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{\sqrt{4x-4x^2}}$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{\sqrt{4x(1-x)}}$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{\sqrt{4x}\,\sqrt{(1-x)}}$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{1-x}} - \frac{1}{2\sqrt{x}\,\sqrt{(1-x)}} = 0$$

The right answer is (b)

Q.4 $\frac{d}{dx}\left(\int_{6x-1}^{0} \sqrt{4t+9}\, dt\right)$ is equal to

(a) $-6\sqrt{24x+5}$ (b) $6\sqrt{24x+5}$ (c) $-\sqrt{24x+5}$ (d) $\sqrt{24x+5}$

The answer : Using Fundamental theorem of calculus

$$\frac{d}{dx}\left(\int_{6x-1}^{0} \sqrt{4t+9}\, dt\right) = 0 - \sqrt{4(6x-1)+9}\ (6)$$

$$= -6\sqrt{24x-4+9} = -6\sqrt{24x+5}$$

The right answer is (a)

Q.5 If $\log\left(\frac{x}{x-1}\right) = 1$ then x is equal to

(a) $\frac{10}{9}$ (b) $\frac{9}{10}$ (c) $\frac{9}{11}$ (d) $\frac{11}{9}$

The answer : $\log\left(\frac{x}{x-1}\right) = 1 \quad \Rightarrow \quad \log\left(\frac{x}{x-1}\right) = \log(10)$

$$\Rightarrow \quad \frac{x}{x-1} = 10 \quad \Rightarrow \quad x = 10(x-1)$$

$$\Rightarrow \quad x = 10x - 10 \quad \Rightarrow \quad 9x = 10 \quad \Rightarrow \quad x = \frac{10}{9}$$

The right answer is (a)

Full questions

Q.6 Evaluate the integral $\int 3^x \left(3^x + 3^{-x}\right)^2 \ dx$ **[4 marks]**

The answer :

$$\int 3^x \left(3^x + 3^{-x}\right)^2 \ dx = \int 3^x \left(3^{2x} + 2 \ 3^x \ 3^{-x} + 3^{-2x}\right) \ dx$$

$$= \int 3^x \left(3^{2x} + 2 + 3^{-2x}\right) \ dx = \int \left(3^{3x} + 2 \ 3^x + 3^{-x}\right) \ dx$$

$$= \frac{1}{3} \int 3^{3x} \ (3) \ dx + 2 \int 3^x \ dx - \int 3^{-x} \ (-1) \ dx$$

$$= \frac{1}{3} \ \frac{3^{3x}}{\ln 3} + 2 \ \frac{3^x}{\ln 3} - \frac{3^{-x}}{\ln 3} + c$$

Q.7 If $f(x) = \left(x^2 + x + 1\right)^{\sin x}$, then find $f'(x)$. **[4 marks]**

The answer :

$$f(x) = \left(x^2 + x + 1\right)^{\sin x} \quad \Rightarrow \quad \ln |f(x)| = \ln \left| \left(x^2 + x + 1\right)^{\sin x} \right|$$

$$\Rightarrow \ln |f(x)| = \sin x \ \ln(x^2 + x + 1)$$

Differentiate both sides

$$\frac{f'(x)}{f(x)} = \cos x \ \ln(x^2 + x + 1) + \sin x \ \frac{2x + 1}{x^2 + x + 1}$$

$$f'(x) = f(x) \ \left[\cos x \ \ln(x^2 + x + 1) + \frac{(2x + 1) \sin x}{x^2 + x + 1}\right]$$

$$f'(x) = \left(x^2 + x + 1\right)^{\sin x} \ \left[\cos x \ \ln(x^2 + x + 1) + \frac{(2x + 1) \sin x}{x^2 + x + 1}\right]$$

Q.8 Evaluate the integral $\int \frac{\tan\left(e^{-2x}\right)}{e^{2x}} \ dx$. **[4 marks]**

The answer : $\int \frac{\tan\left(e^{-2x}\right)}{e^{2x}} \ dx = \int \tan\left(e^{-2x}\right) \ e^{-2x} \ dx$

$$= -\frac{1}{2} \int \tan\left(e^{-2x}\right) \ e^{-2x} \ (-2) \ dx = -\frac{1}{2} \ln \left|\sec\left(e^{-2x}\right)\right| + c$$

Q.9 Evaluate the integral $\int \frac{sech^2(3x)}{\sqrt{1 + 9 \tanh^2(3x)}} \ dx$. **[4 marks]**

The answer :

$$\int \frac{sech^2(3x)}{\sqrt{1 + 9 \tanh^2(3x)}} \ dx = \frac{1}{9} \int \frac{3 \ sech^2(3x) \ (3)}{\sqrt{(1)^2 + (3 \tanh(3x))^2}} \ dx$$

$$= \frac{1}{9} \ \sinh^{-1}\left(3\tanh(3x)\right) + c$$

Q.10 Let $f(x) = x^2$

a) Approximate the area under the graph of f from 0 to 2 by subdividing the interval $[0, 2]$ into n equal parts, using a circumscribed rectanglular polygon (the right-hand endpoint).

b) Deduce the area under the graph of f corresponding to the interval $[0, 2]$.

The answer :

a) $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$

$x_i = x_0 + i \ \Delta x = 0 + i\left(\dfrac{2}{n}\right) = \dfrac{2i}{n}$, for every $i = 1, 2, \cdots, n$

The Riemann sum $R_n = \displaystyle\sum_{i=1}^{n} f\left(c_i\right) \ \Delta x$

Where c_i is the right-hand endpoint , $ci = x_i = \dfrac{2i}{n}$

$$R_n = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\frac{2}{n} = \sum_{i=1}^{n}\frac{4i^2}{n^2}\ \frac{2}{n}$$

$$R_n = \sum_{i=1}^{n}\frac{8}{n^3}\ i^2 = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\ \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{8}{6}\ \frac{n(n+1)(2n+1)}{n^3}$$

b) The area under the graph of f is equal to $\lim\limits_{n\to\infty} R_n$

Area$= \lim\limits_{n\to\infty}\left(\dfrac{8}{6}\ \dfrac{n(n+1)(2n+1)}{n^3}\right) = \dfrac{8}{3}\ (2) = \dfrac{8}{3}$
