M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[1]

Solution of the First Mid-Term Exam

First semester 1438-1439 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} -2 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Compute (if possible) : **A+B** and **BC**

Solution :

A+B is impossible.

$$\mathbf{BC} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 0+(-2) \\ -4+0 & 0+0 \\ 2+0 & 0+6 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \\ 2 & 6 \end{pmatrix}$$

Q.2 Compute The determinant $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ 0 & 4 & 0 & 0 & 4 \\ 2 & 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (12+0+0) - (8+0+0) = 12 - 8 = 4$$

Solution (2) : By the definition (using second row)

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 4 \times \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 4(3-2) = 4$$

[1] E-mail : alfadhel@ksu.edu.sa

Q.3 Solve by Gauss the linear system :

$$\begin{cases} x - 2y + z = 5 \\ y + 3z = 5 \\ -x + 3y - z = -6 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ -1 & 3 & -1 & -6 \end{array}\right) \xrightarrow{R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$-3z = -6 \implies z = 2$

$y + 3z = 5 \implies y + 6 = 5 \implies y = -1$

$x - 2y + z = 5 \implies x - 2(-1) + 2 = 5 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Q.4 Find the standard equation of the parabola with focus $F(5,1)$ and vertex $V(6,1)$, then sketch it.

Solution :

From the position of the focus and the vertex the parabola opens to the left.

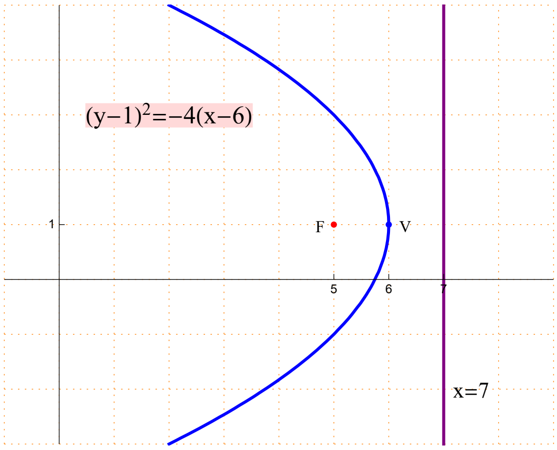
The equation of the parabola has the form $(y-k)^2 = -4a(x-h)$

The vertex is $V(6,1)$, hence $h = 6$, $k = 1$.

"a" is the distance between V and F, hence $a = 1$.

The standard equation of the parabola is $(y-1)^2 = -4(x-6)$

The equation of the directrix is $x = 7$.



Q.5 Find all the elements of the conic section $y^2 - 4x^2 + 10y + 8x + 17 = 0$ and sketch it.

Solution :

$y^2 - 4x^2 + 10y + 8x + 17 = 0$

$y^2 + 10y - 4x^2 + 8x = -17$

$y^2 + 10y - 4(x^2 - 2x) = -17$

By completing the square.

$(y^2 + 10y + 25) - 4(x^2 - 2x + 1) = -17 + 25 - 4$

$(y + 5)^2 - 4(x - 1)^2 = 4$

$$\frac{(y+5)^2}{4} - \frac{4(x-1)^2}{4} = 1$$

$$\frac{(y+5)^2}{4} - \frac{(x-1)^2}{1} = 1$$

The conic section is Hyperbola.

The center is $P = (1, -5)$

$a^2 = 1 \implies a = 1$

$b^2 = 4 \implies b = 2$

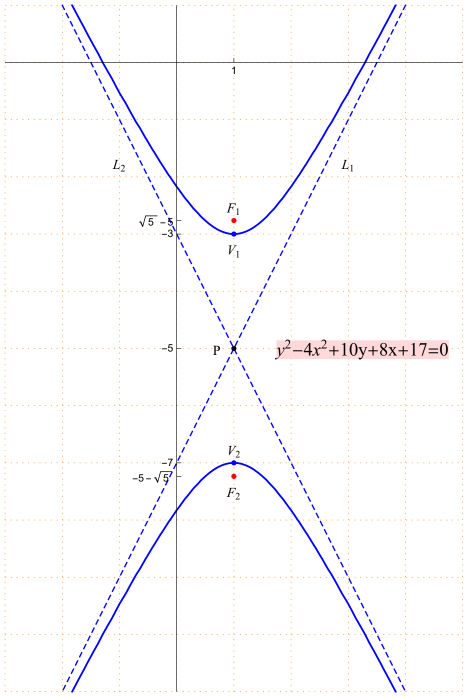
$c^2 = a^2 + b^2 = 1 + 4 = 5 \implies c = \sqrt{5}$

The vertices are $V_1 = (1, -3)$ and $V_2 = (1, -7)$

The foci are $F_1 = \left(1, -5 + \sqrt{5}\right)$ and $F_2 = \left(1, -5 - \sqrt{5}\right)$

The equations of the asymptotes are $L_1 : \ (y + 5) = 2(x - 1)$

and $L_2 : \ (y + 5) = -2(x - 1)$


1
L_2
L_1
F_1
$\sqrt{5}-5$
-3
V_1
-5
P
$y^2-4x^2+10y+8x+17=0$
V_2
-7
$-5-\sqrt{5}$
F_2

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Second Mid-Term Exam

First semester 1438-1439 H

Q.1 Compute the integrals :

(a) $\displaystyle\int 2x(x^2+1)^7 \ dx$

(b) $\displaystyle\int x^2 \cos\left(x^3\right) \ dx$

(c) $\displaystyle\int x^2 \ln x \ dx$

(d) $\displaystyle\int (x+1)e^x \ dx$

(e) $\displaystyle\int \frac{1}{x^2+6x+10} \ dx$

(f) $\displaystyle\int \frac{x+2}{(x-2)(x-4)} \ dx$

Solution :

(a) $\displaystyle\int 2x(x^2+1)^7 dx = \int (x^2+1)^7(2x)dx = \frac{(x^2+1)^8}{8} + c$

Using the formula $\displaystyle\int [f(x)]^n f'(x) \ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

(b) $\displaystyle\int x^2 \cos\left(x^3\right) \ dx = \frac{1}{3}\int \cos\left(x^3\right) \ (3x^2) \ dx = \frac{1}{3}\sin\left(x^3\right) + c$

Using the formula $\displaystyle\int \cos\left(f(x)\right) f'(x) \ dx = \sin\left(f(x)\right) + c$

(c) $\displaystyle\int x^2 \ln x \ dx$

Using integration by parts

$u = \ln x \qquad dv = x^2 \ dx$

$du = \dfrac{1}{x} \ dx \qquad v = \dfrac{x^3}{3}$

$\displaystyle\int x^2 \ln x \ dx = \frac{x^3}{3}\ln x - \int \frac{1}{x}\frac{x^3}{3} \ dx$

$\displaystyle= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 \ dx = \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + c$

(d) $\int (x+1)e^x \; dx$

Using integration by parts :

$$u = x+1 \qquad dv = e^x \; dx$$
$$du = 1 \; dx \qquad v = e^x$$

$$\int (x+1)e^x \; dx = (x+1)e^x - \int e^x \; dx = (x+1)e^x - e^x + c = xe^x + c$$

(e) $\displaystyle\int \frac{1}{x^2+6x+10} \; dx = \int \frac{1}{(x^2+6x+9)+1} \; dx = \int \frac{1}{(x+3)^2+1} \; dx$

$= \tan^{-1}(x+3) + c$

Using the formula $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2} \; dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$, where $a > 0$

(f) $\displaystyle\int \frac{x+2}{(x-2)(x-4)} \; dx$

Using the method of partial fractions

$$\frac{x+2}{(x-2)(x-4)} = \frac{A_1}{x-2} + \frac{A_2}{x-4}$$

$x + 2 = A_1(x-4) + A_2(x-2)$

Put $x = 2$:

$2 + 2 = A_1(2-4) \implies 4 = -2A_1 \implies A_1 = -2$

Put $x = 4$:

$4 + 2 = A_2(4-2) \implies 6 = 2A_2 \implies A_2 = 3$

$$\int \frac{x+2}{(x-2)(x-4)} \; dx = \int \left(\frac{-2}{x-2} + \frac{3}{x-4}\right) \; dx$$

$$= -2\int \frac{1}{x-2} \; dx + 3\int \frac{1}{x-4} \; dx = -2\ln|x-2| + 3\ln|x-4| + c$$

Q.2 (a) Sketch the region R_1 determined by the curves

$$y = x^2 - 1 \; , \; y = -1 \; , \; x = 1$$

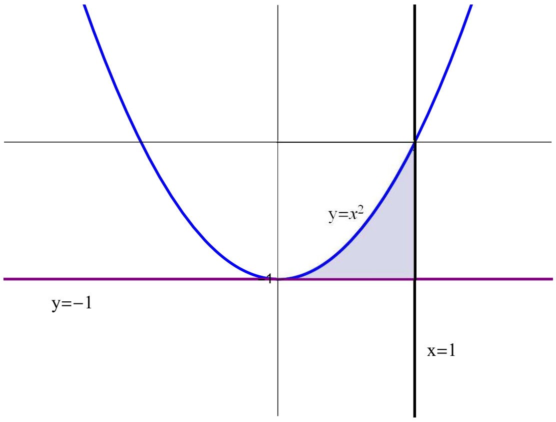
(b) Find the area of the region R_1 described in part (a) .

Solution :

(a) $y = x^2 - 1$ is a parabola opens upwards with vertex $(0, -1)$

$y = -1$ is a straight line parallel to the x-axis and passes through $(0, -1)$

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$



(b) Area $= \int_0^1 \left[(x^2 - 1) - (-1)\right]\ dx = \int_0^1 x^2\ dx$

$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Q.3 (a) Sketch the region R_2 determined by the curves

$$y = x^2\ ,\ x = 2\ ,\ y = 0$$

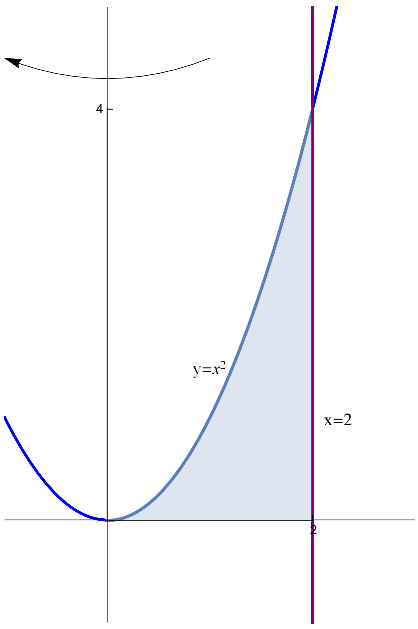
(b) Find the volume of the solid generated by rotating the region R_2 in part (a) about the y-axis .

Solution :

(a) $y = 0$ is the x-axis

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$

$x = 2$ is a straight line parallel to the y-axis and passes through $(2, 0)$



(b) Using Cylindrical Shells method :

$$\text{Volume} = 2\pi \int_0^2 x\left(x^2\right)\ dx = 2\pi \int_0^2 x^3\ dx$$

$$= 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi \left[\frac{2^4}{4} - \frac{0^4}{4}\right] = 2\pi \left[\frac{16}{4}\right] = 8\pi$$

Another solution : Using Washer Method

$y = x^2 \implies x = \sqrt{y}$

$$\text{Volume} = \pi \int_0^4 \left[(2)^2 - (\sqrt{y})^2\right]\ dy = \pi \int_0^4 (4 - y)\ dy$$

$$= \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi \left[\left(4 \times 4 - \frac{4^2}{2}\right) - (0 - 0)\right] = \pi\,(16 - 8) = 8\pi$$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Final Exam
First semester 1438-1439 H

Q.1 **(a)** Compute (if possible) **AB** for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 6 \\ 0 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 5 & 0 & 1 \end{vmatrix}$.

(c) Solve by Gauss Method : $\begin{cases} x - y + z = 1 \\ 2x - 5y + z = -3 \\ 3x - 6y - z = -8 \end{cases}$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 6 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+4+3 & 0+2+3 & 1+8+3 \\ 3+2+6 & 0+1+6 & 3+4+6 \\ 0+4+0 & 0+2+0 & 0+8+0 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 12 \\ 11 & 7 & 13 \\ 4 & 2 & 8 \end{pmatrix}$$

(b) **Solution (1):** Using Sarrus Method

$$\begin{matrix} 0 & 1 & 2 & 0 & 1 \\ 2 & 3 & 4 & 2 & 3 \\ 5 & 0 & 1 & 5 & 0 \end{matrix}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 5 & 0 & 1 \end{vmatrix} = (0+20+0) - (30+0+2) = 20 - 32 = -12$$

Solution (2) : Using the definition (using the first row) :

$$\begin{vmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 5 & 0 & 1 \end{vmatrix} = 0 \times \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}$$

$$= 0 - (2-20) + 2(0-15) = 0 - (-18) - 30 = 18 - 30 = -12$$

(c) Using Gauss Method :

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & 1 & 3 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -3 & -4 & -11 \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$-3z = -6 \implies z = \dfrac{-6}{-3} = 2$

$-3y - z = -5 \implies -3y - 2 = -5 \implies -3y = -3 \implies y = \dfrac{-3}{-3} = 1$

$x - y + z = 1 \implies x - 1 + 2 = 1 \implies x + 1 = 1 \implies x = 0$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the ellipse with foci $(-2, 3)$ and $(4, 3)$, and with vertex $(5, 3)$.

(b) Find the elements of the conic section $9x^2 - 4y^2 + 18x - 24y = 63$.

Solution :

(a) The two foci and the vertex are located on a line parallel to the x-axis.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a > b$.

$P(h, k) = \left(\dfrac{-2+4}{2}, \dfrac{3+3}{2}\right) = (1, 3)$, hence $h = 0$ and $k = 0$

a is the distance between the vertex $(5, 3)$ and P , hence $a = 4$

c is the distance between one of the foci and P , hence $c = 3$

$c^2 = a^2 - b^2 \implies 9 = 16 - b^2 \implies b^2 = 16 - 9 = 7 \implies b = \sqrt{7}$

The standard equation of the ellipse is $\dfrac{(x-1)^2}{16} - \dfrac{(y-3)^2}{7} = 1$

The other vertex is $(-3, 3)$

The end-points of the minor axis are $\left(1, 3 - \sqrt{7}\right)$ and $\left(1, 3 + \sqrt{7}\right)$.

(b) $9x^2 - 4y^2 + 18x - 24y = 63$

$9x^2 + 18x - 4y^2 - 24y = 63$

$9(x^2 + 2x) - 4(y^2 + 6y) = 63$

By completing the square

$9(x^2 + 2x + 1) - 4(y^2 + 6y + 9) = 63 + 9 - 36$

$9(x + 1)^2 - 4(y + 3)^2 = 36$

$\frac{9(x+1)^2}{36} - \frac{4(y+3)^2}{36} = 1$

$\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

The conic section is a hyperbola.

The center is $P(-1,-3)$.

$a^2 = 4 \implies a = 2.$

$b^2 = 9 \implies b = 3.$

$c^2 = a^2 + b^2 = 4 + 9 = 13 \implies c = \sqrt{13}.$

The vertices are $V_1(-3,-3)$ and $V_2(1,-3)$

The foci are $F_1\left(-1-\sqrt{13},-3\right)$ and $F_2\left(-1+\sqrt{13},-3\right)$.

The equations of the asymptotes are :

$L_1 \;:\; y+3 = \frac{3}{2}\,(x+1)$ and $L_2 \;:\; y+3 = -\frac{3}{2}\,(x+1)$

Q.3 **(a)** Compute the integrals :

(i) $\int \frac{x+1}{(x-2)(x-3)}\,dx$ (ii) $\int x\ln x\,dx$ (iii) $\int x\sin\left(x^2\right)\,dx$

(b) Find the area of the region bounded by the curves :

$y = \sqrt{x}$ and $y = x^2$.

(c) The region R between the curves $y = 0$, $x = 1$ and $y = \sqrt{x}$ is rotated about the x-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\int \frac{x+1}{(x-2)(x-3)}\,dx$

Using the method of partial fractions

$\frac{x+1}{(x-2)(x-3)} = \frac{A_1}{x-2} + \frac{A_2}{x-3}$

$x+1 = A_1(x-3) + A_2(x-2)$

Put $x = 2$ then $2+1 = A_1(2-3) \implies 3 = -A_1 \implies A_1 = -3$

Put $x = 3$ then $3+1 = A_2(3-2) \implies A_2 = 4$

$\int \frac{x+1}{(x-2)(x-3)}\,dx = \int \left(\frac{-3}{x-2} + \frac{4}{x-3}\right)\,dx$

$= -3\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x-3}\,dx = -3\ln|x-2| + 4\ln|x-3| + c$

(ii) $\displaystyle\int x \ln x \; dx$

Using integration by parts

$$u = \ln x \qquad dv = x \; dx$$
$$du = \frac{1}{x} \; dx \qquad v = \frac{x^2}{2}$$

$$\int x \ln x \; dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \; \frac{1}{x} x \; dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \; dx = \frac{x^2}{2} \ln x - \frac{1}{2} \; \frac{x^2}{2} + c$$

(iii) $\displaystyle\int x \sin\left(x^2\right) \; dx$

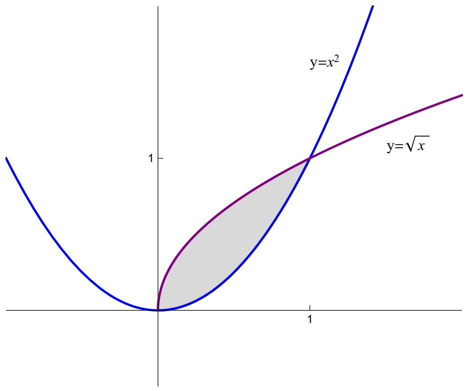
$$\int x \sin\left(x^2\right) \; dx = \frac{1}{2} \int \sin\left(x^2\right) \; (2x) \; dx$$

$$= \frac{1}{2}\left(-\cos\left(x^2\right)\right) + c = -\frac{1}{2} \cos\left(x^2\right) + c$$

Using the formula $\displaystyle\int \sin\left(f(x)\right) \; f'(x) \; dx = -\cos\left(f(x)\right) + c$

(b) $y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

$y = \sqrt{x}$ is the upper-half of the parabola $x = y^2$ which opens to the right with vertex $(0, 0)$,



Points of intersection of $y = x^2$ and $y = \sqrt{x}$:

$$x^2 = \sqrt{x} \implies x^4 = x \implies x^4 - x = 0 \implies x(x^3 - 1) = 0$$

$$\implies x = 0 \; , \; x^3 - 1 = 0 \implies x = 0 \; , \; x = 1$$

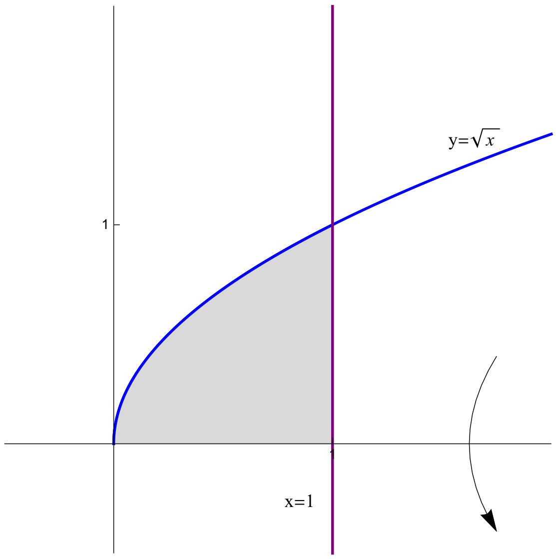
$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right) \; dx = \int_0^1 \left(x^{\frac{1}{2}} - x^2\right) \; dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3} \; x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{2}{3}\,(1)^{\frac{3}{2}} - \frac{(1)^3}{3}\right) - \left(\frac{2}{3}\,(0)^{\frac{3}{2}} - \frac{(0)^3}{3}\right) = \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0) = \frac{1}{3}$$

(c) $y = 0$ is the x-axis .

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$.

$y = \sqrt{x}$ is the upper-half of the parabola $x = y^2$ which opens to the right with vertex $(0, 0)$,



Using Disk Method :

$$\text{Volume} = \pi \int_0^1 \left(\sqrt{x}\right)^2 \; dx = \pi \int_0^1 x \; dx = \pi \left[\frac{x^2}{2}\right]_0^1$$

$$= \pi \left[\frac{(1)^2}{2} - \frac{(0)^2}{2}\right] = \pi \left(\frac{1}{2} - 0\right) = \frac{\pi}{2}$$

Q.4 **(a)** Find f_x , f_y and f_z for the function $f(x, y, z) = xy^3z + \ln\left(xz^4\right)$.

(b) Solve the differential equation $\dfrac{dy}{dx} - 3x^2y^2 = 0$.

Solution :

(a) $f_x = y^3z(1) + \dfrac{z^4(1)}{xz^4} = y^3z + \dfrac{1}{x}$

$f_y = xz\left(3y^2\right) + 0 = 3xy^2z$

$f_z = xy^3(1) + \dfrac{x\left(4z^3\right)}{xz^4} = xy^3 + \dfrac{4}{z}$

(b) $\frac{dy}{dx} - 3x^2y^2 = 0$

$\frac{dy}{dx} = 3x^2y^2$

$\frac{1}{y^2}\ dy = 3x^2\ dx$

It is a separable differential equation.

$\int \frac{1}{y^2}\ dy = \int 3x^2\ dx$

$\int y^{-2}\ dy = \int 3x^2\ dx$

$\frac{y^{-1}}{-1} = x^3 + c$

$\frac{-1}{y} = x^3 + c$

$\frac{1}{y} = -x^3 - c$

$y = \frac{1}{-x^3 - c} = \frac{-1}{x^3 + c}$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the First Mid-Term Exam
Second semester 1438-1439 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix}$.

Compute (if possible) : $\mathbf{AB}$ and $\mathbf{B} + \mathbf{C}$

Solution :

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4+3 & 1+6+3 \\ 3+4+2 & 1+6+2 \\ 6-2+1 & 2-3+1 \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 9 & 9 \\ 5 & 0 \end{pmatrix}$$

$$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 7 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3+7 & 1+1 \\ 2+3 & 3+6 \\ 1+1 & 1+3 \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 5 & 9 \\ 2 & 4 \end{pmatrix}$$

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

Solution : Using Sarrus Method

$$\begin{matrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 & 3 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = (1+4+6) - (2+3+4) = 11 - 9 = 2$$

Q.3 Solve by Gauss the linear system :

$$\begin{cases} x - 2y + z = 0 \\ x - 3y - z = -2 \\ 2x + 2y - z = 4 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -3 & -1 & -2 \\ 2 & 2 & -1 & 4 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -2 \\ 2 & 2 & -1 & 4 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & 6 & -3 & 4 \end{array}\right) \xrightarrow{6R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -15 & -8 \end{array}\right)$$

$$-15z = -8 \implies z = \frac{-8}{-15} = \frac{8}{15}$$

$$-y - 2z = -2 \implies -y - 2\left(\frac{8}{15}\right) = -2 \implies -y - \frac{16}{15} = -2$$

$$\implies -y = -2 + \frac{16}{15} \implies -y = \frac{-30+16}{16} = -\frac{14}{15} \implies y = \frac{14}{15}$$

$$x - 2y + z = 0 \implies x - 2\left(\frac{14}{15}\right) + \frac{8}{15} = 0 \implies x - \frac{28}{15} + \frac{8}{15} = 0$$

$$\implies x = \frac{28}{15} - \frac{8}{15} = \frac{20}{15} = \frac{4}{3}$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{14}{15} \\ \frac{8}{15} \end{pmatrix}$

Q.4 Find all the elements of the conic section $x^2 - 4y^2 + 2x + 8y - 7 = 0$ and sketch it.

Solution :

$x^2 - 4y^2 + 2x + 8y - 7 = 0$

$x^2 + 2x - 4y^2 + 8y = 7$

$x^2 + 2x - 4(y^2 - 2y) = 7$

By completing the square.

$(x^2 + 2x + 1) - 4(y^2 - 2y + 1) = 7 + 1 - 4$

$(x+1)^2 - 4(y-1)^2 = 4$

$$\frac{(x+1)^2}{4} - \frac{4(y-1)^2}{4} = 1$$

$$\frac{(x+1)^2}{4} - \frac{(y-1)^2}{1} = 1$$

The conic section is Hyperbola.

The center is $P(-1, 1)$

$a^2 = 4 \implies a = 2$

$b^2 = 1 \implies b = 1$

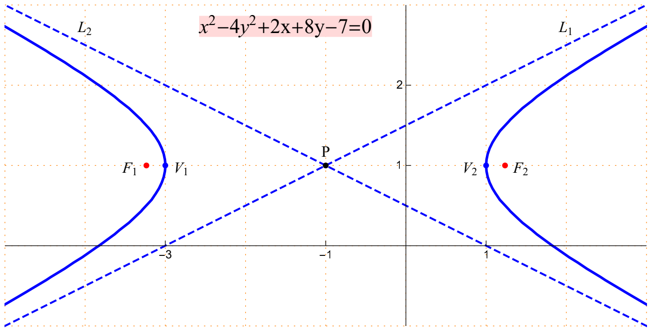
$c^2 = a^2 + b^2 = 4 + 1 = 5 \implies c = \sqrt{5}$

The vertices are $V_1(-3,1)$ and $V_2(1,1)$

The foci are $F_1\left(-1-\sqrt{5},1\right)$ and $F_2\left(-1+\sqrt{5},1\right)$

The equations of the asymptotes are $L_1:\ (y-1)=\dfrac{1}{2}(x+1)$

and $L_2:\ (y-1)=-\dfrac{1}{2}(x+1)$



Q.5 Find the standard equation of the ellipse with foci $F_1(5,1)$ and $F_2(5,7)$ with vertex $V(5,8)$, then sketch it.

Solution :

The two foci and the vertex are located on a line parallel to the y-axis.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $b>a$.

$P(h,k)=\left(\dfrac{5+5}{2},\dfrac{1+7}{2}\right)=(5,4)$, hence $h=5$ and $k=4$

c is the distance between one of the foci and P , hence $c=3$

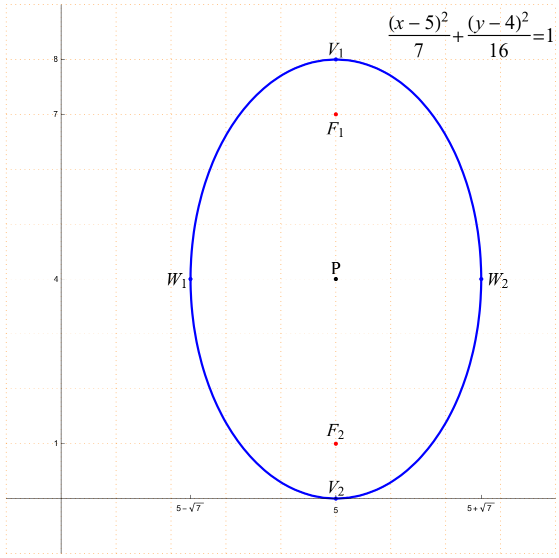
b is the distance between the vertex $(5,8)$ and P , hence $b=4$

$c^2=b^2-a^2 \implies 9=16-a^2 \implies a^2=16-9=7 \implies a=\sqrt{7}$

The standard equation of the ellipse is $\dfrac{(x-5)^2}{7}+\dfrac{(y-4)^2}{16}=1$

The other vertex is $(5,0)$

The end-points of the minor axis are $\left(5-\sqrt{7},4\right)$ and $\left(5+\sqrt{7},4\right)$.


$\frac{(x-5)^2}{7}+\frac{(y-4)^2}{16}=1$
V_1
F_1
P
W_1
W_2
F_2
V_2
8
7
4
1
$5-\sqrt{7}$
5
$5+\sqrt{7}$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Second Mid-Term Exam

Second semester 1438-1439 H

Q.1 Compute the integrals :

(a) $\displaystyle\int \frac{2x}{x^2+1}\,dx$

(b) $\displaystyle\int x\cos x\,dx$

(c) $\displaystyle\int x\ln x\,dx$

(d) $\displaystyle\int xe^{x^2}\,dx$

(e) $\displaystyle\int \frac{x}{(x-2)^2}\,dx$

(f) $\displaystyle\int \frac{1}{(x-1)(x-2)}\,dx$

Solution :

(a) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1)+c$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$

(b) $\displaystyle\int x\cos x\,dx$

Using integration by parts

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int x\ \cos x\,dx = x\ \sin x - \int \sin x\,dx$$

$$= x\ \sin x - (-\cos x) + c = x\ \sin x + \cos x + c$$

(c) $\displaystyle\int x\ln x\,dx$

Using integration by parts

$u = \ln x \qquad dv = x\,dx$

$du = \dfrac{1}{x}\,dx \qquad v = \dfrac{x^2}{2}$

$$\int x^2 \ln x \; dx = \frac{x^2}{2} \ln x - \int \frac{1}{x} \frac{x^2}{2} \; dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \; dx = \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + c$$

(d) $\displaystyle\int x e^{x^2} \; dx = \frac{1}{2} \int e^{x^2} (2x) \; dx = \frac{1}{2} e^{x^2} + c$

Using the formula $\displaystyle\int e^{f(x)} f'(x) \; dx = e^{f(x)} + c$

(e) $\displaystyle\int \frac{x}{(x-2)^2} \; dx$

Using the method of partial fractions

$$\frac{x}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} = \frac{A_1(x-2) + A_2}{(x-2)^2}$$

$x = A_1(x-2) + A_2 = A_2 x - 2A_1 + A_2$

By comparing the coefficients of the two polynomials in each side :

$A_1 = 1$

$-2A_1 + A_2 = 0 \implies -2 + A_2 = 0 \implies A_2 = 2$

$$\int \frac{x}{(x-2)^2} \; dx = \int \left(\frac{1}{x-2} + \frac{2}{(x-2)^2} \right) \; dx$$

$$= \int \frac{1}{x-2} \; dx + 2 \int (x-2)^{-2} \; dx$$

$$= \ln|x-2| + 2 \; \frac{(x-2)^{-1}}{-1} + c = \ln|x-2| - \frac{2}{x-2} + c$$

(f) $\displaystyle\int \frac{1}{(x-1)(x-2)} \; dx$

Using the method of partial fractions

$$\frac{1}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} = \frac{A_1(x-2) + A_2(x-1)}{(x-1)(x-2)}$$

$1 = A_1(x-2) + A_2(x-1)$

Put $x = 1$:

$1 = A_1(1-2) \implies 1 = -A_1 \implies A_1 = -1$

Put $x = 2$:

$1 = A_2(2-1) \implies A_2 = 1$

$$\int \frac{1}{(x-1)(x-2)}\, dx = \int \left(\frac{-1}{x-1} + \frac{1}{x-2} \right)\, dx$$

$$= -\int \frac{1}{x-1}\, dx + \int \frac{1}{x-2}\, dx = -\ln|x-1| + \ln|x-2| + c$$

Q.2 (a) Sketch the region **R** determined by the curves

$$y = x^2 \ , \ y = 0 \ , \ x = 2$$

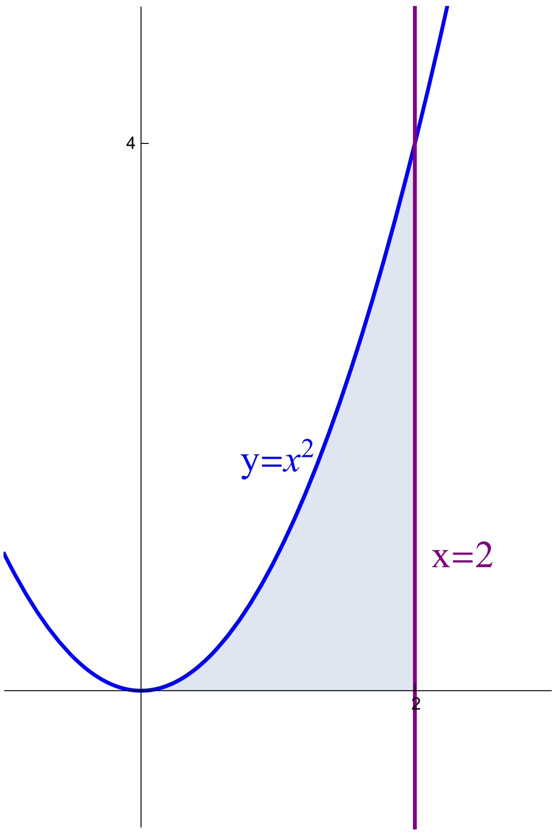
(b) Find the area of the region **R** described in part (a) .

Solution :

(a) $y = x^2$ is a parabola opens upwards with vertex $(0, 0)$

$y = 0$ is the x-axis

$x = 2$ is a straight line parallel to the y-axis and passes through $(2, 0)$



(b) $\text{Area} = \displaystyle\int_0^2 x^2\ dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$

Q.3 (a) Sketch the region **R** determined by the curves

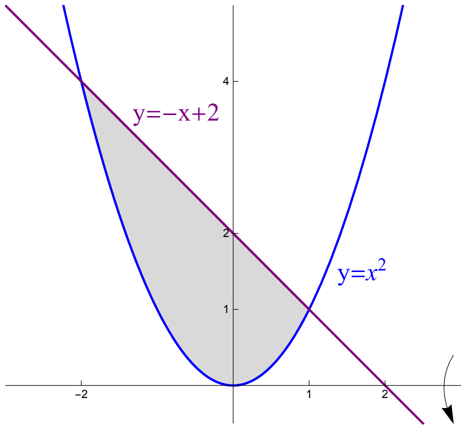
$$y = x^2 \ , \ y = -x + 2$$

(b) Find the volume of the solid generated by rotating the region **R** in part (a) about the x-axis .

Solution :

(a) $y = x^2$ is a parabola opens upwards with vertex $(0, 0)$

$y = -x + 2$ is a straight line passes through $(0, 2)$ with slope equals -1



(b) Points of intersection of $y = x^2$ and $y = -x + 2$:

$x^2 = -x + 2 \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0$

$\implies x = -2 \ , \ x = 1$

Using Washer method :

$$\text{Volume} = \pi \int_{-2}^{1} \left[(-x+2)^2 - \left(x^2\right)^2 \right] \, dx = \pi \int_{-2}^{1} \left[(x^2 - 4x + 4) - x^4 \right] \, dx$$

$$= \pi \int_{-2}^{1} \left[-x^4 + x^2 - 4x + 4 \right] \, dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x \right]_{-2}^{1}$$

$$= \pi \left[\left(-\frac{1^5}{5} + \frac{1^3}{3} - 2(1^2) + 4(1) \right) - \left(-\frac{(-2)^5}{5} + \frac{(-2)^3}{3} - 2((-2)^2) + 4(-2) \right) \right]$$

$$= \pi \left[-\frac{1}{5} + \frac{1}{3} - 2 + 4 - \left(\frac{32}{5} - \frac{8}{3} - 8 - 8 \right) \right] = \pi \left(-\frac{33}{5} + 3 + 2 + 16 \right)$$

$$= \pi \left(21 - \frac{33}{5} \right) = \frac{72\pi}{5}$$