M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[1]

Solution of the First Mid-Term Exam

First semester 1437-1438 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix}$.

Compute (if possible) : **AB** and **B+C**

Solution :

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 1+4+3 \\ 0+0+0 & 0+4+2 \\ 2+0+0 & 2+0+1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 6 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{B+C} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 3 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+1 \\ 0+3 & 2+6 \\ 0+1 & 1+3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \\ 1 & 4 \end{pmatrix}$$

Q.2 Compute The determinant $\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{matrix} 1 & -2 & 1 & 1 & -2 \\ 2 & 1 & -1 & 2 & 1 \\ -2 & 0 & 1 & -2 & 0 \end{matrix}$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = (1+(-4)+0)-(-2+0+(-4)) = -3-(-6) = -3+6 = 3$$

Solution (2) : By the definition (using third row)

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = -2 \times \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= -2(2-1) - 0 + 1(1-(-4)) = -2(1) + 1(5) = -2+5 = 3$$

[1]E-mail : alfadhel@ksu.edu.sa

Q.3 Solve by Gauss-Jordan the linear system :

$$\begin{cases} x & - & 2y & + & z & = & 0 \\ 2x & - & 3y & - & z & = & -2 \\ -2x & & & + & z & = & -1 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & -2 \\ -2 & 0 & 1 & -1 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ -2 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -4 & 3 & -1 \end{array}\right) \xrightarrow{4R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -9 & -9 \end{array}\right)$$

$$\xrightarrow{\frac{1}{-9}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{3R_3+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3+R_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{2R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Q.4 Find all the elements of the conic section $y^2 - 4x^2 + 2y + 8x - 7 = 0$ and sketch it.

Solution :

$y^2 - 4x^2 + 2y + 8x - 7 = 0$

$y^2 + 2y - 4x + 8x = 7$

$(y^2 + 2y) - 4(x^2 - 2x) = 7$

By completing the square

$(y^2 + 2y + 1) - 4(x^2 - 2x + 1) = 7 + 1 - 4$

$(y + 1)^2 - 4(x - 1)^2 = 4$

$\dfrac{(y+1)^2}{4} + \dfrac{4(x-1)^2}{4} = 1$

$\dfrac{(y+1)^2}{4} + \dfrac{(x-1)^2}{1} = 1$

The conic section is a Hyperbola .

The center is $P(1, -1)$.

$a^2 = 1 \implies a = 1$

$b^2 = 4 \implies b = 2$

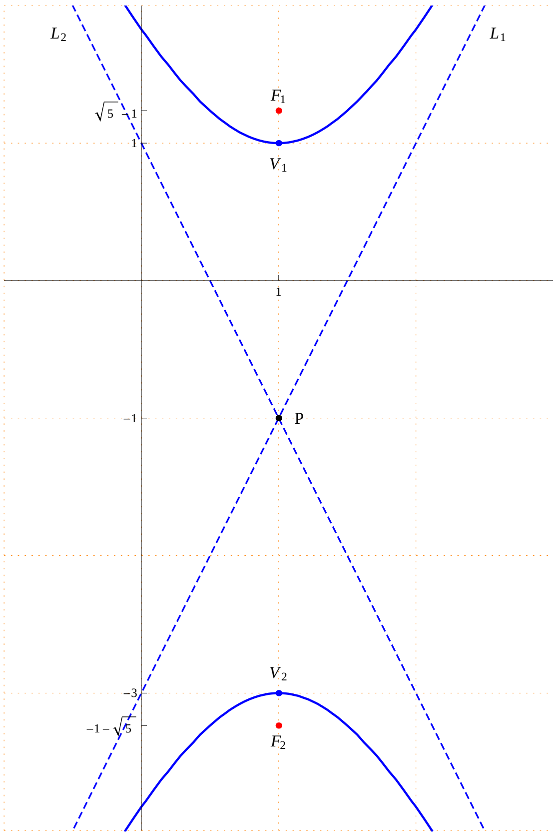
$c^2 = a^2 + b^2 = 1 + 4 = 5 \implies c = \sqrt{5}$

The vertices are $V_1(1, 1)$ and $V_2(1, -3)$

The foci are $F_1\left(1, -1 + \sqrt{5}\right)$ and $F_2\left(1, -1 - \sqrt{5}\right)$

The equations of the asymptotes are $L_1: \ (y+1) = 2(x-1)$

and $L_2: \ (y+1) = -2(x-1)$



Q.5 Find the standard equation of the parabola with focus $F(5, 1)$ and with directrix $x = -1$, and sketch it.

Solution :

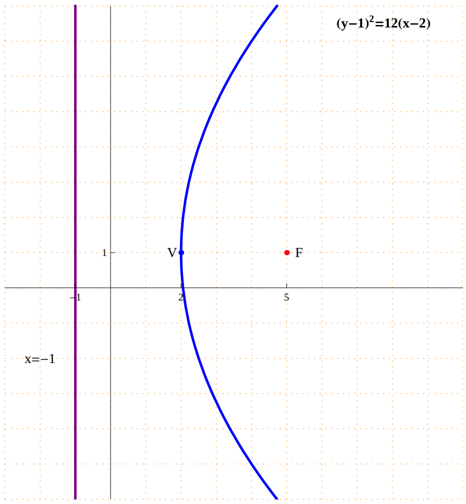
From the positions of the focus and directrix the parabola opens to the right.

The equation of the parabola has the form $(y-k)^2 = 4a(x-h)$.

The vertex lies between the focus and the directrix , hence the vertex is $V(2,1)$

a is the distance between the focus and the vertex , hence $a = 3$

The standard equation of the parabola is $(y-1)^2 = 12(x-2)$



M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Second Mid-Term Exam
First semester 1437-1438 H

Q.1 Compute the integrals :

(a) $\displaystyle\int_0^1 16x(x^2-1)^7\ dx$

(b) $\displaystyle\int \frac{2x-1}{(x-2)(x-3)}\ dx$

(c) $\displaystyle\int (x+1)\sin x\ dx$

(d) $\displaystyle\int x^3 \ln|x|\ dx$

(e) $\displaystyle\int \frac{2x}{(x+1)^2}\ dx$

Solution :

(a) $\displaystyle\int_0^1 16x(x^2-1)^7\ dx = 8\int_0^1 (x^2-1)^7\ 2x\ dx = 8\left[\frac{(x^2-1)^8}{8}\right]_0^1$

$\displaystyle = 8\left[\frac{(1-1)^8}{8} - \frac{(0-1)^8}{8}\right] = 8\left(0-\frac{1}{8}\right) = 0-1=-1$

Using the formula $\displaystyle\int [f(x)]^n f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

(b) $\displaystyle\int \frac{2x-1}{(x-2)(x-3)}\ dx$

Using the method of partial fractions

$\displaystyle\frac{2x-1}{(x-2)(x-3)} = \frac{A_1}{x-2} + \frac{A_2}{x-3}$

$2x-1 = A_1(x-3)+A_2(x-2)$

Put $x=2$:

$2(2)-1 = A_1(2-3) \implies A_1=-3$

Put $x=3$:

$2(3)-1 = A_2(3-2) \implies A_2 = 5$

$\displaystyle\int \frac{2x-1}{(x-2)(x-3)}\ dx = \int\left(\frac{-3}{x-2}+\frac{5}{x-3}\right)\ dx$

$\displaystyle = -3\int\frac{1}{x-2}\ dx + 5\int\frac{1}{x-3}\ dx = -3\ln|x-2| + 5\ln|x-3| + c$

(c) $\int (x+1)\sin x \; dx$

Using integration by parts

$$\begin{array}{ll} u = x+1 & dv = \sin x \; dx \\ du = dx & v = -\cos x \end{array}$$

$$\int (x+1)\sin x \; dx = (x+1)(-\cos x) - \int -\cos x \; dx$$

$$= -(x+1)\cos x + \int \cos x \; dx = -(x+1)\cos x + \sin x + c$$

(d) $\int x^3 \ln|x| \; dx$

Using integration by parts :

$$\begin{array}{ll} u = \ln|x| & dv = x^3 \; dx \\ du = \frac{1}{x} \; dx & v = \frac{x^4}{4} \end{array}$$

$$\int \ln|x| \; dx = \frac{x^4}{4}\ln|x| - \int \frac{x^4}{4}\,\frac{1}{x} \; dx$$

$$= \frac{x^4}{4}\ln|x| - \frac{1}{4}\int x^3 \; dx = \frac{x^4}{4}\ln|x| - \frac{1}{4}\,\frac{x^4}{4} + c$$

(e) $\int \frac{2x}{(x+1)^2} \; dx$

Using the method of partial fractions

$$\frac{2x}{(x+1)^2} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2}$$

$$\frac{2x}{(x+1)^2} = \frac{A_1(x+1)}{(x+1)^2} + \frac{A_2}{(x+1)^2}$$

$$2x = A_1(x+1) + A_2 = A_1 x + (A_1 + A_2)$$

By comparing the coefficients of both sides :

$$\begin{array}{lll} A_1 = 2 & \longrightarrow & (1) \\ A_1 + A_2 = 0 & \longrightarrow & (2) \end{array}$$

From equations (1) and (2) : $A_1 = 2$ and $A_2 = -2$

$$\int \frac{2x}{(x+1)^2} \; dx = \int \left(\frac{2}{x+1} + \frac{-2}{(x+1)^2} \right) \; dx$$

$$= 2\int \frac{1}{x+1} \; dx - 2\int (x+1)^{-2} \; dx = 2\ln|x+1| - 2\frac{(x+1)^{-1}}{-1} + c$$

$= 2\ln|x+1| + \dfrac{2}{x+1} + c$

Q.2 Find the area of the region bounded by the curves :

$y = x^2 + 1$ and $y = 2$

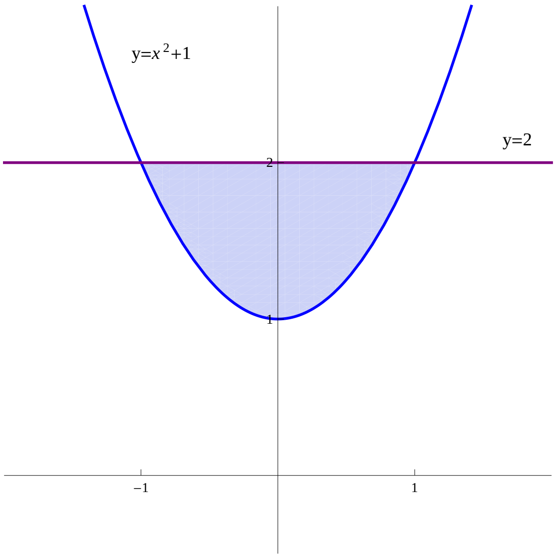
Solution :

$y = 2$ is a straight line parallel to the x-axis and passes through $(0, 2)$

$y = x^2 + 1$ is a parabola opens upwards with vertex $(0, 1)$

Points of intersection of $y = x^2 + 1$ and $y = 2$:

$x^2 + 1 = 2 \implies x^2 = 1 \implies x = \pm 1$



$$\text{Area} = \int_{-1}^{1} \left[2 - (x^2 + 1)\right]\ dx = \int_{-1}^{1} (2 - x^2 - 1)\ dx = \int_{-1}^{1} (1 - x^2)\ dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) = 1 - \frac{1}{3} + 1 - \frac{1}{3}$$

$$= 2 - \frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}$$

Q.3 Find the volume of the solid of revolution generated by rotation about the x-axis of the region **R** limited by the following curves :

$y = 0$, $x = -1$, $x = 1$ and $y = 2x^2$

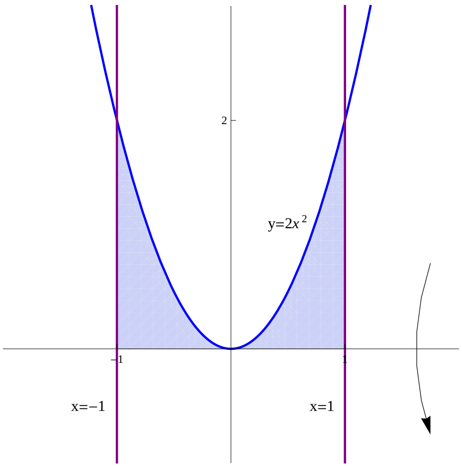
Solution :

$y = 0$ is the x-axis

$x = -1$ is a straight line parallel to the y-axis and passes through $(-1, 0)$

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$

$y = 2x^2$ is a parabola opens upwards with vertex $(0, 0)$



Using Disk method :

$$\text{Volume} = \pi \int_{-1}^{1} \left(2x^2\right)^2 \ dx = \pi \int_{-1}^{1} 4x^4 \ dx = 4\pi \int_{-1}^{1} x^4 \ dx$$

$$= 4\pi \left[\frac{x^5}{5}\right]_{-1}^{1} = 4\pi \left[\frac{1}{5} - \left(\frac{-1}{5}\right)\right] = 4\pi \ \left(\frac{2}{5}\right) = \frac{8\pi}{5}$$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Final Exam
First semester 1437-1438 H

Q.1 **(a)** Compute (if possible) **AB** and **BA** for $\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

and $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix}$.

(c) Solve by Gauss Method : $\begin{cases} x & - & y & + & z & = & 2 \\ x & - & 2y & + & z & = & 0 \\ 2x & - & y & + & z & = & 3 \end{cases}$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 0+0+2 & 0+3+4 \\ 2+0+4 & 6+3+8 \\ 2+0+2 & 6+3+4 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 6 & 17 \\ 4 & 13 \end{pmatrix}$$

BA is impossible , because the number of columns of **B** does not equal the number of rows of **A**

(b) **Solution (1):** Using Sarrus Method

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & 1 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (3+4+3) - (18+1+2) = 10 - 21 = -11$$

Solution (2) : Using the definition (using the first row) :

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (3-1) - 2(1-2) + 3(1-6) = 2 + 2 - 15 = -11$$

(c) Using Gauss Method :

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & 1 & 3 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & -1 & -1 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

$-z = -3 \implies z = 3$

$-y = -2 \implies y = 2$

$x - y + z = 2 \implies x - 2 + 3 = 2 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the hyperbola with foci $F_1(2,0)$, $F_2(-2,0)$ and a vertex $V_1(1,0)$ then sketch it.

(b) Find the elements of the conic section $x^2 + 4y^2 - 8y + 4x + 4 = 0$ and sketch it.

Solution :

(a) The two foci and the vertex are located on the x-axis.

The standard equation of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

$P(h,k) = \left(\dfrac{-2+2}{2}, \dfrac{0+0}{2}\right) = (0,0)$, hence $h = 0$ and $k = 0$

a is the distance between V_1 and P , hence $a = 1$

c is the distance between F_1 and P , hence $c = 2$

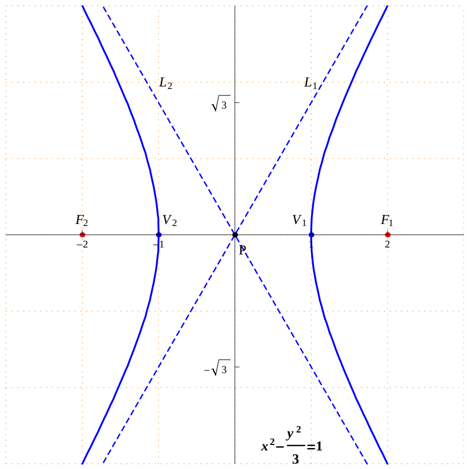
$c^2 = a^2 + b^2 \implies 4 = 1 + b^2 \implies b^2 = 3 \implies b = \sqrt{3}$

The standard equation of the hyperbola is $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$

The other vertex is $V_2 = (0-1, 0) = (-1, 0)$

The equations of the asymptotes are :

$L_1 \ : \ y = \sqrt{3}x$ and $L_2 \ : \ y = -\sqrt{3}x$



(b) $x^2 + 4y^2 - 8y + 4x + 4 = 0$

$x^2 - 4x + 4y^2 - 8y = -4$

$x^2 - 4x + 4(y^2 - 2y) = -4$

By completing the square

$(x^2 - 4x + 4) + 4(y^2 - 2y + 1) = -4 + 4 + 4$

$(x - 2)^2 + 4(y - 1)^2 = 4$

$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{1} = 1$

The conic section is an ellipse.

The center is $P(2, 1)$.

$a^2 = 4 \implies a = 2.$

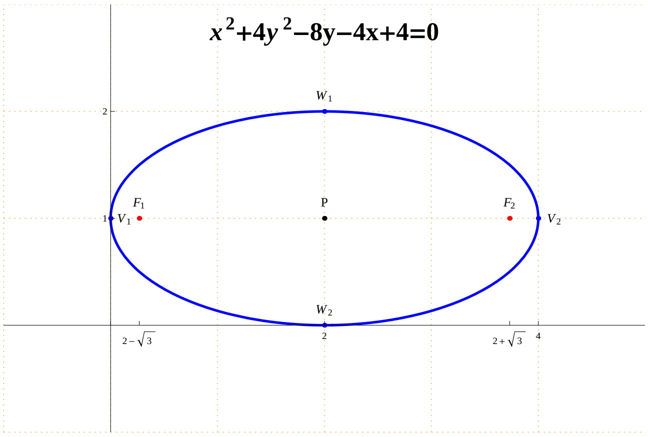
$b^2 = 1 \implies b = 1.$

$c^2 = a^2 - b^2 = 4 - 1 = 3 \implies c = \sqrt{3}.$

The vertices are $V_1(0, 1)$ and $V_2(4, 1)$

The foci are $F_1\left(2 - \sqrt{3}, 1\right)$ and $F_2\left(2 + \sqrt{3}, 1\right)$.

The end-points of the minor axis are $W_1(2, 2)$ and $W_2(2, 0)$.



Q.3 **(a)** Compute the integrals :

(i) $\int 6x(3x^2+9)^{15}\ dx$ (ii) $\int x^{10}\ln x\ dx$ (iii) $\int \dfrac{1}{(x-1)(x-2)}\ dx$

(b) Find the area of the region bounded by the graphs :

$y = 5$ and $y = x^2 + 1$.

(c) The region R between the curves $y = x^2$ and $y = x$ is rotated about the y-axis to form a solid of revolution S . Find the volume of S.

(d) Using polar coordinates find the area of the region lying in the first quadrant bounded by the circles with polar equations $r = 1$ and $r = 2$.

Solution :

(a) (i) $\int 6x(3x^2+9)^{15}\ dx = \dfrac{(3x^2+9)^{16}}{16} + c$

Using the formula $\int [f(x)]^n\ f'(x)\ dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$ where $n \neq -1$.

(ii) $\int x^{10}\ln x\ dx$

Using integration by parts

$$u = \ln x \qquad dv = x^{10}\ dx$$
$$du = \frac{1}{x}\ dx \qquad v = \frac{x^{11}}{11}$$

$$\int x^{10}\ln x\ dx = \frac{x^{11}}{11}\ln x - \int \frac{x^{11}}{11}\ \frac{1}{x} x\ dx$$

$$= \frac{x^{11}}{11}\ln x - \frac{1}{11}\int x^{10}\ dx = \frac{x^{11}}{11}\ln x - \frac{1}{11}\ \frac{x^{11}}{11} + c$$

(iii) $\int \dfrac{1}{(x-1)(x-2)}\ dx$

Using the method of partial fractions

$$\frac{1}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

$$1 = A_1(x-2) + A_2(x-1)$$

Put $x = 1$ then $1 = A_1(1-2) \implies A_1 = -1$

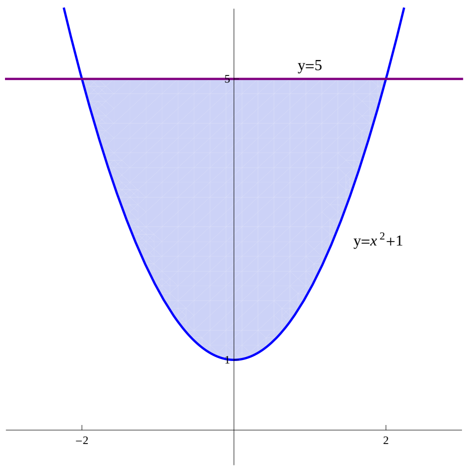
Put $x = 2$ then $1 = A_2(2-1) \implies A_2 = 1$

$$\int \frac{1}{(x-1)(x-2)}\ dx = \int \left(\frac{-1}{x-1} + \frac{1}{x-2} \right)\ dx$$

$$= -\int \frac{1}{x-1}\ dx + \int \frac{1}{x-2}\ dx = -\ln|x-1| + \ln|x-2| + c$$

(b) $y = 5$ is a straight line parallel to the x-axis and passes through $(0, 5)$.

$y = x^2 + 1 \implies y - 1 = x^2$ is a parabola with vertex $(1, 0)$ and opens upwards.



Points of intersection of $y = 5$ and $y = x^2 + 1$:

$x^2 + 1 = 5 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = -2\ ,\ x = 2$

$$\text{Area} = \int_{-2}^{2} \left[5 - (x^2 + 1)\right]\ dx = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(4 \times 2 - \frac{2^3}{3}\right) - \left(4 \times -2 - \frac{(-2)^3}{3}\right) = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

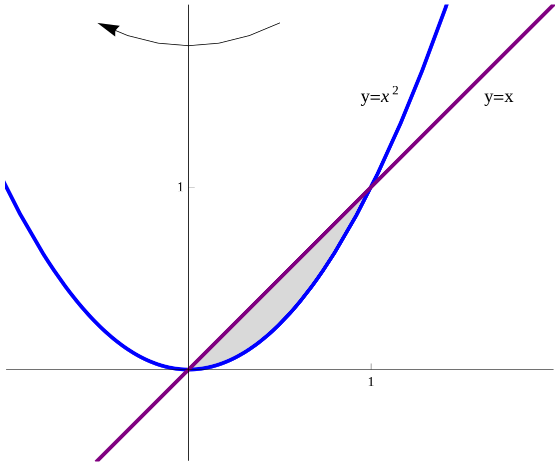
$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{48 - 16}{3} = \frac{32}{3}$$

(c) $y = x$ is a straight line passing through $(0, 0)$ with slope equals 1.

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

Points of intersection of $y = x^2$ and $y = x$:

$x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0\ ,\ x = 1$



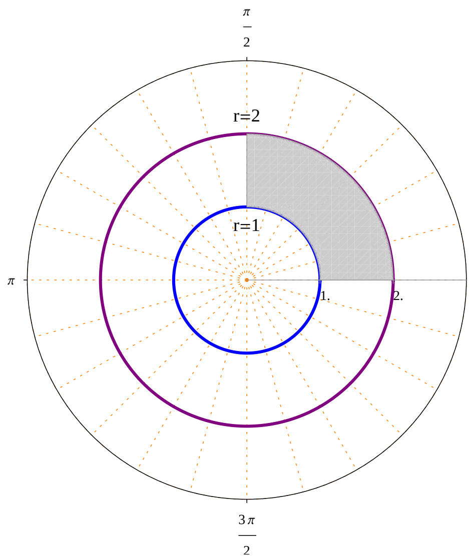
Using Cylindrical shells Method :

Volume $= 2\pi \displaystyle\int_0^1 x\left(x - x^2\right)\, dx = 2\pi \int_0^1 \left(x^2 - x^3\right)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= 2\pi \left[\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0)\right] = 2\pi \left(\frac{4-3}{12}\right) = \frac{\pi}{6}$

(d) $r = 1$ is a circle with center $(0, 0)$ and radius equals 1.

$r = 2$ is a circle with center $(0, 0)$ and radius equals 2.



$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(2)^2 - (1)^2\right]\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (4-1)\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 3\ d\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{3}{2}\left[\theta\right]_0^{\frac{\pi}{2}} = \frac{3}{2}\left(\frac{\pi}{2} - 0\right) = \frac{3\pi}{4}$$

Q.4 **(a)** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit function z defined by the equation $x^3 z^2 \sin(y) + (x+y)e^z = 0$.

(b) Solve the differential equation $\frac{dy}{dx} - 2xy^2 = 0$.

Solution :

(a) Put $F(x,y,z) = x^3 z^2 \sin(y) + (x+y)e^z$

$$F_x = (3x^2)z^2 \sin(y) + (1+0)e^z = 3x^2 z^2 \sin(y) + e^z$$

$$F_y = x^3 z^2 \cos(y) + (0+1)e^z = x^3 z^2 \cos(y) + e^z$$

$$F_z = x^3(2z)\sin(y) + (x+y)e^z = 2x^3 z \sin(y) + (x+y)e^z$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 z^2 \sin(y) + e^z}{2x^3 z \sin(y) + (x+y)e^z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^3 z^2 \cos(y) + e^z}{2x^3 z \sin(y) + (x+y)e^z}$$

(b) $\frac{dy}{dx} - 2xy^2 = 0$

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{1}{y^2}\ dy = 2x\ dx$$

It is a separable differential equation.

$$\int \frac{1}{y^2}\ dy = \int 2x\ dx$$

$$\int y^{-2}\ dy = \int 2x\ dx$$

$$\frac{y^{-1}}{-1} = x^2 + c$$

$$\frac{-1}{y} = x^2 + c$$

$$\frac{1}{y} = -x^2 - c$$

$$y = \frac{1}{-x^2 - c}$$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the First Mid-Term Exam
Second semester 1437-1438 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 3 & -3 \\ 1 & 7 \end{pmatrix}$.

Compute (if possible) : **AB** and **BC**

Solution :

$$\mathbf{AB} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4+1 & 2+0+1 \\ 0+6+2 & 0+0+2 \\ 2+0+1 & 2+0+1 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 8 & 2 \\ 3 & 3 \end{pmatrix}$$

BC is impossible, because the number of columns of **B** does not equal the number of rows of **C**

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{matrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (1+4+0) - (2+0+4) = 5 - 6 = -1$$

Solution (2) : By the definition (using third row)

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 2(2-1) - 0 + 1(1-4) = 2(1) + 1(-3) = 2 - 3 = -1$$

Q.3 Solve by Gauss method the linear system :

$$\begin{cases} x - 2y + z = 0 \\ 2x - 3y - z = 3 \\ -x + y + z = -2 \end{cases}$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & 3 \\ -1 & 1 & 1 & -2 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ -1 & 1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ 0 & -1 & 2 & -2 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$-z = 1 \implies z = -1$

$y - 3z = 3 \implies y + 3 = 3 \implies y = 0$

$x - 2y + z = 0 \implies x - 2(0) + (-1) = 0 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Q.4 Find the elements of the conic section $5y^2 - 4x^2 + 10y + 8x - 19 = 0$ and sketch it.

Solution :

$5y^2 - 4x^2 + 10y + 8x - 19 = 0$

$5y^2 + 10y - 4x + 8x = 19$

$5(y^2 + 2y) - 4(x^2 - 2x) = 19$

By completing the square

$5(y^2 + 2y + 1) - 4(x^2 - 2x + 1) = 19 + 5 - 4$

$5(y+1)^2 - 4(x-1)^2 = 20$

$$\frac{5(y+1)^2}{20} + \frac{4(x-1)^2}{20} = 1$$

$$\frac{(y+1)^2}{4} + \frac{(x-1)^2}{5} = 1$$

The conic section is a Hyperbola .

The center is $P(1, -1)$.

$a^2 = 5 \implies a = \sqrt{5}$

$b^2 = 4 \implies b = 2$

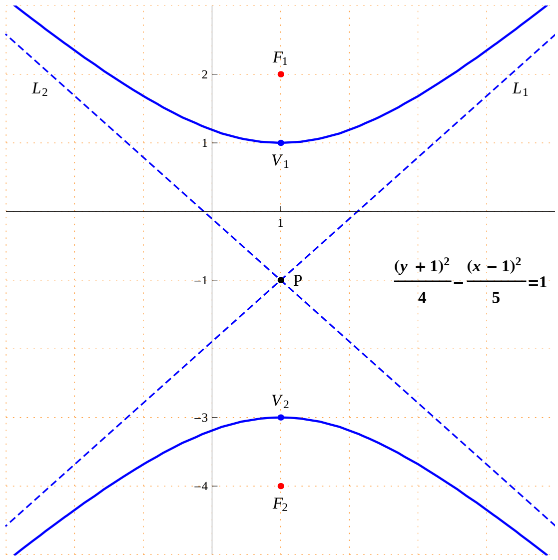
$c^2 = a^2 + b^2 = 5 + 4 = 9 \implies c = 3$

The vertices are $V_1(1,1)$ and $V_2(1,-3)$

The foci are $F_1\,(1,2)$ and $F_2\,(1,-4)$

The equations of the asymptotes are $L_1: \; (y+1) = \frac{2}{\sqrt{5}}(x-1)$

and $L_2: \; (y+1) = -\frac{2}{\sqrt{5}}(x-1)$



Q.5 Find the standard equation of the ellipse with foci $F_1(5,1)$, $F_2(-5,1)$ and with vertex $V_1(6,1)$, and sketch it.

Solution :

The general formula of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$

From the positions of F_1 and F_2 the major axis is parallel to the x-axis, hence $a > b$.

$P(h,k) = \left(\frac{5+(-5)}{2}, \frac{1+1}{2}\right) = (0,1)$

c is the distance between P and one of the foci, hence $c = 5$

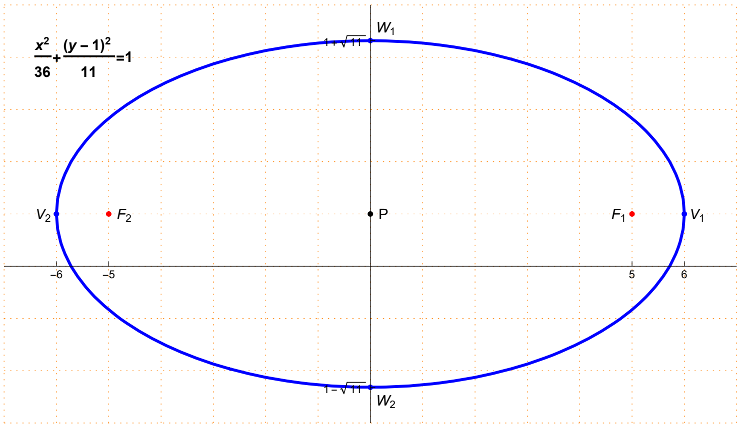
a is the distance between P and the vertex $V_1(6,1)$, hence $a = 6$

$c^2 = a^2 - b^2 \implies 25 = 36 - b^2 \implies b^2 = 36 - 25 = 11 \implies b = \sqrt{11}$

The standard equation of the ellipse is $\dfrac{x^2}{36} + \dfrac{(y-1)^2}{11} = 1$

The other vertex is $V_2(-6, 1)$

The end-points of the minor axis are $W_1\left(0, 1+\sqrt{11}\right)$ and $W_2\left(0, 1-\sqrt{11}\right)$



M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Final Exam
Second semester 1437-1438 H

Q.1 **(a)** Compute (if possible) **AB** for $\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 6 \\ 0 & 1 & 1 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix}$.

(c) Solve by Cramer's rule : $\begin{cases} x - y = 1 \\ 3x - 5y = -1 \end{cases}$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 6 \\ 0 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+12+0 & 0+4+2 & 1+24+2 \\ 3+0+0 & 0+0+3 & 3+0+3 \\ 0+6+0 & 0+2+0 & 0+12+0 \end{pmatrix} = \begin{pmatrix} 13 & 6 & 27 \\ 3 & 3 & 6 \\ 6 & 2 & 12 \end{pmatrix}$$

(b) **Solution (1):** Using Sarrus Method

$$\begin{matrix} 4 & 1 & 2 & 4 & 1 \\ 2 & 3 & 3 & 2 & 3 \\ 4 & 0 & 1 & 4 & 0 \end{matrix}$$

$$\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix} = (12+12+0) - (24+0+2) = 24 - 26 = -2$$

Solution (2) : Using the definition (using the third row) :

$$\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 4 \times \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} - 0 \times \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 4(3-6) - 0 + (12-2) = -12 + 10 = -2$$

(c) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix} , \ \mathbf{A}_x = \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix} , \ \mathbf{A}_y = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} = (1 \times -5) - (-1 \times 3) = -5 - (-3) = -5 + 3 = -2 \neq 0$$

$$|\mathbf{A}_x| = \begin{vmatrix} 1 & -1 \\ -1 & -5 \end{vmatrix} = (1 \times -5) - (-1 \times -1) = -5 - 1 = -6$$

$$|\mathbf{A}_y| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = (1 \times -1) - (1 \times 3) = -1 - 3 = -4$$

$$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} = \frac{-6}{-2} = 3$$

$$y = \frac{|\mathbf{A}_y|}{|\mathbf{A}|} = \frac{-4}{-2} = 2$$

The solution of the linear system is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the hyperbola with foci $(-2, 3)$, $(4, 3)$ and a vertex $(3, 3)$ then sketch it.

(b) Find the elements of the conic section $y^2 - 2y + 4x^2 + 8x + 1 = 0$ and sketch it.

Solution :

(a) The two foci and the vertex are located on a line parallel to the x-axis.

The standard equation of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

$P = (h, k) = \left(\dfrac{-2+4}{2}, \dfrac{3+3}{2} \right) = (1, 3)$, hence $h = 1$ and $k = 3$

a is the distance between $V_1 = (3, 3)$ and P , hence $a = 2$

c is the distance between $F_1 = (4, 3)$ and P , hence $c = 3$

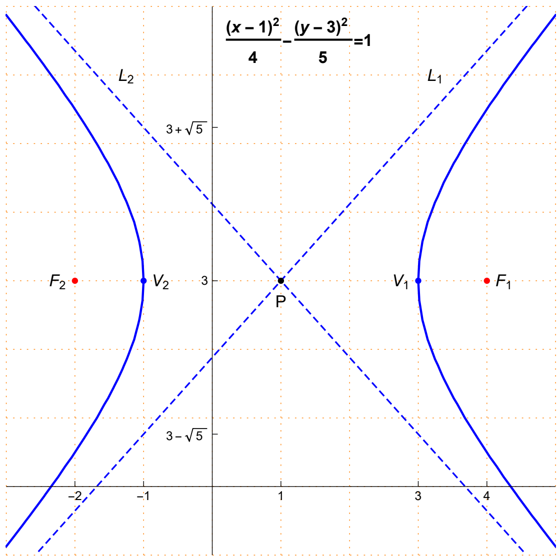
$c^2 = a^2 + b^2 \implies 9 = 4 + b^2 \implies b^2 = 9 - 4 = 5 \implies b = \sqrt{5}$

The standard equation of the hyperbola is $\dfrac{(x-1)^2}{4} - \dfrac{(y-3)^2}{5} = 1$

The other vertex is $V_2 = (1 - 2, 3) = (-1, 3)$

The equations of the asymptotes are :

$L_1 \ : \ y - 3 = \dfrac{\sqrt{5}}{2}(x - 1)$ and $L_2 \ : \ y - 3 = -\dfrac{\sqrt{5}}{2}(x - 1)$



(b) $y^2 - 2y + 4x^2 + 8x + 1 = 0$

$4x^2 + 8x + y^2 - 2y = -1$

$4(x^2 + 2x) + (y^2 - 2y) = -1$

By completing the square

$4(x^2 + 2x + 1) + (y^2 - 2y + 1) = -1 + 4 + 1$

$4(x + 1)^2 + (y - 1)^2 = 4$

$$\frac{4(x+1)^2}{4} + \frac{(y-1)^2}{4} = 1$$

$$(x+1)^2 + \frac{(y-1)^2}{4} = 1$$

The conic section is an ellipse.

The center is $P = (-1, 1)$.

$a^2 = 1 \implies a = 1.$

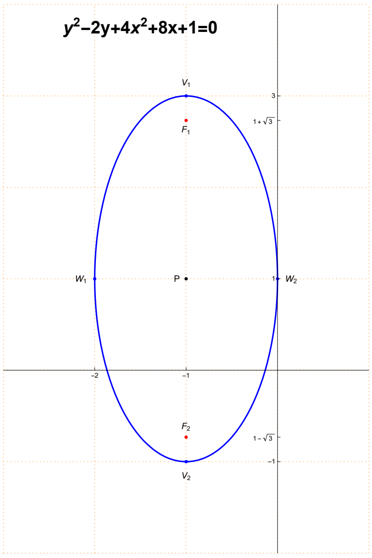
$b^2 = 4 \implies b = 2.$

$c^2 = b^2 - a^2 = 4 - 1 = 3 \implies c = \sqrt{3}.$

The vertices are $V_1 = (-1, 3)$ and $V_2 = (-1, -1)$

The foci are $F_1 = \big(-1, 1 + \sqrt{3}\big)$ and $F_2 = \big(-1, 1 - \sqrt{3}\big)$.

The end-points of the minor axis are $W_1 = (-2, 1)$ and $W_2 = (0, 1)$.



Q.3 **(a)** Compute the integrals :

(i) $\displaystyle\int \frac{x}{(x-1)(x-2)}\,dx$ (ii) $\displaystyle\int \ln x\,dx$ (iii) $\displaystyle\int (2x+1)\cos(x^2+x)\,dx$

(b) Find the area of the region determined by the curves :

$y=\sqrt{x}$ and $y=x$.

(c) The region R between the curves $y=0$, $x=2$ and $y=x^2$ is rotated about the y-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\displaystyle\int \frac{x}{(x-1)(x-2)}\,dx$

Using the method of partial fractions

$$\frac{x}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

$x = A_1(x-2) + A_2(x-1)$

Put $x=1$ then $1 = A_1(1-2) \implies A_1 = -1$

Put $x=2$ then $2 = A_2(2-1) \implies A_2 = 2$

$$\int \frac{x}{(x-1)(x-2)}\,dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-2}\right)\,dx$$

$$= -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx = -\ln|x-1| + 2\ln|x-2| + c$$

(ii) $\displaystyle\int \ln x\,dx$

Using integration by parts

$u = \ln x \qquad dv = 1\ dx$

$du = \frac{1}{x}\ dx \qquad v = x$

$$\int \ln x\ dx = x \ln x - \int x\ \frac{1}{x}\ dx$$

$$= x \ln x - \int 1\ dx = x\,(\ln x) - x + c$$

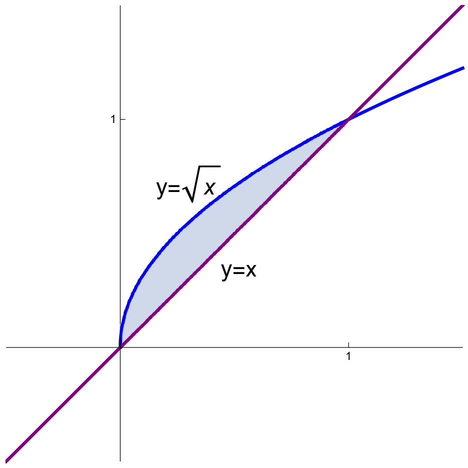
(iii) $\int (2x+1)\cos(x^2+x)\ dx$

$$\int (2x+1)\cos(x^2+x)\ dx = \sin(x^2+x) + c$$

Using the formula $\int \cos\left(f(x)\right)\ f'(x)\ dx = \sin\left(f(x)\right) + c$

(b) $y = x$ is a straight line passing through the origin with slope equals 1.

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0,0)$ and opens to the right.



Points of intersection of $y = x$ and $y = \sqrt{x}$:

$$x = \sqrt{x} \Longrightarrow x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0\ ,\ x = 1$$

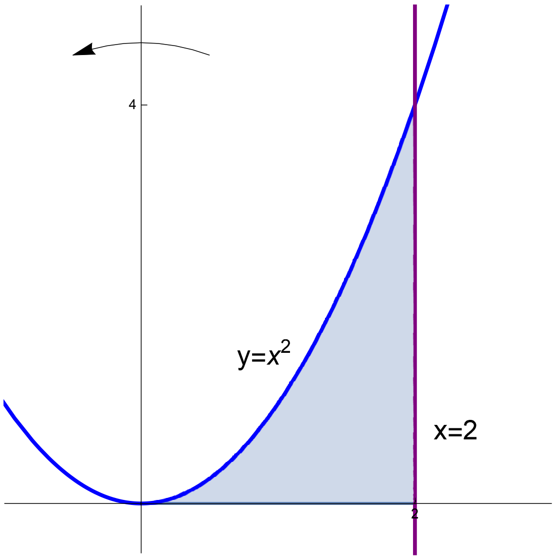
$$\text{Area} = \int_0^1 \left(\sqrt{x} - x\right)\ dx = \int_0^1 \left(x^{\frac{1}{2}} - x\right)\ dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{2}{3}\ x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{2}\right) - (0-0) = \frac{1}{6}$$

(c) $y = 0$ is the x-axis

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$.

$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.



Using Cylindrical shells Method :

Volume $= 2\pi \int_0^2 x\left(x^2\right)\ dx = 2\pi \int_0^2 x^3\ dx = 2\pi \left[\frac{x^4}{4}\right]_0^2$

$= 2\pi\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = 2\pi(4) = 8\pi$

Q.4 Find f_x , f_y and f_z for the function $f(x, y, z) = x^2yz + xz^4 \sin(2x + y^2)$.

Solution :

$f_x = (2x)yz + z^4\left[(1)\sin(2x + y^2) + x\cos(2x + y^2)(2 + 0)\right]$

$= 2xyz + z^4\left[\sin(2x + y^2) + 2x\cos(2x + y^2)\right]$

$f_y = x^2z(1) + xz^4\cos(2x + y^2)(0 + 2y)$

$= x^2z + 2xyz^4\cos(2x + y^2)$

$f_z = x^2y(1) + (4z^3)x\sin(2x + y^2)$

$= x^2y + 4xz^3\sin(2x + y^2)$