M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[1]

Solution of the First Mid-Term Exam

First semester 1434-1435 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 2 & 0 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 3 \end{pmatrix}$

Compute (if possible) : **BA** , **BAC** and **CB**

Solution :

$$\mathbf{BA} = \begin{pmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 2 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5+0+10 & 10+0+0 & 15+0+30 \\ 0+20+0 & 0+0+0 & 0+25+0 \\ 5+0+10 & 10+0+0 & 15+0+30 \end{pmatrix} = \begin{pmatrix} 15 & 10 & 45 \\ 20 & 0 & 25 \\ 15 & 10 & 45 \end{pmatrix}$$

$$\mathbf{BAC} = (\mathbf{BA})\,\mathbf{C} = \begin{pmatrix} 15 & 10 & 45 \\ 20 & 0 & 25 \\ 15 & 10 & 45 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 15+20+0 & 15+40+135 \\ 20+0+0 & 20+0+75 \\ 15+20+0 & 15+40+135 \end{pmatrix} = \begin{pmatrix} 35 & 190 \\ 20 & 95 \\ 35 & 190 \end{pmatrix}$$

CB is not possible because the number of columns of **C** (which is 2) is not equal to the number of rows of **B** (which is 3)

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}$

Solution (1) : Using the properties of the determinants

Multiply C_3 by 2 and multiply C_4 by 3

$$\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \begin{vmatrix} 1 & 2 & 6 & 6 \\ 0 & 4 & 6 & 6 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}$$

$= \frac{1}{2} \times \frac{1}{3} \times 0 = 0$ (because $C_3 = C_4$)

Solution (2) : Interchange R_1 and R_4.

[1]E-mail : alfadhel@ksu.edu.sa

$$\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 2 \end{vmatrix} = -2 \begin{vmatrix} 4 & 3 & 2 \\ 2 & 0 & 0 \\ 2 & 3 & 2 \end{vmatrix}$$

Using Sarrus method

$$\begin{matrix} 4 & 3 & 2 & 4 & 3 \\ 2 & 0 & 0 & 2 & 0 \\ 2 & 3 & 2 & 2 & 3 \end{matrix}$$

$$\begin{vmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix} = -2[(0+0+12) - (0+0+12)] = -2[12-12] = 0$$

Q.3 Find all the elements of the conic section $y^2 = 4y + 2x + 4$ and sketch it.

Solution :

$y^2 = 4y + 2x + 4 \implies y^2 - 4y = 2x + 4$

$\implies y^2 - 4y + 4 = 2x + 4 + 4 \implies (y-2)^2 = 2x + 8$

$\implies (y-2)^2 = 2(x+4)$

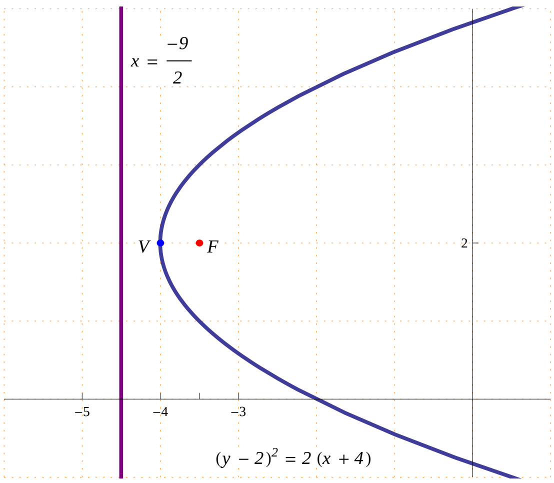
The conic section is a parabola opens to the right.

The vertex is $V(-4, 2)$.

$4a = 2 \implies a = \frac{2}{4} = \frac{1}{2}.$

The focus is $F\left(-4 + \frac{1}{2}\right) = \left(-\frac{7}{2}, 2\right) = (-3.5, 2).$

The equation of the directrix is $x = -4 - \frac{1}{2} \implies x = -\frac{9}{2} = -4.5$



Q.4 Find the standard equation of the hyperbola with foci $(2,0)$ and $(-2,0)$, and with vertex $(1,0)$, and then sketch it.

Solution :

The center of the hyperbola is the mid-point of the two foci .

The center is $P\left(\frac{-2+2}{2}, \frac{0+0}{2}\right) = (0,0)$, hence $h = 0$ and $k = 0$.

Since the transverse axis lies on the x-axis then the standard equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

a is the distance between the center $(0,0)$ and the vertex $(1,0)$, hence $a = 1$.

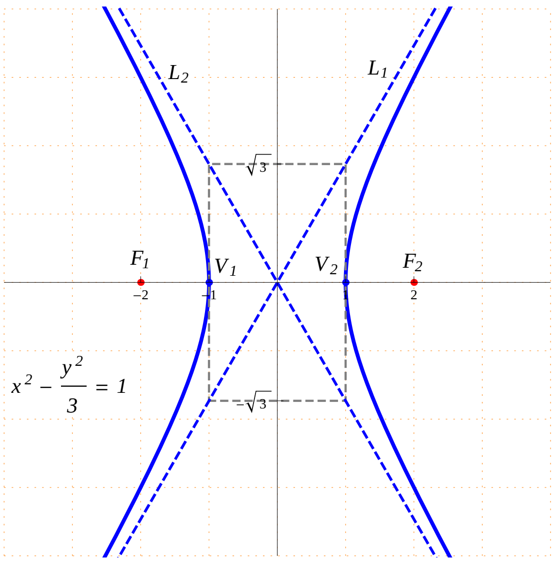
c is the distance between the center $(0,0)$ and one of the foci , hence $c = 2$.

$c^2 = a^2 + b^2 \implies 2^2 = 1^2 + b^2 \implies b^2 = 3 \implies b = \sqrt{3}$.

The standard equation of the hyperbola is $x^2 - \frac{y^2}{3} = 1$.

The other vertex is $(-1,0)$.

The equations of the asymptotes are $L_1 : y = \sqrt{3}x$ and $L_2 : y = -\sqrt{3}x$



Q.5 Solve by Cramer's Rule the following linear system $\begin{cases} x + 4y = 11 \\ 5x - 2y = 11 \end{cases}$

Solution :

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 5 & -2 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 11 & 4 \\ 11 & -2 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 1 & 11 \\ 5 & 11 \end{pmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} = (1 \times -2) - (5 \times 4) = -2 - 20 = -22$$

$$\det(\mathbf{A}_1) = \begin{vmatrix} 11 & 4 \\ 11 & -2 \end{vmatrix} = (11 \times -2) - (11 \times 4) = -22 - 44 = -66$$

$$\det(\mathbf{A}_2) = \begin{vmatrix} 1 & 11 \\ 5 & 11 \end{vmatrix} = (1 \times 11) - (5 \times 11) = 11 - 55 = -44$$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-66}{-22} = 3$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{-44}{-22} = 2$$

The solution of the linear system is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

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Dr. Tariq A. AlFadhel[2]

Solution of the Second Mid-Term Exam

First semester 1434-1435 H

Q.1 Compute the integrals :

(a) $\int 3x^2 \cos(x^3)\ dx$

(b) $\int \dfrac{3}{(x-5)(x-2)}\ dx$

(c) $\int \dfrac{5}{x^2+2x+5}\ dx$

(d) $\int x^{2014} \ln x\ dx$

(e) $\int x \sin x\ dx$

Solution :

(a) Using the formula $\int \cos(f(x))\ f'(x)\ dx = \sin(f(x)) + c$

$$\int 3x^2 \cos(x^3)\ dx = \int \sin(x^3)\ (3x^2)\ dx = \sin(x^3) + c$$

(b) $\int \dfrac{3}{(x-5)(x-2)}\ dx$

Using the method of partial fractions

$$\frac{3}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

$$\frac{3}{(x-5)(x-2)} = \frac{A(x-2)}{(x-5)(x-2)} + \frac{B(x-5)}{(x-2)(x-5)}$$

$3 = A(x-2) + B(x-5)$

Put $x = 5$, then $3 = A(5-2) + B(5-5) \implies 3A = 3 \implies A = 1$

Put $x = 2$, then $3 = A(2-2) + B(2-5) \implies -3B = 3 \implies B = -1$

$$\frac{3}{(x-5)(x-2)} = \frac{1}{x-5} + \frac{-1}{x-2}$$

$$\int \frac{3}{(x-5)(x-2)}\ dx = \int \left(\frac{1}{x-5} - \frac{1}{x-2} \right)\ dx$$

$$= \int \frac{1}{x-5}\ dx - \int \frac{1}{x-2}\ dx = \ln|x-5| - \ln|x-2| + c$$

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[2]E-mail : alfadhel@ksu.edu.sa

(c) Using the formula $\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$

$$\int \frac{5}{x^2+2x+5}\,dx = 5\int \frac{1}{(x^2+2x+1)+4}\,dx$$

$$= 5\int \frac{1}{(x+1)^2+2^2} = 5\times\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c = \frac{5}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

(d) $\int x^{2014}\ln x\;dx$

Using integration by parts

$$u = \ln x \qquad dv = x^{2014}\;dx$$
$$du = \frac{1}{x}\;dx \qquad v = \frac{x^{2015}}{2015}$$

$$\int x^{2014}\ln x\;dx = \frac{x^{2015}}{2015}\ln x - \int \frac{x^{2015}}{2015}\frac{1}{x}\;dx$$

$$= \frac{x^{2015}}{2015}\ln x - \frac{1}{2015}\int x^{2014}\;dx$$

$$= \frac{x^{2015}}{2015}\ln x - \frac{1}{2015}\;\frac{x^{2015}}{2015} + c = \frac{x^{2015}}{2015}\ln x - \frac{x^{2015}}{(2015)^2} + c$$

(e) $\int x\sin x\;dx$

Using integration by parts

$$u = x \qquad dv = \sin x\;dx$$
$$du = dx \qquad v = -\cos x$$

$$\int x\sin x\;dx = -x\cos x - \int -\cos x\;dx$$

$$= -x\cos x + \int \cos x\;dx = -x\cos x + \sin x + c$$

Q.2 (a) Sketch the region **R** determined by the curves

$y = x^2+2$, $x = 1$, $x = 2$ and $y = 1$.

(b) Find the area of the region **R** .

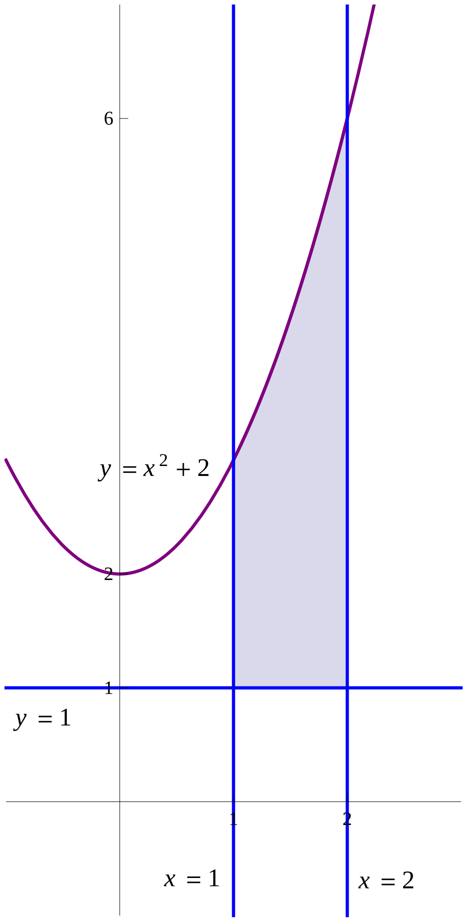
Solution :

(a) $y = x^2+2$ is a parabola with vertex $(0,2)$ and opens upwards.

$y = 1$ is a straight line parallel to the x-axis and passes through $(0,1)$

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$

$x = 2$ is a straight line parallel to the y-axis and passes through $(2, 0)$



(b) Area $= \displaystyle\int_1^2 [(x^2+2) - 1]\, dx = \int_1^2 (x^2+1)\, dx = \left[\frac{x^3}{3} + x\right]_1^2$

$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{1^3}{3} + 1\right) = \frac{8}{3} + 2 - \frac{1}{3} - 1 = \frac{7}{3} + 1 = \frac{10}{3}$$

Q.3 (a) Sketch the region **R** determined by the curves

$y = x^2 + 1$, $x = 2$ and $y = -2x + 4$

(b) The region **R** is rotated about the x-axis to form a solid of revolution **S**. Find the volume of **S**.

Solution :

(a) $y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

$y = -2x + 4$ is a straight line with slope -2 and passes through $(0, 4)$.

$x = 2$ is a straight line parallel to the y-axis and passes through $(2, 0)$

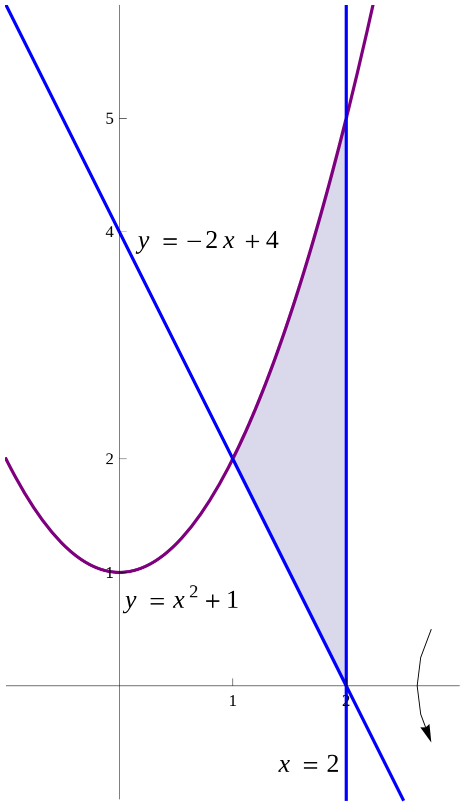
Point of intersection of $y = x^2 + 1$ and $y = -2x + 4$:

$x^2 + 1 = -2x + 4 \implies x^2 + 2x - 3 = 0 \implies (x + 3)(x - 1) = 0$

$\implies x = 1 \ , \ x = -3$

Point of intersection of $y = -2x + 4$ and $x = 2$:

$y = (-2 \times 2) + 4 = -4 + 4 = 0$, the point of intersection is $(2, 0)$



(b) Using Washer method :

Volume $= \pi \int_1^2 \int \left[(x^2 + 1)^2 - (-2x + 4)^2\right] \ dx$

$= \pi \int_1^2 \left[(x^4 + 2x^2 + 1) - (4x^2 - 16x + 16)\right] \ dx$

$= \pi \int_1^2 \left[x^4 + 2x^2 + 1 - 4x^2 + 16x - 16\right] \ dx$

$$= \pi \int_1^2 \left[x^4 - 2x^2 + 16x - 15\right] \ dx$$

$$= \pi \left[\frac{x^5}{5} - 2\frac{x^3}{3} + 8x^2 - 15x\right]_1^2$$

$$= \pi \left[\left(\frac{2^5}{5} - \frac{2 \times 2^3}{3} + 8 \times 2^2 - 15 \times 2\right) - \left(\frac{1^5}{5} - \frac{2 \times 1^3}{3} + 8 \times 1^2 - 15 \times 1\right)\right]$$

$$= \pi \left[\left(\frac{32}{5} - \frac{16}{3} + 32 - 30\right) - \left(\frac{1}{5} - \frac{2}{3} + 8 - 15\right)\right]$$

$$= \pi \left[\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} + 7\right] = \pi \left[\frac{31}{5} - \frac{14}{3} + 9\right]$$

$$= \pi \left[\frac{93 - 70 + 135}{15}\right] = \frac{158}{15}\pi$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[3]

Solution of the Final Exam

First semester 1434-1435 H

Q.1 **(a)** Compute : $\mathbf{BA}-\mathbf{AB}$ for $\mathbf{A}=\begin{pmatrix}2&1&0\\1&4&1\\0&1&3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&3&1\\2&1&2\\1&3&1\end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix}1&2&3&4\\0&1&5&2\\1&2&1&4\\0&1&3&2\end{vmatrix}$.

(c) Solve by Gauss Method the linear system : $\begin{cases}x+y+z=3\\x-y+2z=5\\2x+z=4\end{cases}$

Solution :

(a) $\mathbf{BA}=\begin{pmatrix}2&3&1\\2&1&2\\1&3&1\end{pmatrix}\begin{pmatrix}2&1&0\\1&4&1\\0&1&3\end{pmatrix}$

$$=\begin{pmatrix}4+3+0&2+12+1&0+3+3\\4+1+0&2+4+2&0+1+6\\2+3+0&1+12+1&0+3+3\end{pmatrix}=\begin{pmatrix}7&15&6\\5&8&7\\5&14&6\end{pmatrix}$$

$\mathbf{AB}=\begin{pmatrix}2&1&0\\1&4&1\\0&1&3\end{pmatrix}\begin{pmatrix}2&3&1\\2&1&2\\1&3&1\end{pmatrix}$

$$=\begin{pmatrix}4+2+0&6+1+0&2+2+0\\2+8+1&3+4+3&1+8+1\\0+2+3&0+1+9&0+2+3\end{pmatrix}=\begin{pmatrix}6&7&4\\11&10&10\\5&10&5\end{pmatrix}$$

$$\mathbf{BA}-\mathbf{AB}=\begin{pmatrix}7&15&6\\5&8&7\\5&14&6\end{pmatrix}-\begin{pmatrix}6&7&4\\11&10&10\\5&10&5\end{pmatrix}$$

$$=\begin{pmatrix}7-6&15-7&6-4\\5-11&8-10&7-10\\5-5&14-10&6-5\end{pmatrix}=\begin{pmatrix}1&8&2\\-6&-2&-3\\0&4&1\end{pmatrix}$$

(b) $\begin{vmatrix}1&2&3&4\\0&1&5&2\\1&2&1&4\\0&1&3&2\end{vmatrix}=0$

Because $C_4=2C_2$

[3]E-mail : alfadhel@ksu.edu.sa

(c) Using Gauss method :

The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 5 \\ 2 & 0 & 1 & 4 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 5 \\ 2 & 0 & 1 & 4 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 2 \\ 2 & 0 & 1 & 4 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & -2 & -1 & -2 \end{array}\right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & -2 & -4 \end{array}\right)$$

$-2z = -4 \implies z = 2$

$-2y + z = 2 \implies -2y + 2 = 2 \implies -2y = 0 \implies y = 0$

$x + y + z = 3 \implies x + 0 + 2 = 3 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Q.2 Find The elements of the two following conic sections and sketch their graphs:

(a) $y^2 = 4y + 2x + 4$

(b) $9x^2 - 4y^2 - 54x - 16y + 29 = 0$

Solution :

(a) $y^2 = 4y + 2x + 4$

$y^2 - 4y = 2x + 4$

By completing the square

$y^2 - 4y + 4 = 2x + 4 + 4$

$(y - 2)^2 = 2x + 8$

$(y - 2)^2 = 2(x + 4)$

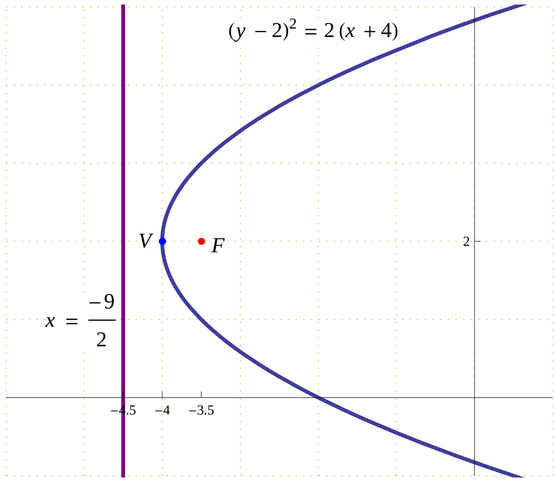
The conic section is a parabola opens to the right.

The vertex is $V = (-4, 2)$.

$4a = 2 \implies a = \dfrac{1}{2}$.

The focus is $F = \left(-4 + \dfrac{1}{2}, 2\right) = \left(-\dfrac{7}{2}, 2\right)$.

The equation of the directrix is $x = -4 - \frac{1}{2} \implies x = -\frac{9}{2}$



(b) $9x^2 - 4y^2 - 54x - 16y + 29 = 0$

$9x^2 - 54x - 4y^2 - 16y = -29$

$9(x^2 - 6x) - 4(y^2 + 4y) = -29$

By completing the square

$9(x^2 - 6x + 9) - 4(y^2 + 4y + 4) = -29 + 81 - 16$

$9(x-3)^2 - 4(y+2)^2 = 36$

$\frac{9(x-3)^2}{36} - \frac{4(y+2)^2}{36} = 1$

$\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$

The conic section is a hyperbola with transverse axis parallel to the x-axis.

The ceneter is $P = (3, -2)$.

$a^2 = 4 \implies a = 2$ and $b^2 = 9 \implies b = 3$.

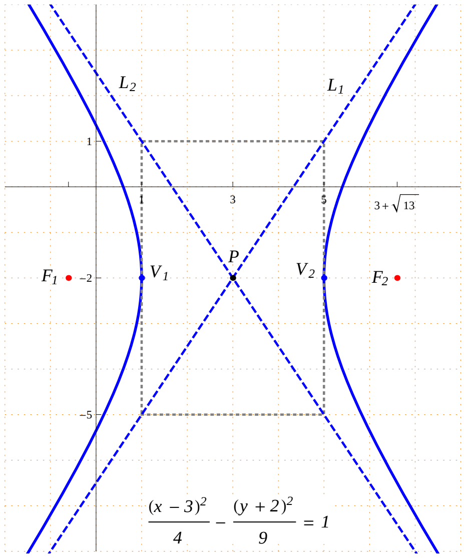
$c^2 = a^2 + b^2 = 4 + 9 = 13 \implies c = \sqrt{13}$.

The vertices are $V_1 = (3-2, -2) = (1, -2)$ and $V_2 = (3+2, -2) = (5, -2)$.

The foci are $F_1 = \left(3 - \sqrt{13}, -2\right)$ and $F_2 = \left(3 + \sqrt{13}, -2\right)$.

The equations of the asymptotes are $L_1 \ : \ y + 2 = \frac{3}{2}\,(x-3)$ and

$L_2 \ : \ y + 2 = -\frac{3}{2}\,(x-3)$



Q.3 **(a)** Compute the integrals :

(i) $\displaystyle\int \frac{3x+1}{x(x+1)}\,dx$ (ii) $\displaystyle\int x\cos x\,dx$ (iii) $\displaystyle\int \frac{2x-4}{x^2-4x+13}\,dx$

(b) Find the area of the surface delimited by the curves :

$y = x^2$ and $y = 2 - x^2$.

(c) The region R between the curves $x = 0$, $y = 1$, $y = x^2 + 2$ and $x = 1$ is rotated about the x-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\displaystyle\int \frac{3x+1}{x(x+1)}dx$

Using the method of partial fractions.

$$\frac{3x+1}{x(x+1)} = \frac{A_1}{x} + \frac{A_2}{x+1}$$

$$\frac{3x+1}{x(x+1)} = \frac{A_1(x+1)}{x(x+1)} + \frac{A_2 x}{x(x+1)}$$

$$3x+1 = A_1(x+1) + A_2x = A_1x + A_1 + A_2x$$

$$3x+1 = (A_1+A_2)\,x + A_1$$

By comparing the coefficients of both sides :

$$\begin{array}{lllll} A_1 & + & A_2 & = & 3 & \longrightarrow (1) \\ A_1 & & & = & 1 & \longrightarrow (2) \end{array}$$

$A_1 = 1$ and $A_1 + A_2 = 3 \implies A_2 = 2$

$$\frac{3x+1}{x(x+1)} = \frac{1}{x} + \frac{2}{x+1}$$

$$\int \frac{3x+1}{x(x+1)}\ dx = \int \left(\frac{1}{x} + \frac{2}{x+1}\right)\ dx$$

$$= \int \frac{1}{x}\ dx + 2\int \frac{1}{x+1}\ dx = \ln|x| + 2\ln|x+1| + c$$

(ii) $\int x\cos x\ dx$

Using integration by parts

$$u = x \qquad dv = \cos x\ dx$$
$$du = dx \qquad v = \sin x$$

$$\int x\cos x\ dx = x\sin x - \int \sin x\ dx$$

$$= x\sin x - (-\cos x) + c = x\sin x + \cos x + c$$

(iii) $\int \frac{2x-4}{x^2-4x+13}\ dx = \ln(x^2-4x+13) + c$

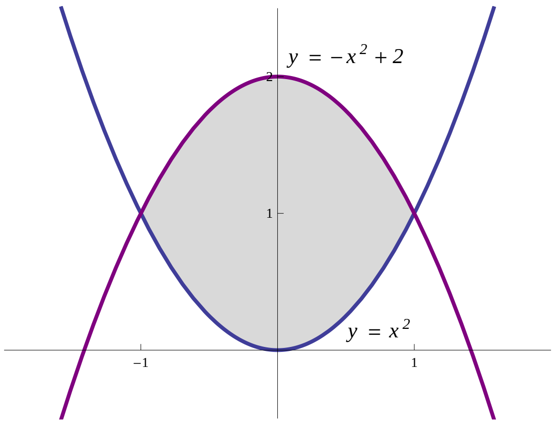
Using the formula $\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + c$

(b) $y = x^2$ is a parabola opens upwards with vertex $(0,0)$.

$y = -x^2+2$ is a parabola opens downwards with vertex $(0,2)$.

Points of intersections of $y = -x^2+2$ and $y = x^2$:

$x^2 = -x^2+4 \quad \Rightarrow \quad 2x^2 = 2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$



$$\text{Area} = \int_{-1}^{1} [(-x^2+2) - x^2]\ dx = \int_{-1}^{1} (-2x^2+2)\ dx$$

$$= \left[\frac{-2x^3}{3} + 2x\right]_{-2}^{2} = \left(\frac{-2\times 1^3}{3} + 2\times 1\right) - \left(\frac{-2\times(-1)^3}{3} + 2\times -1\right)$$

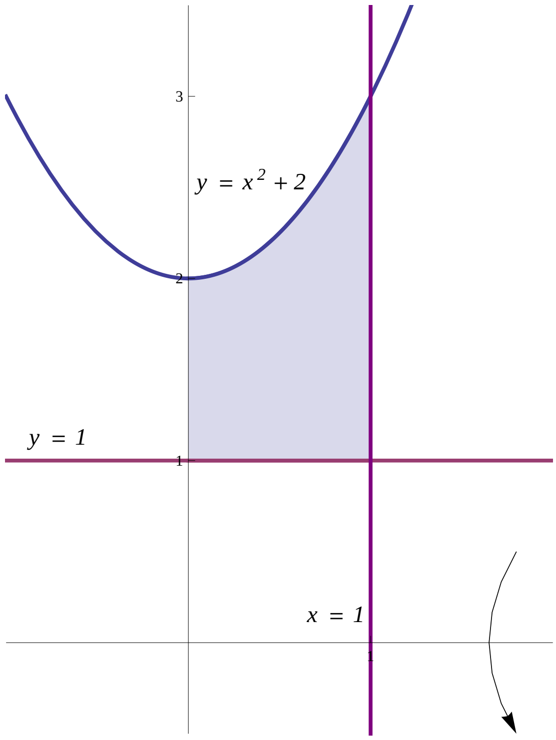
$$= \frac{-2}{3} + 2 - \left(\frac{2}{3} - 2\right) = 2 - \frac{2}{3} + 2 - \frac{2}{3} = 4 - \frac{4}{3} = \frac{8}{3}$$

(c) $x = 0$ is the y-axis.

$y = 1$ is a straight line parallel to the x-axis and passes through $(0, 1)$.

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$.

$y = x^2 + 2$ is a parabola opens upwards with vertex $(0, 2)$.



Using Washer Method :

$$\text{Volume} = \pi \int_0^1 \left[(x^2 + 2)^2 - (1)^2\right]\ dx = \pi \int_0^1 (x^4 + 4x^2 + 4 - 1)\ dx$$

$$= \pi \int_0^1 (x^4 + 4x^2 + 3)\ dx = \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 3x\right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 3\right) - (0 + 0 + 0)\right] = \pi \left(\frac{3 + 20 + 45}{15}\right) = \frac{68}{15}\pi$$

Q.4 **(a)** Find f_x and f_y for the function $f(x, y) = \dfrac{x + y^2}{x + y}$

(b) Solve the differential equation $x\dfrac{dy}{dx} - y = x^2 \cos x$ with $y(\pi) = 0$

Solution :

(a) $f_x = \dfrac{\partial f}{\partial x} = \dfrac{(1+0)(x+y)-(x+y^2)(1+0)}{(x+y)^2} = \dfrac{x+y-(x+y^2)}{(x+y^2)}$

$$f_x = \frac{x+y-x-y^2}{(x+y)^2} = \frac{y-y^2}{(x+y)^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{(0+2y)(x+y)-(x+y^2)(0+1)}{(x+y)^2} = \frac{2y(x+y)-(x+y^2)}{(x+y)^2}$$

$$f_y = \frac{2xy+2y^2-x-y^2}{(x+y)^2} = \frac{2xy-x+y^2}{(x+y)^2}$$

(b) $x\dfrac{dy}{dx} - y = x^2 \cos x$

$$xy' - y = x^2 \cos x$$

$$y' - \frac{1}{x}y = x\cos x$$

$P(x) = -\dfrac{1}{x}$ and $Q(x) = x\cos x$

The integrating factor is :

$$u(x) = e^{\int -\frac{1}{x}\ dx} = e^{-\int \frac{1}{x}\ dx} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1}$$

The general solution of the differential equation is :

$$y(x) = \frac{1}{u(x)}\int u(x)\ Q(x)\ dx = \frac{1}{x^{-1}}\int x^{-1}x\cos x\ dx$$

$$y(x) = x\int \cos x\ dx = x(\sin x + c) = x\sin x + cx$$

Using the initial condition $y(\pi) = 0$:

$$0 = \pi\sin\pi + c\pi \implies 0 = 0 + c\pi \implies c = \frac{0}{\pi} = 0.$$

The particular solution of the differential equation is :

$$y(x) = x\sin x$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[4]

Solution of the First Mid-Term Exam

Second semester 1434-1435 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & -4 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 3 \end{pmatrix}$

Compute (if possible) : **AB** , and **BC**

Solution :

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & -4 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1-6+0 & 1-4+3 \\ -3-12+0 & 3-8+2 \\ -2+0+0 & 2+0+1 \end{pmatrix} = \begin{pmatrix} -7 & 0 \\ -15 & -3 \\ -2 & 3 \end{pmatrix}$$

BC is impossible because the number of columns of **B** (which is 2) is not equal to the number of rows of **C** (which is 3)

Q.2 Compute The determinant $\begin{vmatrix} 3 & 2 & 1 \\ -3 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$

Solution (1) : Using Sarrus Method

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ -3 & 2 & 0 & -3 & -2 \\ 3 & -1 & 0 & 3 & -1 \end{matrix}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ -3 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= (3\times 2\times 0 + 2\times 0\times 3 + 1\times -3\times -1) - (3\times 2\times 1 + -1\times 0\times 3 + 0\times -3\times 2)$$

$$= (0+0+3) - (6+0+0) = 3-6 = -3$$

Solution (2) : Using the properties of the determinants

$$\begin{vmatrix} 3 & 2 & 1 \\ -3 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} \xrightarrow{C_1 \longleftrightarrow C_3} -1 \times \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{vmatrix} \xrightarrow{R_2 \longleftrightarrow R_3}$$

$$-1 \times -1 \times \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 2 & -3 \end{vmatrix} \xrightarrow{2R_2 + R_3} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times -1 \times 3 = -3$$

[4]E-mail : alfadhel@ksu.edu.sa

Q.3 Solve by Gauss elimination : $\begin{cases} \quad - 2y + z = -1 \\ -x + y - z = 0 \\ 4x + z = 1 \end{cases}$

Solution :

$$\left(\begin{array}{ccc|c} 0 & -2 & 1 & -1 \\ -1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 \\ 4 & 0 & 1 & 1 \end{array}\right) \xrightarrow{4R_1 + R_3}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 4 & -3 & 1 \end{array}\right) \xrightarrow{2R_2 + R_3} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$-z = -1 \implies z = \frac{-1}{-1} = 1$$

$$-2y + z = -1 \implies -2y + 1 = -1 \implies -2y = -2 \implies y = \frac{-2}{-2} = 1$$

$$-x + y - z = 0 \implies -x + 1 - 1 = 0 \implies -x = 0 \implies x = \frac{0}{-1} = 0$$

Q.4 Find the elements of the conic section $5x^2 - 4y^2 - 16y - 10x - 31 = 0$ and sketch it.

Solution :

$5x^2 - 4y^2 - 16y - 10x - 31 = 0$

$5x^2 - 10x - 4y^2 - 16y = 31$

$5(x^2 - 2x) - 4(y^2 + 4y) = 31$

$5(x^2 - 2x + 1) - 4(y^2 + 4y + 4) = 31 + 5 - 16$

$5(x-1)^2 - 4(y+2)^2 = 20$

$\dfrac{5(x-1)^2}{20} - \dfrac{4(y+2)^2}{20} = 1$

$\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{5} = 1$

The conic section is a hyperbola.

The center of the hyperbola is $P = (1, -2)$.

The transverse axis is parallel to the x-axis.

$a^2 = 4 \implies a = 2.$

$b^2 = 5 \implies b = \sqrt{5}.$

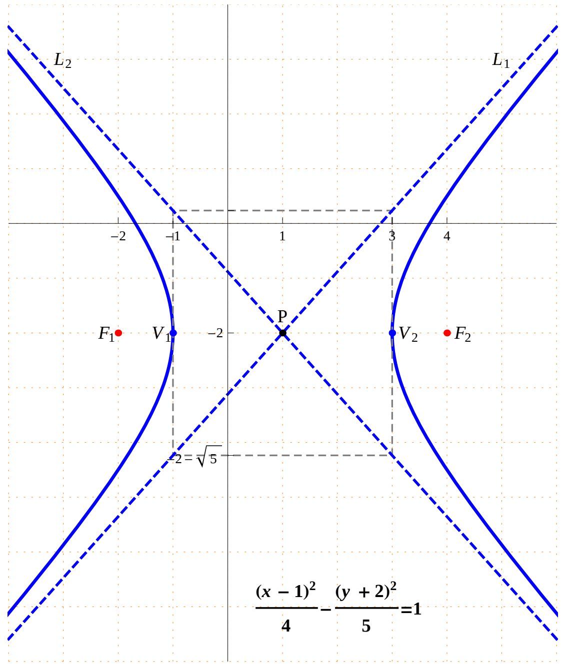
$c^2 = a^2 + b^2 = 4 + 5 = 9 \implies c = 3.$

The vertices are $V_1 = (-1, -2)$ and $V_2 = (3, -2)$.

The foci are $F_1 = (-2, -2)$ and $F_2 = (4, -2)$.

The length of the transverse axis equals $2a = 4$.

The equation of the asymptotes are $y + 2 = \pm \frac{\sqrt{5}}{2}(x - 1)$



Q.5 Find the standard equation of the ellipse with foci $(-4, 2)$ and $(4, 2)$, and the length of its major axis is 10 , and sketch it.

Solution :

The standard equation of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The center of the ellipse is the mid-point of the two foci.

Therefore, $P = (h, k) = \left(\frac{-4+4}{2}, \frac{2+2}{2} \right) = (0, 2)$.

c is the distance between the center and one of the two foci.

Therefore, $c = 4$.

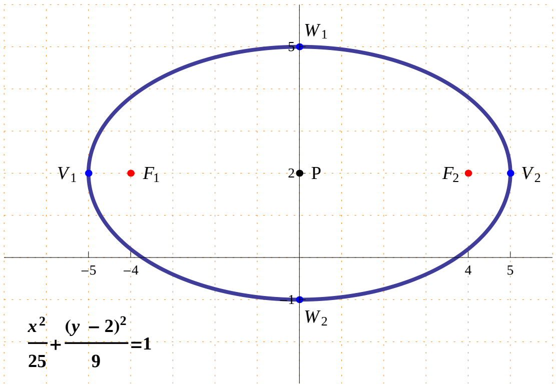
Since the straight line passing through the two foci is paralle to the x-axis, then the length of the major axis is $2a$, hence $2a = 10 \implies a = 5$.

$c^2 = a^2 - b^2 \implies 16 = 25 - b^2 \implies b^2 = 25 - 16 = 9 \implies b = 3.$

The standard equation of the ellipse is $\dfrac{x^2}{25} + \dfrac{(y-2)^2}{9} = 1$

The vertices are $V_1 = (-5, 2)$ and $V_2 = (5, 2)$.

The end-points of the minor axis are $W_1 = (0, 5)$ and $W_2 = (0, -1)$.



M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[5]

Solution of the Second Mid-Term Exam

Second semester 1434-1435 H

Q.1 Compute the integrals :

(a) $\displaystyle\int 3x^2 \ln|x| \; dx$

(b) $\displaystyle\int \frac{x}{(x-1)(x-2)} \; dx$

(c) $\displaystyle\int \frac{1}{x^2+6x+10} \; dx$

(d) $\displaystyle\int (x+1)^{2014} \; dx$

(e) $\displaystyle\int x^2 \cos x \; dx$

Solution :

(a) Using integration by parts

$u = \ln|x| \qquad dv = 3x^2 \; dx$

$du = \dfrac{1}{x} \; dx \qquad v = x^3 \; dx$

$$\int 3x^2 \ln|x| \; dx = x^3 \ln|x| - \int x^3 \; \frac{1}{x} \; dx$$

$$= x^3 \ln|x| - \int x^2 \; dx = x^3 \ln|x| - \frac{x^3}{3} + c$$

(b) $\displaystyle\int \frac{x}{(x-1)(x-2)} \; dx$

Using the method of partial fractions

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\frac{x}{(x-1)(x-2)} = \frac{A(x-2)}{(x-1)(x-2)} + \frac{B(x-1)}{(x-2)(x-5)}$$

$x = A(x-2) + B(x-1)$

Put $x = 1$, then $1 = A(1-2) + B(1-1) \implies -A = 1 \implies A = -1$

Put $x = 2$, then $2 = A(2-2) + B(2-1) \implies B = 2$

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

[5]E-mail : alfadhel@ksu.edu.sa

$$\int \frac{x}{(x-1)(x-2)}\ dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-2}\right)\ dx$$

$$= \int \frac{-1}{x-1}\ dx + \int \frac{2}{x-2}\ dx = -\ln|x-1| + 2\ln|x-2| + c$$

(c) Using the formula $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$

$$\int \frac{1}{x^2+6x+10}\ dx = \int \frac{1}{(x^2+6x+9)+1}\ dx$$

$$= \int \frac{1}{(x+3)^2+1^2} = \tan^{-1}(x+3) + c$$

(d) $\displaystyle\int (x+1)^{2014}\ dx$

Using the formula $\displaystyle\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

$$\int (x+1)^{2014}\ dx = \frac{(x+1)^{2015}}{2015} + c$$

(e) $\displaystyle\int x^2 \cos x\ dx$

Using integration by parts

$u = x^2 \qquad dv = \cos x\ dx$
$du = 2x\ dx \qquad v = \sin x$

$$\int x^2\cos x\ dx = x^2\sin x - \int 2x\sin x\ dx$$

Using integration by parts again

$u = 2x \qquad dv = \sin x\ dx$
$du = 2\ dx \qquad v = -\cos x$

$$\int x^2\cos x\ dx = x^2\sin x - \int 2x\sin x\ dx$$

$$= x^2\sin x - \left(-2x\cos x - \int -2\cos x\ dx\right) = x^2\sin x + 2x\cos x - 2\int \cos x\ dx$$

$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

Q.2 (a) Sketch the region **R** determined by the curves
$y = x^2$, $x = 0$ and $y = -2x + 3$.
(b) Find the area of the region **R** .

Solution :

(a) $y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

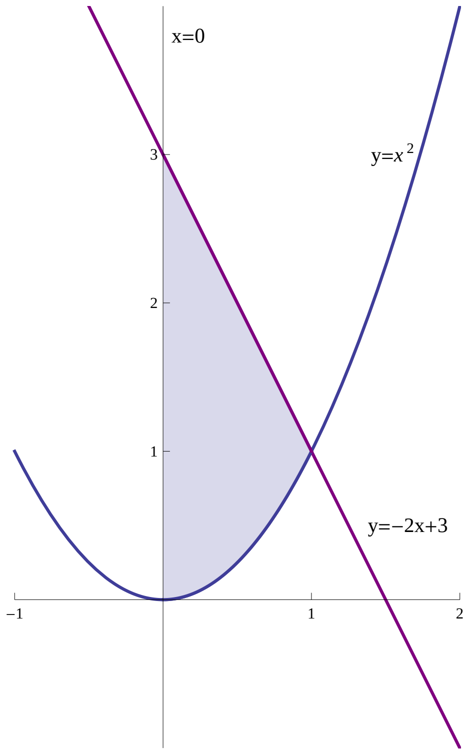
$x = 0$ is the y-axis.

$y = -2x + 3$ is a straight line passes through $(0, 3)$ with slope equals -1.

Points of intersection of $y = x^2$ and $y = -2x + 3$:

$x^2 = -2x + 3 \implies x^2 + 2x - 3 = 0 \implies (x + 3)(x - 1) = 0$

$\implies x = 1 \ , \ x = -3$



(b) Area $= \displaystyle\int_0^1 [(-2x + 3) - x^2] \; dx$

$= \displaystyle\int_0^1 (-x^2 - 2x + 3) \; dx = \left[-\frac{x^3}{3} - x^2 + 3x \right]_0^1$

$= \left(-\dfrac{1}{3} - 1 + 3 \right) - (0 - 0 + 0) = 2 - \dfrac{1}{3} = \dfrac{5}{3}$

Q.3 (a) Sketch the region **R** determined by the curves

$y = \sqrt{x}$, $y = 0$ and $y = -x + 2$

(b) The region **R** is rotated about the y-axis to form a solid of revolution **S**. Find the volume of **S**.

Solution :

(a) $y = 0$ is the x-axis.

$y = -x + 2$ is a straight line passes through $(0, 2)$, with slope equals -1.

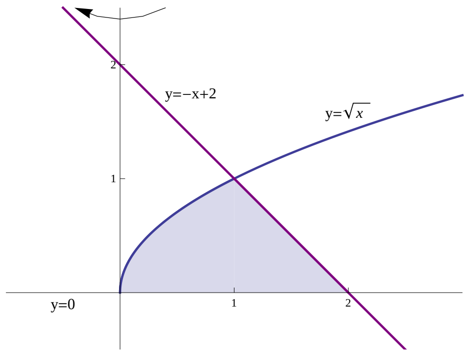
$y = \sqrt{x}$ is the upper-half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right.

Point of intersection of $y = \sqrt{x}$ and $y = -x + 2$:

$y = \sqrt{x} \implies x = y^2$ and $y = -x + 2 \implies x = -y + 2$

$y^2 = -y + 2 \implies y^2 + y - 2 = 0 \implies (y - 1)(y + 2) = 0$

$\implies y = 1$, $y = -2$ (Note that $y = -2$ is not in the desired region)



(b) Using Washer method :

$$\text{Volume} = \pi \int_0^1 \left[(-y + 2)^2 - (y^2)^2\right] \, dy$$

$$= \pi \int_0^1 \left[(y^2 - 4y + 4) - (y^4)\right] \, dy = \pi \int_0^1 (-y^4 + y^2 - 4y + 4) \, dy$$

$$= \pi \left[-\frac{y^5}{5} + \frac{y^3}{3} - 2y^2 + 4y\right]_0^1$$

$$= \pi \left[\left(-\frac{1}{5} + \frac{1}{3} - 2 + 4\right) - (0 + 0 - 0 + 0)\right]$$

$$= \pi \left(2 + \frac{1}{3} - \frac{1}{5}\right) = \pi \, \frac{30 + 5 - 3}{15} = \frac{32\pi}{15}$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel[6]

Solution of the Final Exam

Second semester 1434-1435 H

Q.1 (a) Compute $\mathbf{BA} + \mathbf{AB}$ for $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix}$.

(c) Solve by Gauss-Jordan Method the linear system :

$$\begin{array}{ccccccc} x & + & 3y & - & z & = & 7 \\ x & + & 2y & + & 4z & = & 5 \\ & & 4y & - & 3z & = & 8 \end{array}$$

Solution :

(a) $\mathbf{BA} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0+3+0 & 1+15+1 & 0+3+0 \\ 0+1+0 & 3+5+3 & 0+1+0 \\ 0+3+0 & 1+15+1 & 0+3+0 \end{pmatrix} = \begin{pmatrix} 3 & 17 & 3 \\ 1 & 11 & 1 \\ 3 & 17 & 3 \end{pmatrix}$$

$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0+3+0 & 0+1+0 & 0+3+0 \\ 1+15+1 & 3+5+3 & 1+15+1 \\ 0+3+0 & 0+1+0 & 0+3+0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 3 \\ 17 & 11 & 17 \\ 3 & 1 & 3 \end{pmatrix}$$

$\mathbf{BA} + \mathbf{AB} = \begin{pmatrix} 3 & 17 & 3 \\ 1 & 11 & 1 \\ 3 & 17 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 3 \\ 17 & 11 & 17 \\ 3 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 3+3 & 17+1 & 3+3 \\ 1+17 & 11+11 & 1+17 \\ 3+3 & 17+1 & 3+3 \end{pmatrix} = \begin{pmatrix} 6 & 18 & 6 \\ 18 & 22 & 18 \\ 6 & 18 & 6 \end{pmatrix}$$

(b) **Solutuon (1):** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix} \xrightarrow{-R_1+R_3} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & -2 & -2 & -4 \\ 0 & 1 & 0 & 2 \end{vmatrix}$

[6]E-mail : alfadhel@ksu.edu.sa

$$\xrightarrow{R_2\leftrightarrow R_4} (-1)\times\begin{vmatrix}1&2&3&4\\0&1&0&2\\0&-2&-2&-4\\0&2&0&2\end{vmatrix} \xrightarrow{2R_2+R_3} (-1)\times\begin{vmatrix}1&2&3&4\\0&1&0&2\\0&0&-2&0\\0&2&0&2\end{vmatrix}$$

$$\xrightarrow{-2R_2+R_4} (-1)\times\begin{vmatrix}1&2&3&4\\0&1&0&2\\0&0&-2&0\\0&0&0&-2\end{vmatrix} = (-1)\times(1\times 1\times -2\times -2) = -4$$

Solution (2) : $\begin{vmatrix}1&2&3&4\\0&2&0&2\\1&0&1&0\\0&1&0&2\end{vmatrix} \xrightarrow{-R_1+R_3} \begin{vmatrix}1&2&3&4\\0&2&0&2\\0&-2&-2&-4\\0&1&0&2\end{vmatrix}$

$= 1\times\begin{vmatrix}2&0&2\\-2&-2&-4\\1&0&2\end{vmatrix}$

Using Sarrus Method

$$\begin{matrix}2&0&2&2&0\\-2&-2&-4&-2&-2\\1&0&2&1&0\end{matrix}$$

$$\begin{vmatrix}1&2&3&4\\0&2&0&2\\1&0&1&0\\0&1&0&2\end{vmatrix} = \begin{vmatrix}2&0&2\\-2&-2&-4\\1&0&2\end{vmatrix} = (-8+0+0)-(-4+0+0) = -8+4 = -4$$

(c) Using Gauss-Jordan method :

The augmented matrix is $\left(\begin{array}{ccc|c}1&3&-1&7\\1&2&4&5\\0&4&-3&8\end{array}\right)$

$$\left(\begin{array}{ccc|c}1&3&-1&7\\1&2&4&5\\0&4&-3&8\end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c}1&3&-1&7\\0&-1&5&-2\\0&4&-3&8\end{array}\right)$$

$$\xrightarrow{4R_2+R_3} \left(\begin{array}{ccc|c}1&3&-1&7\\0&-1&5&-2\\0&0&17&0\end{array}\right) \xrightarrow{\frac{1}{17}R_3} \left(\begin{array}{ccc|c}1&3&-1&7\\0&-1&5&-2\\0&0&1&0\end{array}\right)$$

$$\xrightarrow{-5R_3+R_2} \left(\begin{array}{ccc|c}1&3&-1&7\\0&-1&0&-2\\0&0&1&0\end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c}1&3&0&7\\0&-1&0&-2\\0&0&1&0\end{array}\right)$$

$$\xrightarrow{3R_2+R_1} \left(\begin{array}{ccc|c}1&0&0&1\\0&-1&0&-2\\0&0&1&0\end{array}\right)$$

$x = 1$, $y = 2$ and $z = 0$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Q.2 Find The elements of the two following conic sections and sketch their graphs:

(a) $4y^2 + 9x^2 - 18x = 27$

(b) $4x^2 - 9y^2 + 24x - 36y - 36 = 0$

Solution :

(a) $4y^2 + 9x^2 - 18x = 27$

$4y^2 + 9(x^2 - 2x) = 27$

By completing the square

$4y^2 + 9(x^2 - 2x + 1) = 27 + 9$

$4y^2 + 9(x - 1)^2 = 36$

$\frac{4y^2}{36} + \frac{9(x-1)^2}{36} = 1$

$\frac{y^2}{9} + \frac{(x-1)^2}{4} = 1$

The conic section is an ellipse.

The center is $P = (1, 0)$.

$a^2 = 4 \implies a = 2$, $b^2 = 9 \implies b = 3.$

$c^2 = b^2 - a^2 = 9 - 4 = 5 \implies b = \sqrt{5}$.

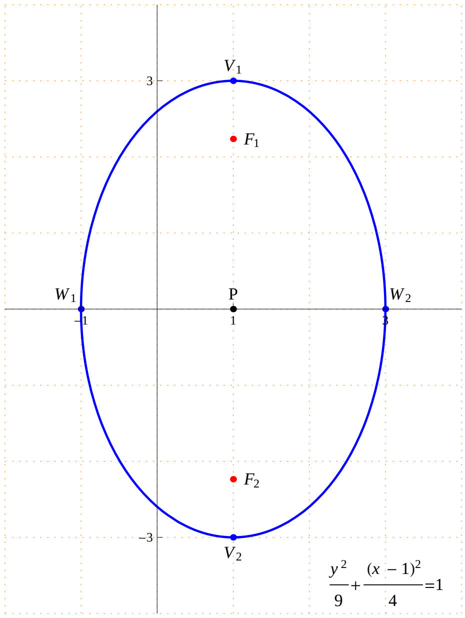
The foci are $F_1 = \left(1, 0 + \sqrt{5}\right) = \left(1, \sqrt{5}\right)$ and $F_2 = \left(1, 0 - \sqrt{5}\right) = \left(1, -\sqrt{5}\right)$.

The vertices are $V_1 = (1, 0 + 3) = (1, 3)$ and $V_2 = (1, 0 - 3) = (1, -3)$.

The length of the major axis is $2b = 6$

The end-points of the minor axis are $W_1 = (1 - 2, 0) = (-1, 0)$ and $W_2 = (1 + 2, 0) = (3, 0)$

The length of the minor axis is $2a = 4$



(b) $4x^2 - 9y^2 + 24x - 36y - 36 = 0$

$4x^2 + 24x - 9y^2 - 36y = 36$

$4(x^2 + 6x) - 9(y^2 + 4y) = 36$

By completing the square

$4(x^2 + 6x + 9) - 9(y^2 + 4y + 4) = 36 + 36 - 36$

$4(x + 3)^2 - 9(y + 2)^2 = 36$

$$\frac{4(x+3)^2}{36} - \frac{9(y+2)^2}{36} = 1$$

$$\frac{(x+3)^2}{9} - \frac{(y+2)^2}{4} = 1$$

The conic section is a hyperbola with transverse axis parallel to the x-axis.

The ceneter is $P = (-3, -2)$.

$a^2 = 9 \implies a = 3$ and $b^2 = 4 \implies b = 2$.

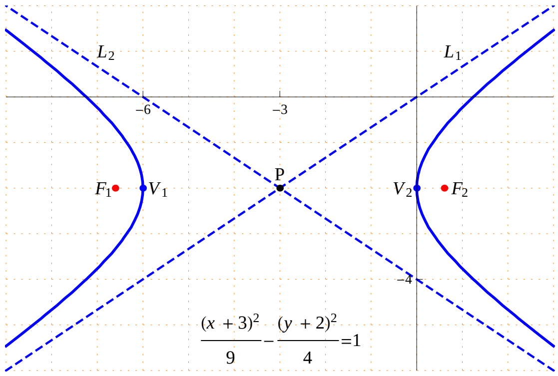
$c^2 = a^2 + b^2 = 9 + 4 = 13 \implies c = \sqrt{13}$.

The vertices are $V_1 = (-3 - 3, -2) = (-6, -2)$ and $V_2 = (-3 + 3, -2) = (0, -2)$.

The foci are $F_1 = \left(-3 - \sqrt{13}, -2\right)$ and $F_2 = \left(-3 + \sqrt{13}, -2\right)$.

The equations of the asymptotes are $L_1 \ : \ y + 2 = \frac{2}{3}\,(x + 3)$ and

$L_2 \ : \ y+2=-\dfrac{2}{3}\,(x+3)$



Q.3 **(a)** Compute the integrals :

(i) $\displaystyle\int \frac{x+1}{x(x+1)}\,dx$ (ii) $\displaystyle\int x\sin x\;dx$ (iii) $\displaystyle\int \frac{4}{x^2-4x+5}\,dx$

(b) Find the area of the surface delimited by the curves :

$y=x^2-1$ and $y=0$.

(c) The region R between the curves $x=0$, $y=0$, $y=x^2+1$ and $x=1$ is rotated about the y-axis to form a solid of revolution S . Find the volume of S.

Solution :

(a) (i) $\displaystyle\int \frac{x+1}{x(x+1)}dx=\int \frac{1}{x}\,dx=\ln|x|+c$

(ii) $\displaystyle\int x\sin x\;dx$

Using integration by parts

$$\begin{array}{ll} u=x & dv=\sin x\;dx \\ du=dx & v=-\cos x \end{array}$$

$$\int x\sin x\;dx=-x\cos x-\int -\cos x\;dx=-x\cos x+\int \cos x\;dx$$

$=-x\cos x+\sin x+c$

(iii) $\displaystyle\int \frac{4}{x^2-4x+5}\,dx$

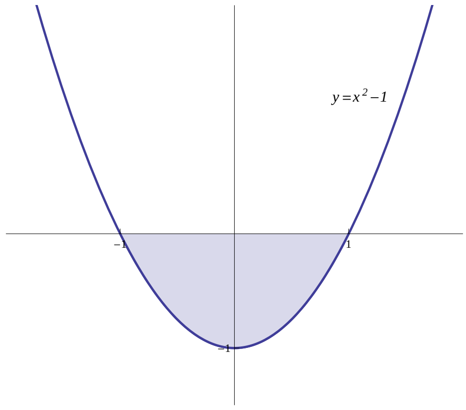
Using the formula $\displaystyle\int \frac{f'(x)}{[f(x)]^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)+c$

$$\int \frac{4}{x^2-4x+5}\,dx=4\int \frac{1}{(x^2-4x+4)+1}\,dx$$

$$= 4\int \frac{1}{(x-2)^2+1^2}\,dx = 4\tan^{-1}(x-2)+c$$

(b) $y = x^2 - 1$ is a parabola opens upwards with vertex $(0,-1)$.

$y = 0$ is the x-axis. .



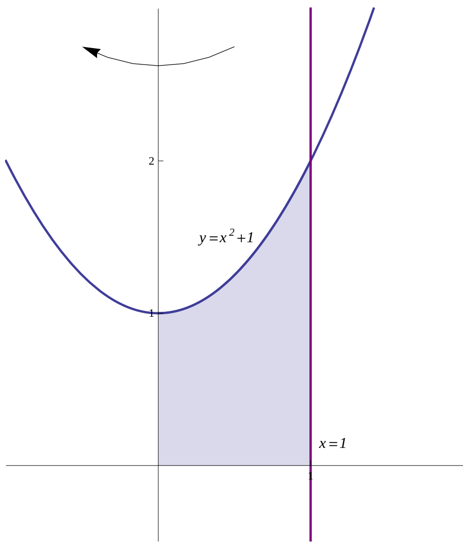
Points of intersections of $y = x^2 - 1$ and $y = 0$:

$x^2 - 1 = 0 \quad \Rightarrow \quad (x-1)(x+1) = 0 \quad \Rightarrow \quad x = -1\ ,\ x = 1$

$$\text{Area} = \int_{-1}^{1} [0 - (x^2-1)]\,dx = \int_{-1}^{1} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$\text{Area} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

(c)



$x = 0$ is the y-axis.

$y = 0$ is the x-axis.

$x = 1$ is a straight line parallel to the y-axis and passes through $(1, 0)$.

$y = x^2 + 1$ is a parabola opens upwards with vertex $(0, 1)$.

Using Cylindrical shells Method :

$$\text{Volume} = 2\pi \int_0^1 x(x^2+1)\ dx = 2\pi \int_0^1 (x^3+x)\ dx = 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$\text{Volume} = 2\pi \left[\left(\frac{1}{4} + \frac{1}{2}\right) - (0+0)\right] = 2\pi \left(\frac{3}{4} - 0\right) = \frac{3\pi}{2}$$

Q.4 **(a)** Find f_x , f_y and f_z for the function $f(x, y) = xy \ln(x^2 + y^3 + z)$

(b) Solve the differential equation $dy - \sin x\ (1 + y^2)\ dx = 0$

Solution :

(a) $$f_x = \frac{\partial f}{\partial x} = (1 \times y) \ln(x^2+y^3+z) + xy \left(\frac{2x+0+0}{x^2+y^3+z}\right)$$

$$f_x = y \ln(x^2+y^3+z) + \frac{2x^2 y}{x^2+y^3+z}$$

$$f_y = \frac{\partial f}{\partial y} = (1 \times x) \ln(x^2+y^3+z) + xy \left(\frac{0+3y^2+0}{x^2+y^3+z}\right)$$

$$f_y = x \ln(x^2+y^3+z) + \frac{3xy^3}{x^2+y^2+z}$$

$$f_z = \frac{\partial f}{\partial z} = xy \left(\frac{0+0+1}{x^2+y^3+z}\right) = \frac{xy}{x^2+y^3+z}$$

(b) $dy - \sin x\ (1 + y^2)\ dx = 0$

The differential equation is separable.

$$dy - \sin x(1+y^2)\ dx = 0 \implies dy = \sin x(1+y^2)\ dx$$

$$\implies \frac{1}{1+y^2}\ dy = \sin x\ dx \implies \int \frac{1}{1+y^2}\ dy = \int \sin x\ dx$$

$$\implies \tan^{-1} y = -\cos x + c \implies y = \tan(-\cos x + c)$$