

M 104 - GENERAL MATHEMATICS -2-

*Dr. Tariq A. AlFadhel*¹

Solution of the First Mid-Term Exam

First semester 1433-1434 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$

Compute (if possible) : \mathbf{BA} , \mathbf{BAC} and \mathbf{CB}

Solution :

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-6+6 & 2-6+0 & 3+0+3 \\ 4+15-12 & 8+15+0 & 12+0-6 \\ 2+0+2 & 4+0+0 & 6+0+1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 6 \\ 7 & 23 & 6 \\ 4 & 4 & 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{BAC} &= (\mathbf{BA})\mathbf{C} = \begin{pmatrix} 1 & -4 & 6 \\ 7 & 23 & 6 \\ 4 & 4 & 7 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1-8+0 & -1-12+24 \\ 7+46+0 & -7+69+24 \\ 4+8+0 & -4+12+28 \end{pmatrix} = \begin{pmatrix} -7 & 11 \\ 53 & 86 \\ 12 & 36 \end{pmatrix} \end{aligned}$$

\mathbf{CB} is not possible because the number of columns of \mathbf{C} (which is 2) is not equal to the number of rows of \mathbf{B} (which is 3)

Q.2 Compute The determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix}$

Solution (1) : Using the properties of the determinants

Multiply C_2 by 3 and C_3 by 2

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{vmatrix} 1 & 6 & 6 & 4 \\ 0 & 6 & 6 & 0 \\ 1 & 6 & 6 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = 0 \text{ (since } C_2 = C_3 \text{)}$$

Solution (2) : Interchange R_1 and R_4 .

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$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{vmatrix} = -3 \begin{vmatrix} 2 & 3 & 0 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

Using Sarrus method

$$\begin{array}{cccc} 2 & 3 & 0 & 2 & 3 \\ 2 & 3 & 5 & 2 & 3 \\ 2 & 3 & 4 & 2 & 3 \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = -3[(24 + 30 + 0) - (0 + 30 + 24)] = -3[54 - 54] = 0$$

Q.3 Find all the elements of the conic section $9x^2 + 16y^2 + 96y - 36x + 36 = 0$ and sketch it.

Solution :

$$9x^2 + 16y^2 + 96y - 36x + 36 = 0 \Rightarrow 9x^2 - 36x + 16y^2 + 96y = -36$$

$$\Rightarrow 9(x^2 - 4x) + 16(y^2 + 6y) = -36$$

$$\Rightarrow 9(x^2 - 4x + 4) + 16(y^2 + 6y + 9) = -36 + 36 + 144$$

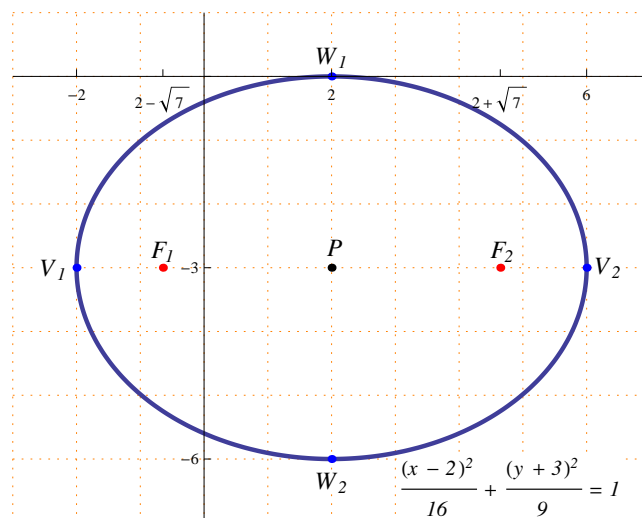
$$\Rightarrow 9(x - 2)^2 + 16(y + 3)^2 = 144 \Rightarrow \frac{9(x - 2)^2}{144} + \frac{16(y + 3)^2}{144} = 1$$

$$\frac{(x - 2)^2}{16} + \frac{(y + 3)^2}{9} = 1$$

The conic section is an Ellipse

$$a^2 = 16 \Rightarrow a = 4 \text{ and } b^2 = 9 \Rightarrow b = 3$$

$$c^2 = a^2 - b^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$$



The center is $P = (2, -3)$

The vertices are $V_1 = (-2, -3)$ and $V_2 = (6, -3)$, and the length of the major axis is $2a = 6$.

The foci are $F_1 = (2 - \sqrt{7}, -3)$ and $F_2 = (2 + \sqrt{7}, -3)$.

The end points of the minor axis are $W_1 = (2, 0)$ and $W_2 = (2, -6)$, and its length is $2b = 4$

Q.4 Find the standard equation of the hyperbola with foci $(4, 3)$ and $(-2, 3)$ and with vertex $(3, 3)$, then sketch it.

Solution :

Since the line passing through the two foci is parallel to the x -axis then the standard equation has the form $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$.

The center of the hyperbola is the middle point between $(-2, 3)$ and $(4, 3)$, hence $(h, k) = (1, 3)$

c is the distance between the center and one of the foci, hence $c = 3$

a is the distance between the center and the vertex $(3, 3)$, hence $a = 2$

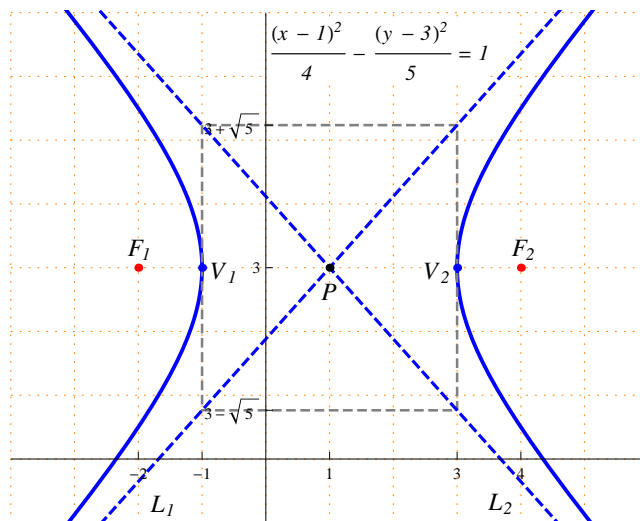
$$c^2 = a^2 + b^2 \Rightarrow 9 = 4 + b^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}$$

The standard equation of the hyperbola is $\frac{(x - 1)^2}{4} - \frac{(y - 3)^2}{5} = 1$.

The other vertex is $(-1, 3)$ and the length of the transverse axis is $2a = 4$.

The equations of the asymptotes are $L_1 : y - 3 = \frac{\sqrt{5}}{2}(x - 1)$

and $L_2 : y - 3 = -\frac{\sqrt{5}}{2}(x - 1)$



Q.5 Find all the elements of the conic section $y^2 - 4y - 4x = 0$

Solution :

$$y^2 - 4y - 4x = 0 \Rightarrow y^2 - 4y = 4x \Rightarrow y^2 - 4y + 4 = 4x + 4 \\ \Rightarrow (y - 2)^2 = 4(x + 1)$$

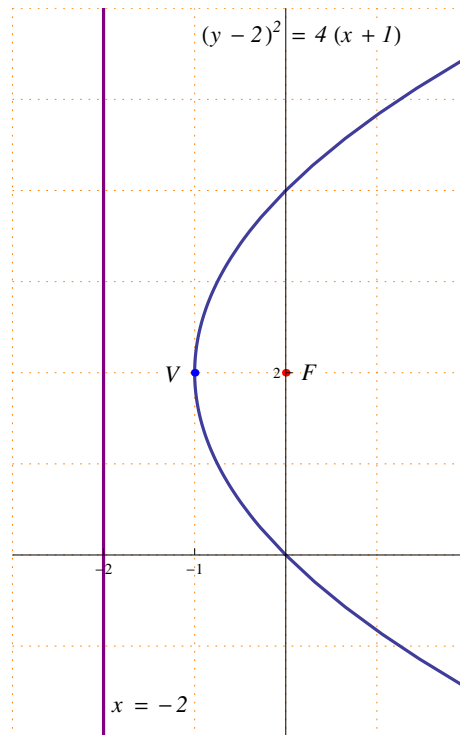
The conic section is parabola opens to the right.

The vertex is $V = (-1, 2)$

$$4a = 4 \Rightarrow a = 1$$

The focus is $F = (0, 2)$

The equation of the directrix is $x = -2$



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Solution of the Second Mid-Term Exam

First semester 1433-1434 H

Q.1 Solve by Gauss elimination method :

$$\begin{aligned}x + y + z &= 4 \\x - y - z &= 6 \\2x + y - z &= 3\end{aligned}$$

Solution : The augmented matrix is

$$\begin{aligned}& \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 6 \\ 2 & 1 & -1 & 3 \end{array} \right) \\& \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 \\ 2 & 1 & -1 & 3 \end{array} \right) \\& \xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 \\ 0 & -1 & -3 & -5 \end{array} \right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -3 & -5 \end{array} \right) \\& \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -6 \end{array} \right) \xrightarrow{-\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right)\end{aligned}$$

$$z = 3$$

$$y + z = -1 \implies y + 3 = -1 \implies y = -4$$

$$x + y + z = 4 \implies x - 4 + 3 = 4 \implies x = 5$$

Q.2 Solve by Cramer's rule :

$$\begin{aligned}x - 2y &= 3 \\3x + 5y &= 20\end{aligned}$$

Solution :

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 3 & -2 \\ 20 & 5 \end{pmatrix} \text{ and } \mathbf{A}_2 = \begin{pmatrix} 1 & 3 \\ 3 & 20 \end{pmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = (1 \times 5) - (3 \times -2) = 5 - (-6) = 5 + 6 = 11 .$$

$$\det(\mathbf{A}_1) = \begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix} = (3 \times 5) - (20 \times -2) = 15 - (-40) = 15 + 40 = 55$$

$$\det(\mathbf{A}_2) = \begin{vmatrix} 1 & 3 \\ 3 & 20 \end{vmatrix} = (1 \times 20) - (3 \times 3) = 20 - 9 = 11 .$$

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$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{55}{11} = 5$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{11}{11} = 1$$

Q.3 Compute the integrals :

$$(a) \int (x^2 + x)^{11}(2x + 1) dx \quad (b) \int \frac{2x - 1}{x^2 + 1} dx$$

$$(c) \int \frac{x + 2}{(x - 2)^2(x - 1)} dx \quad (d) \int x^{12} \ln(x) dx$$

Solution :

$$(a) \int (x^2 + x)^{11}(2x + 1) dx = \frac{(x^2 + x)^{12}}{12} + c .$$

$$(b) \int \frac{2x - 1}{x^2 + 1} dx = \int \frac{2x}{x^2 + 1} dx - \int \frac{1}{x^2 + 1} dx$$

$$= \ln(x^2 + 1) - \tan^{-1} x + c .$$

$$(c) \int \frac{x + 2}{(x - 2)^2(x - 1)} dx$$

Using the method of partial fractions

$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}$$

$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{A(x - 2)^2}{(x - 1)(x - 2)^2} + \frac{B(x - 1)(x - 2)}{(x - 1)(x - 2)^2} + \frac{C(x - 1)}{(x - 1)(x - 2)^2}$$

$$x + 2 = A(x - 2)^2 + B(x - 1)(x - 2) + C(x - 1)$$

$$x + 2 = A(x^2 - 4x + 4) + B(x^2 - 3x + 2) + C(x - 1)$$

$$x + 2 = Ax^2 - 4Ax + 4A + Bx^2 - 3Bx + 2B + Cx - C$$

$$x + 2 = (A + B)x^2 + (-4A - 3B + C)x + (4A + 2B - C)$$

By comparing the coefficients :

$$A + B = 0 \quad \longrightarrow (1)$$

$$-4A - 3B + C = 1 \quad \longrightarrow (2)$$

$$4A + 2B - C = 2 \quad \longrightarrow (3)$$

Adding equation (2) and equation (3) : $-B = 3 \implies B = -3$.

From equation (1) : $A - 3 = 0 \implies A = 3$

From equation (2) : $-12 + 9 + C = 1 \implies C = 4$

$$\frac{x + 2}{(x - 1)(x - 2)^2} = \frac{3}{x - 1} + \frac{-3}{x - 2} + \frac{4}{(x - 2)^2}$$

$$\int \frac{x + 2}{(x - 2)^2(x - 1)} dx = \int \left(\frac{3}{x - 1} + \frac{-3}{x - 2} + \frac{4}{(x - 2)^2} \right) dx$$

$$= \int \frac{3}{x - 1} dx + \int \frac{-3}{x - 2} dx + \int \frac{4}{(x - 2)^2} dx$$

$$= 3 \int \frac{1}{x - 1} dx - 3 \int \frac{1}{x - 2} dx + 4 \int (x - 2)^{-2} dx$$

$$= 3 \ln |x - 1| - 3 \ln |x - 2| + 4 \frac{(x - 2)^{-1}}{-1} + c$$

$$= 3 \ln|x-1| - 3 \ln|x-2| - \frac{4}{x-2} + c$$

$$(d) \int x^{12} \ln(x) dx$$

Using integration by parts

$$u = \ln(x) \quad dv = x^{12} dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^{13}}{13}$$

$$\int x^{12} \ln(x) dx = \frac{x^{13}}{13} \ln(x) - \int \frac{1}{x} \frac{x^{13}}{13} dx$$

$$= \frac{x^{13}}{13} \ln(x) - \frac{1}{13} \int x^{12} dx = \frac{x^{13}}{13} \ln(x) - \frac{1}{13} \frac{x^{13}}{13} + c$$

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Solution of the Final Exam

First semester 1433-1434 H

Q.1 (a) Compute : $2\mathbf{BA} - \mathbf{AB}$ for $\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 7 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 1 & 2 \\ 1 & 9 & 1 \end{pmatrix}$

(b) Compute the determinant $\begin{vmatrix} 11 & 1 & 7 \\ 1 & 2 & 8 \\ 4 & -1 & 0 \end{vmatrix}$.

(c) Solve by Gauss Elimination Method : $\begin{matrix} x & - & 2y & + & z & = & 8 \\ -2x & + & y & - & 3z & = & -13 \\ 4x & - & y & + & z & = & 9 \end{matrix}$

Solution :

(a) $\mathbf{BA} = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 1 & 2 \\ 1 & 9 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 1 & 7 & 1 \\ 0 & 1 & 3 \end{pmatrix}$
 $= \begin{pmatrix} 2+7+0 & 2+49+1 & 0+7+3 \\ 4+1+0 & 4+7+2 & 0+1+6 \\ 2+9+0 & 2+63+1 & 0+9+3 \end{pmatrix} = \begin{pmatrix} 9 & 52 & 10 \\ 5 & 13 & 7 \\ 11 & 66 & 12 \end{pmatrix}$
 $2\mathbf{BA} = 2 \begin{pmatrix} 9 & 52 & 10 \\ 5 & 13 & 7 \\ 11 & 66 & 12 \end{pmatrix} = \begin{pmatrix} 2 \times 9 & 2 \times 52 & 2 \times 10 \\ 2 \times 5 & 2 \times 13 & 2 \times 7 \\ 2 \times 11 & 2 \times 66 & 2 \times 12 \end{pmatrix} = \begin{pmatrix} 18 & 104 & 20 \\ 10 & 26 & 14 \\ 22 & 132 & 24 \end{pmatrix}$
 $\mathbf{AB} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 7 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 7 & 1 \\ 2 & 1 & 2 \\ 1 & 9 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 2+4+0 & 14+2+0 & 2+4+0 \\ 1+14+1 & 7+7+9 & 1+14+1 \\ 0+2+3 & 0+1+27 & 0+2+3 \end{pmatrix} = \begin{pmatrix} 6 & 16 & 6 \\ 16 & 23 & 16 \\ 5 & 28 & 5 \end{pmatrix}$
 $2\mathbf{BA} - \mathbf{AB} = \begin{pmatrix} 18 & 104 & 20 \\ 10 & 26 & 14 \\ 22 & 132 & 24 \end{pmatrix} - \begin{pmatrix} 6 & 16 & 6 \\ 16 & 23 & 16 \\ 5 & 28 & 5 \end{pmatrix}$
 $= \begin{pmatrix} 18-6 & 104-16 & 20-6 \\ 10-16 & 26-23 & 14-16 \\ 22-5 & 132-28 & 24-5 \end{pmatrix} = \begin{pmatrix} 12 & 88 & 14 \\ -6 & 3 & -2 \\ 17 & 104 & 19 \end{pmatrix}$

(b) Using Sarrus method $\begin{matrix} 11 & 1 & 7 & 11 & 1 \\ 1 & 2 & 8 & 1 & 2 \\ 4 & -1 & 0 & 4 & -1 \end{matrix}$

$$\begin{vmatrix} 11 & 1 & 7 \\ 1 & 2 & 8 \\ 4 & -1 & 0 \end{vmatrix} = (0+32-7) - (56-88+0) = 25 - (-32) = 25+32 = 57$$

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(c) The augmented matrix is
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ -2 & 1 & -3 & -13 \\ 4 & -1 & 1 & 9 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ -2 & 1 & -3 & -13 \\ 4 & -1 & 1 & 9 \end{array} \right) \xrightarrow{2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 4 & -1 & 1 & 9 \end{array} \right)$$

$$\xrightarrow{-4R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 7 & -3 & -23 \end{array} \right) \xrightarrow{3R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 21 & -9 & -69 \end{array} \right)$$

$$\xrightarrow{7R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 0 & -16 & -48 \end{array} \right) \xrightarrow{-\frac{1}{16}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$z = 3$

$$-3y - z = 3 \implies -3y - 3 = 3 \implies -3y = 6 \implies y = -2$$

$$x - 2y + z = 8 \implies x - 2(-2) + 3 = 8 \implies x + 4 + 3 = 8 \implies x = 1$$

The solution is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Q.2 Find all the elements of the two following conic sections and sketch their graphs:

(a) $x^2 + 9y^2 - 72y - 4x + 139 = 0$

(b) $9x^2 - 4y^2 - 16y - 18x - 43 = 0$

Solution :

(a) $x^2 + 9y^2 - 72y - 4x + 139 = 0$

$$(x^2 - 4x) + (9y^2 - 72y) = -139$$

$$(x^2 - 4x) + 9(y^2 - 8y) = -139$$

$$(x^2 - 4x + 4) + 9(y^2 - 8y + 16) = -139 + 4 + 144$$

$$(x - 2)^2 + 9(y - 4)^2 = 9$$

$$\frac{(x - 2)^2}{9} + \frac{9(y - 4)^2}{9} = 1$$

$$\frac{(x - 2)^2}{9} + (y - 4)^2 = 1$$

The conic section is an ellipse

The center is $P(2, 4)$

$$a^2 = 9 \implies a = 3$$

$$b^2 = 1 \implies b = 1$$

$$c^2 = a^2 - b^2 = 9 - 1 = 8 \implies c = \sqrt{8} = 2\sqrt{2}$$

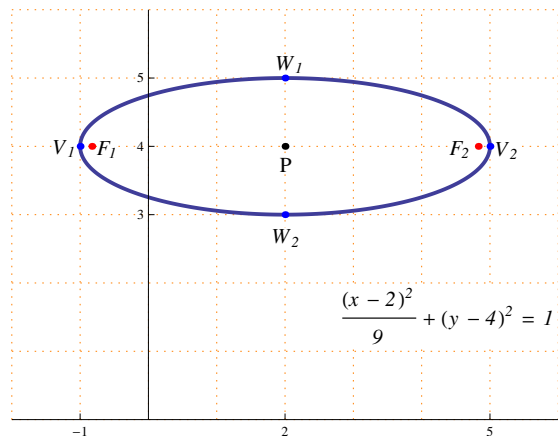
The vertices are $V_1(-1, 4)$ and $V_2(5, 4)$.

The major axis is parallel to the x -axis and its length is $2a = 6$

The foci are $F_1(2 - 2\sqrt{2}, 4)$ and $F_2(2 + 2\sqrt{2}, 4)$

The end-points of the minor axis are $W_1(2, 5)$ and $W_2(2, 3)$

The minor axis is parallel to the y -axis and its length is $2b = 2$



(b) $9x^2 - 4y^2 - 16y - 18x - 43 = 0$

$$(9x^2 - 18x) - (4y^2 + 16y) = 43$$

$$9(x^2 - 2x) - 4(y^2 + 4y) = 43$$

$$9(x^2 - 2x + 1) - 4(y^2 + 4y + 4) = 43 + 9 - 16$$

$$9(x-1)^2 - 4(y+2)^2 = 36$$

$$\frac{9(x-1)^2}{36} - \frac{4(y+2)^2}{36} = 1$$

$$\frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1$$

The conic section is a hyperbola

The center is $P(1, -2)$

$$a^2 = 4 \Rightarrow a = 2$$

$$b^2 = 9 \Rightarrow b = 3$$

$$c^2 = a^2 + b^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$$

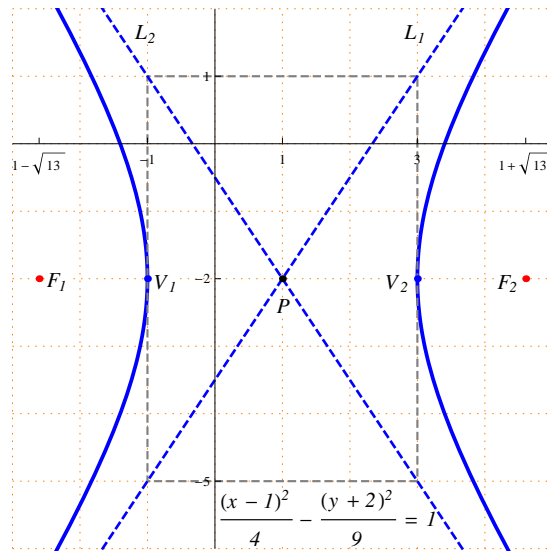
The transverse axis is parallel to the x -axis and its length is $2a = 4$

The vertices are $v_1(-1, -2)$ and $V_2(3, -2)$

The foci are $F_1(1 - \sqrt{13}, -2)$ and $F_2(1 + \sqrt{13}, -2)$.

The equations of the asymptotes are $L_1 : y = \frac{3}{2}(x-1) - 2$

and $L_2 : y = -\frac{3}{2}(x-1) - 2$



Q.3 (a) Compute the integrals :

(i) $\int \frac{1}{x(x+1)} dx$ (ii) $\int (x-1) \cos x dx$ (iii) $\int \frac{2x-2}{x^2-4x+5} dx$

(b) Find the area of the surface delimited by the curves :

$y = 0$, $x = 0$, $x = 4$ and $y = \sqrt{x} + 5$.

(c) The region R between the curves $y = 2$, $y = x^2+2$ and $y = -x+4$ is rotated about the y -axis to form a solid of revolution S . Find the volume of S .

Solution :

(a) (i) $\int \frac{1}{x(x+1)} dx$

Using the method of partial fractions

$$\frac{1}{x(x+1)} = \frac{A_1}{x} + \frac{A_2}{x+1}$$

$$\frac{1}{x(x+1)} = \frac{A_1(x+1)}{x(x+1)} + \frac{A_2x}{x(x+1)}$$

$$1 = A_1(x+1) + A_2x = A_1x + A_1 + A_2x$$

$$1 = (A_1 + A_2)x + A_1$$

By comparing the coefficients :

$$A_1 + A_2 = 0 \quad \rightarrow (1)$$

$$A_1 = 1 \quad \rightarrow (2)$$

From equation (1) : $A_1 + A_2 = 0 \implies 1 + A_2 = 0 \implies A_2 = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$$

$$\int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx = \ln|x| - \ln|x+1| + c$$

$$(ii) \int (x-1) \cos x dx$$

Using integration by parts

$$\begin{aligned} u &= x-1 & dv &= \cos x dx \\ du &= dx & v &= \sin x \end{aligned}$$

$$\begin{aligned} \int (x-1) \cos x dx &= (x-1) \sin x - \int \sin x dx \\ &= (x-1) \sin x - (-\cos x) + c = (x-1) \sin x + \cos x + c \end{aligned}$$

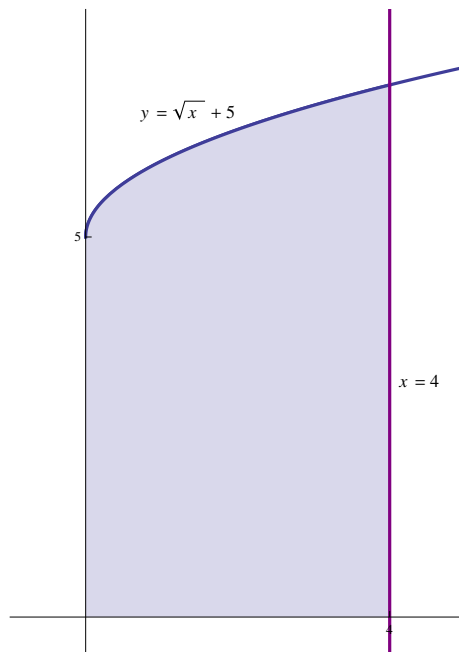
$$\begin{aligned} (iii) \int \frac{2x-2}{x^2-4x+5} dx \\ &= \int \frac{2x-4}{x^2-4x+5} dx + \int \frac{2}{x^2-4x+5} dx \\ &= \int \frac{2x-4}{x^2-4x+5} dx + 2 \int \frac{1}{(x^2-4x+4)+1} dx \\ &= \int \frac{2x-4}{x^2-4x+5} dx + 2 \int \frac{1}{(x-2)^2+(1)^2} dx \\ &= \ln|x^2-4x+5| + 2 \tan^{-1}(x-2) + c \end{aligned}$$

(b) $y = 0$ is the x -axis and $x = 0$ is the y -axis .

$x = 4$ is a straight line parallel to the y -axis and passing through $(4,0)$.

$$y = \sqrt{x} + 5 \implies y - 5 = \sqrt{x} \implies (y - 5)^2 = x .$$

$y = \sqrt{x} + 5$ is the upper half of the parabola $(y - 5)^2 = x$ which opens to the right and with vertex $(0, 5)$.



$$\text{Area} = \int_0^4 (\sqrt{x} + 5) \, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5x \right]_0^4 = \left[\frac{2}{3}x^{\frac{3}{2}} + 5x \right]_0^4$$

$$\text{Area} = \left(\frac{2}{3}(4)^{\frac{3}{2}} + 20 \right) - (0 + 0) = \frac{2}{3} \times 8 + 20 = \frac{16}{3} + \frac{60}{3} = \frac{76}{3}$$

(c) $y = 2$ is a straight line parallel to the x -axis and passing through $(0, 2)$.

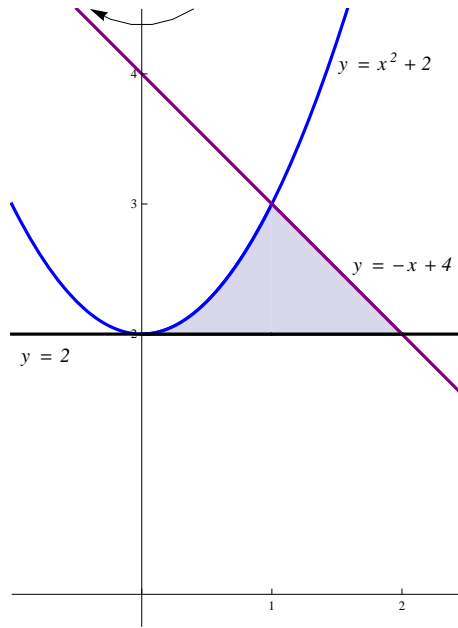
$y = -x + 4$ is a straight line with slope -1 and passing through $(0, 4)$.

$y = x^2 + 2 \Rightarrow x^2 = (y - 2)$ is a parabola with vertex $(0, 2)$ and opens upwards

Points of intersection of $y = -x + 4$ and $y = x^2 + 2$

$$x^2 + 2 = -x + 4 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x - 1)(x + 2) = 0$$

$$\Rightarrow x = 1, x = -2 \Rightarrow y = 3, y = 6$$



$$y = -x + 4 \Rightarrow x = -y + 4$$

$$y = x^2 + 2 \Rightarrow x = \sqrt{y - 2}$$

Using Washer Method :

$$\begin{aligned} \text{Volume} &= \pi \int_2^3 \left[(-y + 4)^2 - (\sqrt{y - 2})^2 \right] dy \\ &= \pi \int_2^3 [(y^2 - 8y + 16) - (y - 2)] dy = \pi \int_2^3 (y^2 - 8y + 16 - y + 2) dy \\ &= \pi \int_2^3 (y^2 - 9y + 18) dy = \pi \left[\frac{y^3}{3} - 9\frac{y^2}{2} + 18y \right]_2^3 \\ &= \pi \left[\left(\frac{27}{3} - 9 \times \frac{9}{2} + 18 \times 3 \right) - \left(\frac{8}{3} - 9 \times \frac{4}{2} + 18 \times 2 \right) \right] \\ &= \pi \left(\frac{19}{3} - \frac{45}{2} + 18 \right) = \pi \left(\frac{38 - 135 + 108}{6} \right) = \frac{11}{6} \pi \end{aligned}$$

Q.4 (a) Find f_x and f_y for the function $f(x, y) = x^2y^2 + xy e^y + \frac{x + y}{x^2 + y^2}$

(b) Solve the differential equation $x \frac{dy}{dx} - y = 2013x^2 e^x$

Solution :

$$\begin{aligned} \text{(a)} \quad f_x &= \frac{\partial f}{\partial x} = 2xy^2 + ye^y + \frac{(1)(x^2 + y^2) - (x + y)(2x)}{(x^2 + y^2)^2} \\ f_y &= \frac{\partial f}{\partial y} = 2yx^2 + x(e^y + ye^y) + \frac{(1)(x^2 + y^2) - (x + y)(2y)}{(x^2 + y^2)^2} \end{aligned}$$

$$\text{(b)} \quad x \frac{dy}{dx} - y = 2013x^2 e^x$$

$$xy' - y = 2013x^2 e^x$$

$$y' - \frac{1}{x}y = 2013xe^x$$

$$P(x) = -\frac{1}{x} \text{ and } Q(x) = 2013xe^x .$$

The integratin factor is

$$u(x) = e^{\int -\frac{1}{x} dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1} = \frac{1}{x}$$

The solution of the differential equation is $y = \frac{1}{x^{-1}} \int \frac{1}{x}(2013xe^x) dx$

$$y = x \int 2013e^x dx = x(2013e^x + c) = 2013xe^x + cx$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel⁴

Solution of the First Mid-Term Exam

Second semester 1433-1434 H

Q.1 Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 0 \end{pmatrix}$

Compute (if possible) : \mathbf{BC} , \mathbf{ABC} and \mathbf{CB}

Solution :

$$\mathbf{BC} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1+4+15 & 1+6+0 \\ 2+10-15 & 2+15+0 \\ 1+0+10 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 20 & 7 \\ -3 & 17 \\ 11 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{ABC} &= \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 20 & 7 \\ -3 & 17 \\ 11 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 20+6+33 & 7-34+3 \\ -20-9+44 & -7+51+4 \\ 40+0+11 & 14+0+1 \end{pmatrix} = \begin{pmatrix} 59 & -24 \\ 15 & 48 \\ 51 & 15 \end{pmatrix} \end{aligned}$$

\mathbf{CB} is not possible because the number of columns of \mathbf{C} (which is 2) is not equal to the number of rows of \mathbf{B} (which is 3)

Q.2 Compute the determinant $\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix}$

Solution 1 : Using the properties of determinants

$$\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} = 0 \text{ (because } R_3 = -3R_1 \text{)}$$

Solution 2 : Using Sarrus Method

$$\begin{array}{ccccc} 1 & -2 & 4 & 1 & -2 \\ 3 & 1 & 2 & 3 & 1 \\ -3 & 6 & -12 & -3 & 6 \end{array}$$

$$\begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} = (-12 + 12 + 72) - (-12 + 12 + 72) = 0$$

Solution 3 : Using the definition of the determinant

$$\begin{aligned} \begin{vmatrix} 1 & -2 & -4 \\ 3 & 1 & 2 \\ -3 & 6 & -12 \end{vmatrix} &= 1 \times \begin{vmatrix} 1 & 2 \\ 6 & -12 \end{vmatrix} - (-2) \times \begin{vmatrix} 3 & 2 \\ -3 & -12 \end{vmatrix} + 4 \times \begin{vmatrix} 3 & 1 \\ -3 & 6 \end{vmatrix} \\ &= 1 \times (-12 - 12) + 2 \times (-36 + 6) + 4 \times (18 + 3) = -24 - 60 + 84 = 0 \end{aligned}$$

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Q.3 Find all the elements of the conic section $9x^2 - 4y^2 - 16y - 54x + 29 = 0$ and sketch it.

Solution :

$$9x^2 - 4y^2 - 16y - 54x + 29 = 0 \Rightarrow 9x^2 - 54x - 4y^2 - 16y = -29$$

$$\Rightarrow 9(x^2 - 6x) - 4(y^2 + 4y) = -29$$

$$\Rightarrow 9(x^2 - 6x + 9) - 4(y^2 + 4y + 4) = -29 + 81 - 16$$

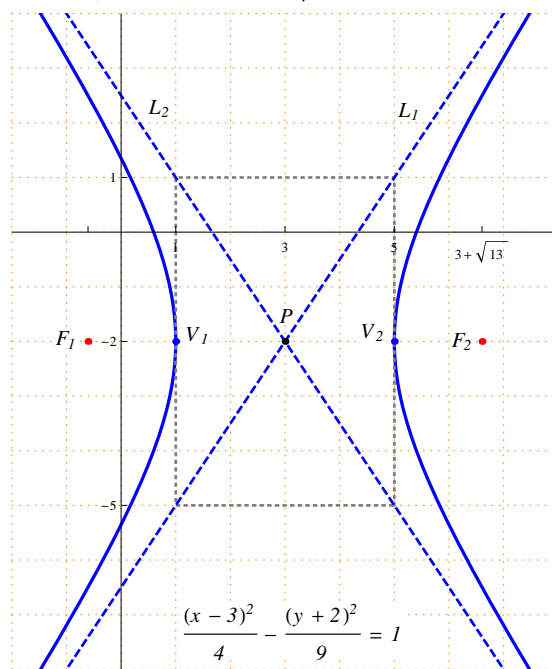
$$\Rightarrow 9(x - 3)^2 - 4(y + 2)^2 = 36 \Rightarrow \frac{9(x - 3)^2}{36} - \frac{4(y + 2)^2}{36} = 1$$

$$\Rightarrow \frac{(x - 3)^2}{4} - \frac{(y + 2)^2}{9} = 1$$

The conic section is Hyperbola.

$$a^2 = 4 \Rightarrow a = 2 \text{ and } b^2 = 9 \Rightarrow b = 3$$

$$c^2 = a^2 + b^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}.$$



The center is $P = (3, -2)$

The vertices are $V_1 = (1, -2)$ and $V_2 = (5, -2)$

The Foci are $F_1 = (3 - \sqrt{13}, -2)$ and $F_2 = (3 + \sqrt{13}, -2)$

The transverse axis is parallel to the x -axis and its length is $2a = 4$.

The equations of the asymptotes are $L_1 : y + 2 = \frac{3}{2}(x - 3)$

and $L_2 : y + 2 = -\frac{3}{2}(x - 3)$

Q.4 Find the standard equation of the parabola with vertex $(-4, 2)$ and with directrix $y = 5$ then sketch it.

Solution :

From the given information the parabola opens downwards.

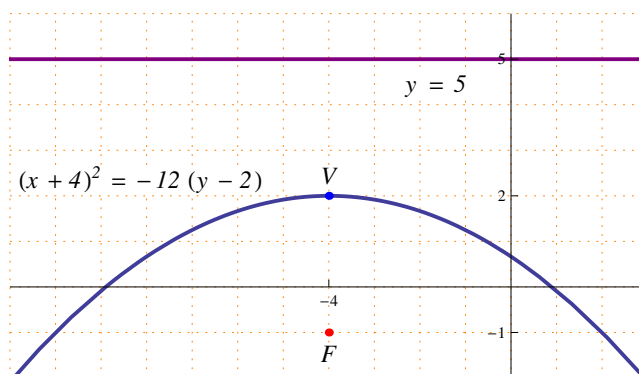
The formula for that parabola is $(x - h)^2 = -4a (y - k)$

Since the vertex is $(-4, 2)$ then $(h, k) = (-4, 2)$

a is the distance between the vertex and the directrix , hence $a = 3$

The standard equation of the parabola is $(x + 4)^2 = -12(y - 2)$

The focus is $F = (-4, -1)$



Q.5 Solve by Gauss method the linear system

$$\begin{aligned} 7x - y + z &= 4 \\ 3x + 2y + 2z &= 17 \\ x + y - z &= 4 \end{aligned}$$

Solution :

$$\begin{aligned} &\left(\begin{array}{ccc|c} 7 & -1 & 1 & 4 \\ 3 & 2 & 2 & 17 \\ 1 & 1 & -1 & 4 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 3 & 2 & 2 & 17 \\ 7 & -1 & 1 & 4 \end{array} \right) \\ &\xrightarrow{-3R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 7 & -1 & 1 & 4 \end{array} \right) \xrightarrow{-7R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 0 & -8 & 8 & -24 \end{array} \right) \\ &\xrightarrow{-8R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & -32 & -64 \end{array} \right) \xrightarrow{-R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & -32 & -64 \end{array} \right) \\ &\xrightarrow{-\frac{1}{32}R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 1 & 2 \end{array} \right) \end{aligned}$$

Therefore , $z = 2$

$$y - 5z = -5 \Rightarrow y - 10 = -5 \Rightarrow y = -5 + 10 = 5$$

$$x + y - z = 4 \Rightarrow x + 5 - 2 = 4 \Rightarrow x + 3 = 4 \Rightarrow x = 1$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel⁵

Solution of the Second Mid-Term Exam

Second semester 1433-1434 H

Q.1 Compute the integrals :

(a) $\int (3x^2 + 1) \sin(x^3 + x + 1) dx$

(b) $\int \frac{x + 3}{(x - 3)(x - 2)} dx$

(c) $\int \frac{5}{x^2 + 1} dx$

(d) $\int (x^2 + 1) \ln x dx$

(e) $\int x^2 \sin x dx$

Solution :

(a) $\int (3x^2 + 1) \sin(x^3 + x + 1) dx = \int \sin(x^3 + x + 1) (3x^2 + 1) dx$
 $= -\cos(x^3 + x + 1) + c$

(b) $\int \frac{x + 3}{(x - 3)(x - 2)} dx$

Using the method of partial fractions

$$\frac{x + 3}{(x - 3)(x - 2)} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

$$\frac{x + 3}{(x - 3)(x - 2)} = \frac{A(x - 2)}{(x - 3)(x - 2)} + \frac{B(x - 3)}{(x - 2)(x - 3)}$$

$$x + 3 = A(x - 2) + B(x - 3)$$

Put $x = 3$, then $3 + 3 = A(3 - 2) + B(3 - 3) \implies A = 6$

Put $x = 2$, then $2 + 3 = A(2 - 2) + B(2 - 3) \implies -B = 5 \implies B = -5$

$$\frac{x + 3}{(x - 3)(x - 2)} = \frac{6}{x - 3} + \frac{-5}{x - 2}$$

$$\int \frac{x + 3}{(x - 3)(x - 2)} dx = \int \left(\frac{6}{x - 3} - \frac{5}{x - 2} \right) dx$$

$$= \int \frac{6}{x - 3} dx - \int \frac{5}{x - 2} dx = 6 \int \frac{1}{x - 3} dx - 5 \int \frac{1}{x - 2} dx$$

$$= 6 \ln |x - 3| - 5 \ln |x - 2| + c$$

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$$(c) \int \frac{5}{x^2+1} dx = 5 \int \frac{1}{x^2+1} dx = 5 \tan^{-1} x + c$$

$$(d) \int (x^2+1) \ln x dx$$

Using integration by parts

$$\begin{aligned} u &= \ln x & dv &= (x^2+1) dx \\ du &= \frac{1}{x} dx & v &= \frac{x^3}{3} + x \end{aligned}$$

$$\begin{aligned} \int (x^2+1) \ln x dx &= \left(\frac{x^3}{3} + x \right) \ln x - \int \left(\frac{x^3}{3} + x \right) \frac{1}{x} dx \\ &= \left(\frac{x^3}{3} + x \right) \ln x - \int \left(\frac{x^2}{3} + 1 \right) dx \\ &= \left(\frac{x^3}{3} + x \right) \ln x - \frac{1}{3} \int x^2 dx - \int 1 dx = \left(\frac{x^3}{3} + x \right) \ln x - \frac{1}{3} \frac{x^3}{3} - x + c \end{aligned}$$

$$(e) \int x^2 \sin x dx$$

Using integration by parts

$$\begin{aligned} u &= x^2 & dv &= \sin x dx \\ du &= 2x dx & v &= -\cos x \end{aligned}$$

$$\int x^2 \sin x dx = -x^2 \cos x - \int 2x(-\cos x) dx = -x^2 \cos x + 2 \int x \cos x dx$$

Using integration by parts again

$$\begin{aligned} u &= x & dv &= \cos x dx \\ du &= dx & v &= \sin x \end{aligned}$$

$$\begin{aligned} \int x^2 \sin x dx &= -x^2 \cos x + 2 \left(x \sin x - \int \sin x dx \right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \end{aligned}$$

Q.2 Find the area of the plane region bounded by the curves $x+y=2$, $y=2$ and $y=2x-4$.

Solution :

$y=2$, $y=2x-4$ and $y=-x+2$ are three straight lines.

Point of intersection of $y=2$ and $y=-x+2$:

$$-x+2=2 \Rightarrow x=0$$

$y=2$ and $y=-x+2$ intersect at the point $(0,2)$.

Point of intersection of $y=2$ and $y=2x-4$:

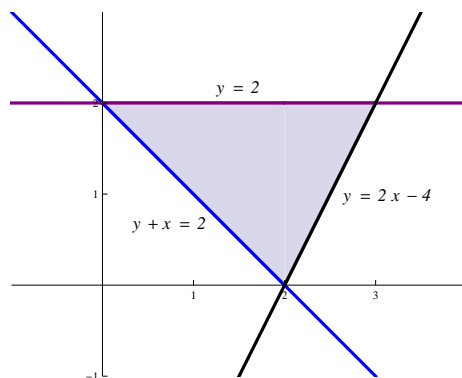
$$2x - 4 = 2 \Rightarrow x = 3$$

$y = 2$ and $y = 2x - 4$ intersect at the point $(3, 2)$

Point of intersection of $y = -x + 2$ and $y = 2x - 4$:

$$2x - 4 = -x + 2 \Rightarrow 3x = 6 \Rightarrow x = 2$$

$y = -x + 2$ and $y = 2x - 4$ intersect at the point $(2, 0)$.



$$y + x = 2 \Rightarrow x = -y + 2 \text{ and } y = 2x - 4 \Rightarrow 2x = y + 4 \Rightarrow x = \frac{1}{2}y + 2$$

$$\text{Area} = \int_0^2 \left[\left(\frac{1}{2}y + 2 \right) - (-y + 2) \right] dy$$

$$\text{Area} = \int_0^2 \left(\frac{1}{2}y + y \right) dy = \int_0^2 \frac{3}{2}y dy$$

$$\text{Area} = \frac{3}{2} \left[\frac{y^2}{2} \right]_0^2 = \frac{3}{2} \left[\frac{2^2}{2} - \frac{0^2}{2} \right] = \frac{3}{2} \times 2 = 3$$

Q.3 The region R between the lines $x + y = 1$, $x = 1$ and $y = 2x + 1$ is rotated about the y -axis to form a solid of revolution S . Find the volume of S .

Solution :

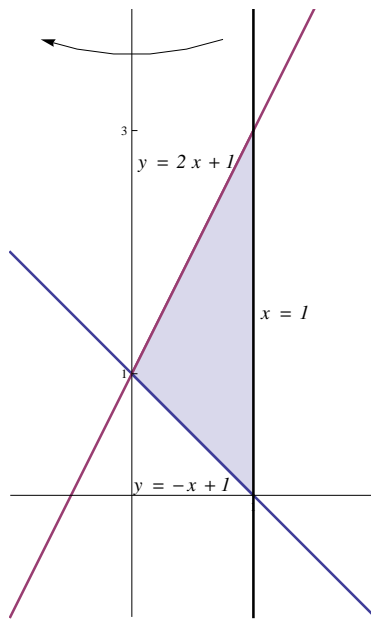
$y = -x + 1$ is a straight line with slope -1 and passing through $(0, 1)$.

$y = 2x + 1$ is a straight line with slope 2 and passing through $(0, 1)$.

$x = 1$ is a straight line parallel to the y -axis and passing through $(1, 0)$.

Point of intersection of $y = -x + 1$ and $y = -x + 2$:

$$2x + 1 = -x + 1 \Rightarrow 3x = 0 \Rightarrow x = 0.$$



Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 x[(2x + 1) - (-x + 1)] dx = 2\pi \int_0^1 x(3x) dx = 2\pi \int_0^1 3x^2 dx$$

$$\text{Volume} = 2\pi [x^3]_0^1 = 2\pi[1 - 0] = 2\pi$$

M 104 - GENERAL MATHEMATICS -2-

*Dr. Tariq A. AlFadhel*⁶

Solution of the Final Exam

Second semester 1433-1434 H

Q.1 (a) Compute (if possible) : $2\mathbf{AB}$ and \mathbf{BA} for

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

(b) Compute the determinant $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 0 \end{vmatrix}$.

(c) Solve : $\begin{cases} x & - & 2y & = & 5 \\ 4x & - & 3y & = & 10 \end{cases}$

Solution :

$$\begin{aligned} \text{(a) } 2\mathbf{AB} &= (2\mathbf{A})\mathbf{B} = \begin{pmatrix} 2 \times 0 & 2 \times 2 & 2 \times 0 \\ 2 \times 3 & 2 \times 2 & 2 \times 3 \\ 2 \times 0 & 2 \times 2 & 2 \times 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 4 & 0 \\ 6 & 4 & 6 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0+0+0 & 0+4+0 & 0+0+0 \\ 6+0+6 & 12+4+12 & 6+0+6 \\ 0+0+0 & 0+4+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 12 & 28 & 12 \\ 0 & 4 & 0 \end{pmatrix} \\ \mathbf{BA} &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 3 & 2 & 3 \\ 0 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0+6+0 & 2+4+2 & 0+6+0 \\ 0+3+0 & 0+2+0 & 0+3+0 \\ 0+6+0 & 2+4+2 & 0+6+0 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 6 \\ 3 & 2 & 3 \\ 6 & 8 & 6 \end{pmatrix} \end{aligned}$$

(b) Using Sarrus method $\begin{matrix} 1 & 0 & 3 & 1 & 0 \\ 4 & 3 & 2 & 4 & 2 \\ 5 & 6 & 0 & 5 & 6 \end{matrix}$

$$\begin{vmatrix} 1 & 0 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 0 \end{vmatrix} = (0 + 0 + 72) - (45 + 12 + 0) = 72 - 57 = 15$$

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(c) Using Gauss-Jordan method :

The augmented matrix is $\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -3 & 10 \end{array} \right)$

$$\left(\begin{array}{cc|c} 1 & -2 & 5 \\ 4 & -3 & 10 \end{array} \right) \xrightarrow{-4R_1+R_2} \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 5 & -10 \end{array} \right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -2 \end{array} \right) \xrightarrow{2R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array} \right)$$

The solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & -3 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 5 & -2 \\ 10 & -3 \end{pmatrix} \text{ and } \mathbf{A}_2 = \begin{pmatrix} 1 & 5 \\ 4 & 10 \end{pmatrix}$$

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} 1 & -2 \\ 4 & -3 \end{vmatrix} = (1 \times -3) - (4 \times -2) = -3 + 8 = 5$$

$$\det(\mathbf{A}_1) = |\mathbf{A}_1| = \begin{vmatrix} 5 & -2 \\ 10 & -3 \end{vmatrix} = (5 \times -3) - (10 \times -2) = -15 + 20 = 5$$

$$\det(\mathbf{A}_2) = |\mathbf{A}_2| = \begin{vmatrix} 1 & 5 \\ 4 & 10 \end{vmatrix} = (1 \times 10) - (4 \times 5) = 10 - 20 = -10$$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{5}{5} = 1$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{-10}{5} = -2$$

The solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Q.2 (a) Find the standard equation of the ellipse with foci $(-2, 3)$ and $(6, 3)$, and the length of its major axis is 10 and then sketch it.

(b) Find the elements of the conic section $x^2 - 4y - 2x + 13 = 0$.

Solution :

(a) The standard equation of an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

The center of the ellipse is the mid-point of the two foci

$$P(h, k) = \left(\frac{-2+6}{2}, \frac{3+3}{2} \right) = (2, 3).$$

The major axis passes through the two foci and it is parallel to the x -axis

The length of the major axis is 10 , means that $2a = 10 \Rightarrow a = 5$

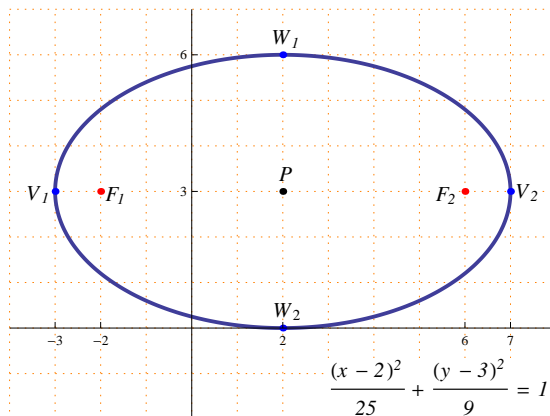
c is the distance between one of the foci and the center, hence $c = 4$

$$c^2 = a^2 - b^2 \Rightarrow 16 = 25 - b^2 \Rightarrow b^2 = 9 \Rightarrow b = 3$$

The standard equation of the ellipse is $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{9} = 1$

The vertices of the ellipse are $V_1(-3, 3)$ and $V_2(7, 3)$

The end-points of the minor axis are $W_1(2, 6)$ and $W_2(2, 0)$



(b) $x^2 - 4y - 2x + 13 = 0 \Rightarrow x^2 - 2x = 4y - 13$

$$\Rightarrow x^2 - 2x + 1 = 4y - 13 + 1 \Rightarrow (x-1)^2 = 4y - 12$$

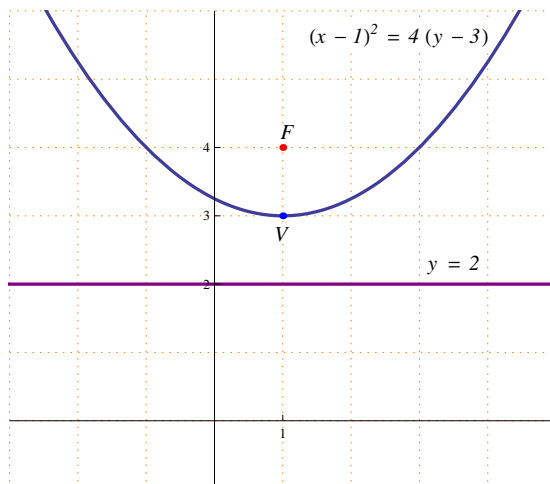
$$\Rightarrow (x-1)^2 = 4(y-3)$$

The conic section is a parabola with vertex $V(1, 3)$ and opens upwards

$$4a = 4 \Rightarrow a = 1$$

The focus is $F(1, 4)$

The equation of the directrix is $y = 2$



Q.3 (a) Compute the integrals :

$$(i) \int \frac{1}{x^2 + 9} dx \quad (ii) \int x \cos x dx \quad (iii) \int \frac{x+1}{(x-1)^2} dx$$

(b) Find the area of the surface determined by the curves :

$$y = -x^2 + 4 \text{ and } y = 0 .$$

(c) The region R between the curves $y = -x + 2$ and $y = x^2$ is rotated about the x -axis to form a solid of revolution S . Find the volume of S .

Solution :

$$(a) (i) \int \frac{1}{x^2 + 9} dx = \int \frac{1}{x^2 + 3^2} dx = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + c$$

$$(ii) \int x \cos x dx$$

Using integration by parts

$$\begin{aligned} u &= x & dv &= \cos x dx \\ du &= dx & v &= \sin x \end{aligned}$$

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

$$= x \sin x - (-\cos x) + c = x \sin x + \cos x + c$$

$$(iii) \int \frac{x+1}{(x-1)^2} dx$$

Using the method of partial fractions

$$\frac{x+1}{(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$\frac{x+1}{(x-1)^2} = \frac{A_1(x-1)}{(x-1)^2} + \frac{A_2}{(x-1)^2} = \frac{A_1(x-1) + A_2}{(x-1)^2}$$

$$x+1 = A_1(x-1) + A_2 = A_1x - A_1 + A_2$$

By comparing the coefficients of both sides :

$$\begin{aligned} A_1 &= 1 & \longrightarrow (1) \\ -A_1 + A_2 &= 1 & \longrightarrow (2) \end{aligned}$$

From equation (1) : $A_1 = 1$.

From equation (2) : $-1 + A_2 = 1 \implies A_2 = 2$

$$\frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

$$\int \frac{x+1}{(x-1)^2} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx$$

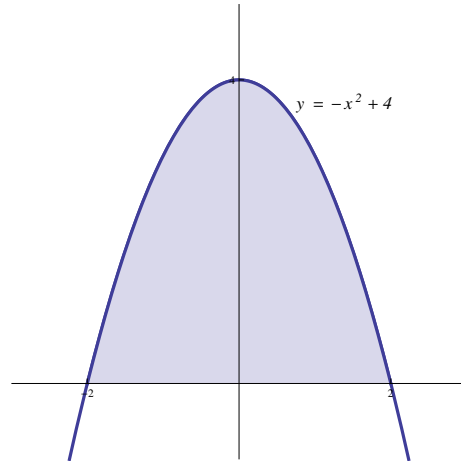
$$\begin{aligned}
&= \int \frac{1}{x-1} dx + \int \frac{2}{(x-1)^2} dx = \int \frac{1}{x-1} dx + 2 \int (x-1)^{-2} dx \\
&= \ln|x-1| + 2 \frac{(x-1)^{-1}}{-1} + c = \ln|x-1| - \frac{2}{x-1} + c
\end{aligned}$$

(b) $y = 0$ is the x -axis .

$y = -x^2 + 4 \implies x^2 = -(y-4)$ is a parabola with vertex $(0, 4)$ and opens downwards.

Point of intersections of $y = -x^2 + 4$ and $y = 0$:

$$-x^2 + 4 = 0 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = -2, x = 2$$



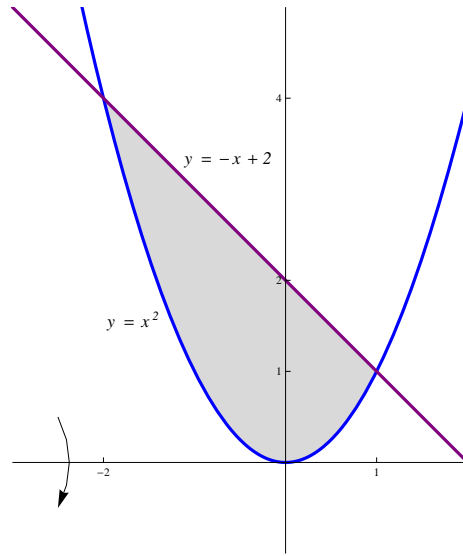
$$\begin{aligned}
\text{Area} &= \int_{-2}^2 (-x^2 + 4) dx = \left[-\frac{x^3}{3} + 4x \right]_{-2}^2 \\
&= \left(-\frac{8}{3} + 8 \right) - \left(\frac{8}{3} - 8 \right) = -\frac{8}{3} + 8 - \frac{8}{3} + 8 = 16 - \frac{16}{3} = \frac{48 - 16}{3} = \frac{32}{3}
\end{aligned}$$

(c) $y = -x + 2$ is a straight line with slope -1 and passing through $(0, 2)$.

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards .

Points of intersection of $y = -x + 2$ and $y = x^2$

$$x^2 = -x + 2 \implies x^2 + x - 2 = 0 \implies (x-1)(x+2) = 0 \implies x = 1, x = -2$$



Using Washer Method :

$$\begin{aligned}
 \text{Volume} &= \pi \int_{-2}^1 [(-x+2)^2 - (x^2)^2] dx = \pi \int_{-2}^1 (x^2 - 4x + 4 - x^4) dx \\
 &= \pi \int_{-2}^1 (-x^4 + x^2 - 4x + 4) dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x \right]_{-2}^1 \\
 &= \pi \left[\left(-\frac{1}{5} + \frac{1}{3} - 2 + 4 \right) - \left(\frac{32}{5} - \frac{8}{3} - 8 - 8 \right) \right] \\
 &= \pi \left(-\frac{1}{5} + \frac{1}{3} + 2 - \frac{32}{5} + \frac{8}{3} + 16 \right) = \pi \left(-\frac{33}{5} + \frac{9}{3} + 2 + 16 \right) \\
 &= \pi \left(-\frac{33}{5} + 21 \right) = \pi \left(\frac{-33 + 105}{5} \right) = \frac{72}{5} \pi
 \end{aligned}$$

Q.4 (a) Find f_x and f_y for the function $f(x, y) = x^2y^3 + xy \ln(y+x)$

(b) Solve the differential equation $y' + y^3e^x = 0$

Solution :

$$\begin{aligned}
 \text{(a) } f_x &= \frac{\partial f}{\partial x} = 2xy^3 + y \ln(x+y) + xy \frac{1}{x+y} = 2xy^3 + y \ln(x+y) + \frac{xy}{x+y} \\
 f_y &= \frac{\partial f}{\partial y} = x^2(3y^2) + x \ln(x+y) + xy \frac{1}{x+y} = 3x^2y^2 + x \ln(x+y) + \frac{xy}{x+y}
 \end{aligned}$$

(b) $y' + y^3e^x = 0$

$$y' = -y^3e^x$$

$$\frac{dy}{dx} = -y^3 e^x$$

$$-\frac{1}{y^3} dy = e^x dx$$

$$-y^{-3} dy = e^x dx$$

$$\int -y^{-3} dy = \int e^x dx$$

$$-\frac{y^{-2}}{-2} = e^x + c$$

$$\frac{1}{2y^2} = e^x + c$$

$$\frac{1}{y^2} = 2(e^x + c)$$

$$y^2 = \frac{1}{2(e^x + c)}$$

$$y = \sqrt{\frac{1}{2(e^x + c)}}$$