King Saud University: Mathematics Department M-254
Semester II (1st Midterm Exam Solution) 1439-1440 H
Maximum Marks=25 Time Allowed: 90 Mins.

Questions : (5 + 5 + 5 + 5 + 5)

Q1: Consider the two schemes:

$$(i) \quad x_{n+1} = \sqrt{\frac{5}{x_n}}, \qquad (ii) \quad x_{n+1} = \frac{1}{3}\left[2x_n + \frac{5}{x_n^2}\right], \; n = 0, 1, 2, \ldots$$

Which iteration will converge faster to $\sqrt[3]{5}$? Explain your answer. Use the faster scheme to find the second approximation, starting with $x_0 = 1.5$.

Solution. It can be easily verify as follows. From the first sequence, we have

$$g_1(x) = 5^{1/2}x^{-1/2} \quad \text{and} \quad g_1'(x) = 5^{1/2}\left[-\frac{1}{2}x^{-3/2}\right],$$

which implies that

$$|g_1'(5^{1/3})| = \left|-\frac{5^{1/2}}{2}[(5^{1/3})^{-3/2}]\right| = \frac{1}{2} = 0.5 < 1.$$

Similarly, from the second sequence, we have

$$g_2(x) = \frac{1}{3}\left[2x + \frac{5}{x^2}\right] \quad \text{and} \quad g_2'(x) = \frac{1}{3}\left[2 - \frac{10}{x^3}\right],$$

gives

$$|g_2'(5^{1/3})| = \frac{1}{3}\left[2 - \frac{10}{(5^{1/3})^3}\right] = \frac{1}{3}[2 - 2] = 0.0.$$

We note that both sequences are converging to $\sqrt[3]{5}$ but the second sequence (ii) will converges faster than the first sequence **(i)** because the value of $|g_2'(\sqrt[3]{5})|$ is smaller than by $|g_1'(\sqrt[3]{5})|$.
Given $x_0 = 1.5$, using faster convergent sequence (first one), we have

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{5}{x_0^2}\right] = \frac{1}{3}\left[2(1.5) + \frac{5}{(1.5)^2}\right] = 1.7407,$$

$$x_2 = \frac{1}{3}\left[2x_1 + \frac{5}{x_1^2}\right] = \frac{1}{3}\left[2(1.7407) + \frac{5}{(1.7407)^2}\right] = 1.7105,$$

the required second approximation. •

Q2: Successive approximations x_n to the desired root of an equation $f(x) = 0$ are generated by the scheme

$$x_{n+1} = \frac{1 + 3x_n^2}{4 + x_n^3}, \qquad n \geq 0.$$

Find $f(x_n)$ and $f'(x_n)$ and then use the Newton's method to find the approximation of the root accurate to 10^{-2}, starting with $x_0 = 0.5$.

Solution. Given

$$x = \frac{1 + 3x^2}{4 + x^3} = g(x),$$

and

$$x - g(x) = x - \frac{1 + 3x^2}{4 + x^3} = \frac{x^4 - 3x^2 + 4x - 1}{4 + x^3}.$$

Since

$$f(x) = x - g(x) = 0,$$

therefore, we have

$$f(x_n) = x_n^4 - 3x_n^2 + 4x_n - 1 \quad \text{and} \quad f'(x_n) = 4x_n^3 - 6x_n + 4.$$

Using these functions values in the Newton's iterative formula,

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n^2 + 4x_n - 1}{4x_n^3 - 6x_n + 4}.$$

Finding the first approximation of the root using the initial approximation $x_0 = 0.5$, we get

$$x_1 = x_0 - \frac{x_0^4 - 3x_0^2 + 4x_0 - 1}{4x_0^3 - 6x_0 + 4} = 0.5 - \frac{0.3125}{1.5} = 0.2917.$$

Similarly, the other approximations can be obtained as

$$x_2 = 0.2917 - \frac{(-0.0813)}{2.3491} = 0.3263; \quad x_3 = 0.3263 - \frac{(-0.0029)}{2.1812} = 0.3276.$$

Notice that $|x_3 - x_2| = |0.3276 - 0.3263| = 0.0013.$ •

Q3: Show that the secant method for finding approximation of the *pth* root of a positive number N is

$$x_{n+1} = \frac{(x_n - x_{n-1})N + (x_n^{p-1} - x_{n-1}^{p-1})x_n x_{n-1}}{x_n^p - x_{n-1}^p}, \quad n \geq 1.$$

Carry out the first two approximations for the square root of 9, using $x_0 = 2, x_1 = 2.5$ and also compute absolute error.

Solution. We shall compute $x = N^{1/p}$ by finding a positive root for the nonlinear equation

$$x^p - N = 0,$$

where $N > 0$ is the number whose root is to be found. If $f(x) = 0$, then $x = \alpha = N^{1/p}$ is the exact zero of the function

$$f(x) = x^p - N.$$

Since the secant formula is

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1.$$

Hence, by using the secant iterative formula, we have

$$x_2 = x_n - \frac{(x_n - x_{n-1})(x_n^p - N)}{(x_n^p - N) - (x_{n-1}^p - N)} = x_n - \frac{(x_n - x_{n-1})(x_n^p - N)}{(x_n^p - x_{n-1}^p)}.$$

After, simplifying, we have

$$x_{n+1} = \frac{(x_n - x_{n-1})N + (x_n^{p-1} - x_{n-1}^{p-1})x_n x_{n-1}}{x_n^p - x_{n-1}^p}, \quad n \geq 1,$$

the secant formula for approximation of the pth root of number N. Now using this formula for approximation of the square root ($p = 2$) of $N = 9$, taking $x_0 = 2$ and $x_1 = 2.5$, we have

$$x_2 = 3.1111 \qquad \text{and} \qquad x_3 = 2.9901.$$

Hence

$$\text{Absolute Error} = |9^{1/2} - x_3| = |3 - 2.9901| = 0.0099,$$

is the possible absolute error. •

Q4: Find the values of a, b and c such that the iterative scheme

$$x_{n+1} = ax_n + \frac{bN}{x_n^2} + \frac{cN^2}{x_n^5}, \qquad n \geq 0,$$

converges at least cubically to $\alpha = N^{\frac{1}{3}}$.

Solution. Given the iterative scheme converges at least cubically means $g' = g'' = 0$ at $\alpha = N^{\frac{1}{3}}$. Let

$$g(x) = ax + \frac{bN}{x^2} + \frac{cN^2}{x^5}, \quad g(N^{\frac{1}{3}}) = 1 = a + b + c,$$

$$g'(x) = a - \frac{2bN}{x^3} - \frac{5cN^2}{x^6}, \quad g'(N^{\frac{1}{3}}) = 0 = a - 2b - 5c,$$

$$g''(x) = 0 + \frac{6bN}{x^4} + \frac{30cN^2}{x^7}, \quad g''(N^{\frac{1}{3}}) = 0 = 3b + 15c,$$

Solving these three equations for unknowns a, b and c, we obtain $a = 1,\ b = \frac{1}{9}$ and $c = -\frac{1}{9}$. Thus

$$x_{n+1} = x_n + \frac{N}{9x_n^2} - \frac{N^2}{9x_n^5}, \qquad n \geq 0,$$

the required iterative scheme which converges at least cubically to $\alpha = N^{\frac{1}{3}}$. •

Q5: If $x = \alpha$ is a root of $f(x) = 0$, with $f(\alpha) = f'(\alpha) = f''(\alpha) = f'''(\alpha) = f^{(4)}(\alpha) = 0$ but $f^{(5)}(\alpha) \neq 0$, then show that the rate of convergence of the Modified Newton's method is at least quadratic.

Solution. The first modified Newton's iteration function is define as follows:

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$

Since the function $f(x)$ has multiple root of multiplicity 5, so

$$f(x) = (x-\alpha)^5 h(x),$$

and its derivative is

$$f'(x) = 5(x-\alpha)^4 h(x) + (x-\alpha)^5 h'(x).$$

Substituting the values of the $f(x)$ and $f'(x)$, we get

$$g(x) = x - \frac{5(x-\alpha)^5 h(x)}{(5(x-\alpha)^4 h(x) + (x-\alpha)^5 h'(x))},$$

or

$$g(x) = x - \frac{5(x-\alpha)h(x)}{(5h(x) + (x-\alpha)h'(x))}.$$

Then

$$\begin{aligned} g'(x) = 1 \quad - \quad & 5\{([5h(x) + (x-\alpha)][h(x) + (x-\alpha)h'(x)] - [(x-\alpha)h(x)] \\ & [5h'(x) + h'(x) + (x-\alpha)h''(x)])\}/([5h(x) + (x-\alpha)h'(x)]^2). \end{aligned}$$

At $x = \alpha$, and since $f(\alpha) = 0$, we have

$$g'(\alpha) = 1 - \frac{[5^2h^2(\alpha)]}{[5h(\alpha)]^2}, \qquad g'(\alpha) = 0.$$

Therefore, the modified Newton's method converges to a multiple root α and the convergence is at least quadratically. •