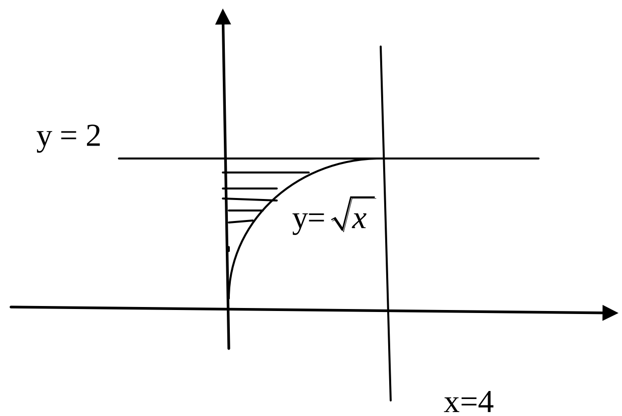
King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second-Mid Term Examination
(First Semester 1431/1432)

Max. Marks: 20 Time: 90 minutes

Marking Scheme: Q:1(3), Q:2(3), Q:3(3), Q:4(3), Q:5(4), Q:6(4)

Q. No: 1 Reverse the order of integration and evaluate the resulting integral $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dydx$.

Solution: Here region R of integration is $0 \le x \le 4,\ \sqrt{x} \le y \le 2.$



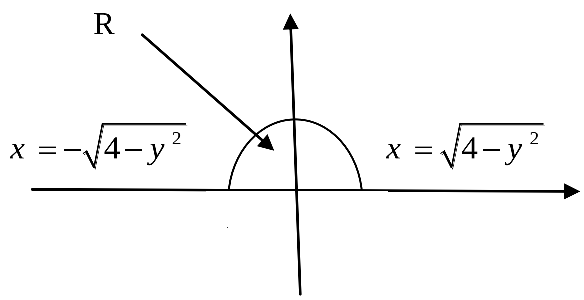
If we reverse the order of integration the we have to use the region R in the following Form $0 \le y \le 2,\ 0 \le x \le y^2$.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dydx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1} dxdy = \int_0^4 \frac{1}{y^3+1} \left[x\right]_0^{y^2} dy$$

$$= \int_0^2 \frac{1}{y^3+1} \left[y^2\right] dy = \int_0^2 \frac{y^2}{y^3+1} dy = \frac{1}{3}\left[\ln(y^3+1)\right]_0^2 = \frac{1}{3}\ln(9).$$

Q. No: 2 Evaluate the integral by changing to polar coordinates $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 dxdy$ **.**

Solution: The region of R integration is $0 \le y \le 2, \quad -\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$.



R in polar coordinates is equal to $0 \le r \le 2, \quad 0 \le \theta \le \pi$.

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 dxdy = \int_0^{\pi} \int_0^2 (r\cos\theta)^2 (r\sin\theta)^2 r dr d\theta = \int_0^{\pi} \int_0^2 r^5 \cos\theta \sin\theta dr d\theta$$

$$= \int_0^{\pi} \left[\frac{r^6}{6}\right]_0^2 \cos^2\theta \sin^2\theta d\theta = \frac{64}{6}\int_0^{\pi} (\cos\theta\sin\theta)^2 d\theta = \frac{32}{3}\int_0^{\pi}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

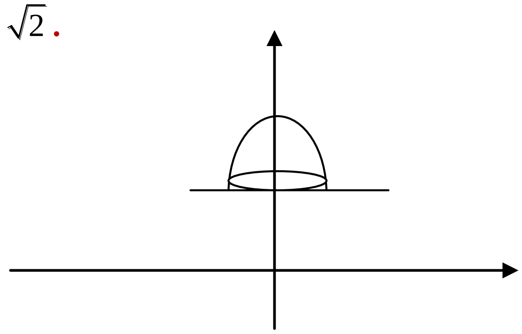
$$= \frac{32}{3}\int_0^{\pi}\frac{1}{4}(\sin 2\theta)^2 d\theta = \frac{8}{3}\int_0^{\pi}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{4}{3}\left\{[\theta]_0^{\pi} - \left[\frac{\sin 4\theta}{8}\right]_0^{\pi}\right\} = \frac{4\pi}{3}.$$

Q. No: 3 Find the surface area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 2$.

Solution: The region of integration in cylindrical coordinate system is:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le \sqrt{2}.$$

Because $z = 2$ **and** $z = 4 - x^2 - y^2 \Rightarrow 4 - x^2 - y^2 = 2 \Rightarrow x^2 + y^2 = 2$ which is **a circle of radius** $\sqrt{2}$ **.**



So the region of integration is $0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}$ **.**

Here $z = 4 - x^2 - y^2 = f(x,y) \Rightarrow f_x = -2x, f_y = -2y$.

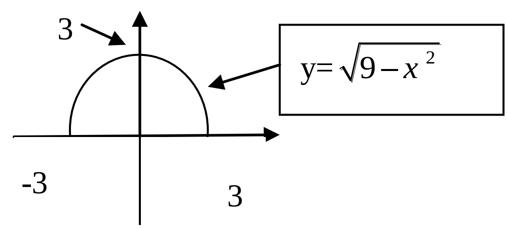
Surface Area= $\iint_R \sqrt{1+f_x^2+f_y^2}\,dA = \iint_R \sqrt{1+(-2x)^2+(-2y)^2}\,dA = \iint_R \sqrt{1+4(x^2+y^2)}\,dA$

Put $1+4r^2 = t \Rightarrow 8rdr = dt$ **and get the following**

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\,rdrd\theta = \frac{1}{8}\int_0^{2\pi}\left[\frac{\left(1+4r^2\right)^{3/2}}{3/2}\right]_0^{\sqrt{2}} d\theta = \frac{2\pi}{12}[27-1] = \frac{13\pi}{3}.$$

Q. No: 4 A lamina having area mass density $\delta(x,y) = |x|$ at the point P(x,y) and has the shape of the region bounded by the graphs of the equations $y = \sqrt{9-x^2}$, $y = 0$. Find the mass of the lamina.

Solution: Region of integration is

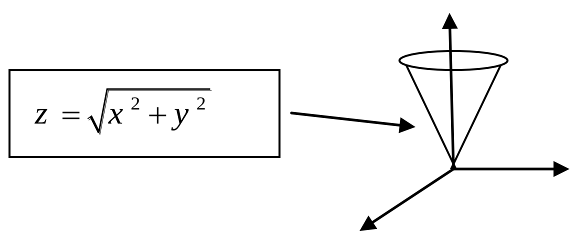


Mass of the lamina= $\iint_R \delta dA = \int_{-3}^{+3}\int_0^{\sqrt{9-x^2}} |x|\,dydx = \int_{-3} |x|\sqrt{9-x^2}dx = 2\int_0^3 x\sqrt{9-x^2}dx$

$$= -\left[\frac{\left(9-x^2\right)^{3/2}}{3/2}\right]_0^3 = -\frac{2}{3}\left[0-(9)^{3/2}\right] = \frac{54}{3} = 18.$$

Q. No: 5 Evaluate the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx$ by changing it to **cylindrical coordinates**.

Solution: The region is



Region in cylindrical is

$$0 \le \theta \le 2\pi, 0 \le r \le 2, \text{ and } 0 \le z \le r.$$

So $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx = \int_0^{2\pi}\int_0^{2}\int_0^{r} r^2 rdzdrd\theta = \int_0^{2\pi}\int_0^{2}(2-r)r^3 drd\theta$

$$= \int_0^{2\pi}\left[2\frac{r^4}{4} - \frac{r^5}{5}\right]_0^2 d\theta = \frac{16}{5}\pi$$

Q. No: 6 Evaluate $\iiint_Q (x^2+y^2+z^2)dV$ **,** where Q is the solid region that lies outside the sphere $x^2+y^2+z^2=1$ and inside the sphere $x^2+y^2+z^2=4$ by using **spherical coordinates.**

Solution: Region of integration Q is between two spheres $x^2+y^2+z^2=1$ **and** $x^2+y^2+z^2=4$

Using spherical coordinates: The region Q is $1\le\rho\le 2, 0\le\varphi\le\pi,$ and $0\le\theta\le 2\pi.$

$$\iiint_Q (x^2+y^2+z^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho^2\rho^2 \sin\varphi d\rho d\varphi d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\varphi\left[\frac{\rho^5}{5}\right]_1^2 d\varphi d\theta = \frac{31}{5}\int_0^{2\pi}\int_0^{\pi} \sin\varphi d\varphi d\theta = \frac{31}{5}\int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi} d\theta = \frac{124}{5}\pi.$$