King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus)
(I-Semester 1431/1432)

Max. Marks: 50 Time: 3 hrs

Marking Scheme: Q.No:1[3+3+5], Q.No:2[4+5+5], Q.No:3[5+4+4] , Q.No:[6,6]

Q. No: 1 (a) Discuss the convergence of the sequence $\left\{ n^2 \ln\left(1+\frac{1}{n}\right)\right\}$.

Solution: $\lim_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n^2}}$ it is a $\frac{0}{0}-form$ **.** By L'Hopital rule

$$= \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)}{-2\frac{1}{x^3}} = \lim_{n\to\infty} \frac{x}{2(1+1/x)} = \lim_{x\to\infty} \frac{x^2}{2(x+1)} = \infty$$ **.** Hence divergent**.**

(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$ converges or diverges.

Solution: Here $a_n = \frac{n^n}{2^{n^2}}$ **.** Applying **root test,** we have $\sqrt[n]{a_n} = \left(\frac{n^n}{2^{n^2}}\right)^{\frac{1}{n}} = \frac{n}{2^n}$

Now, $\lim_{x\to\infty} \frac{x}{2^x}$ **is** $\frac{\infty}{\infty}-form$

$$\lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$$ is convergent**.**

(c) Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+1)^n$ **.**

Solution: $$\lim_{n\to\infty} \left| \frac{(n+1)^2 (x+1)^{n+1}}{2^{n+1}} \frac{2^n}{n^2 (x+1)^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2}|x+1| = \frac{|x+1|}{2}$$

Series is convergent if $\frac{|x+1|}{2} < 1 \Leftrightarrow -1 < \frac{x+1}{2} < 1 \Leftrightarrow -2 < x+1 < 2$

$\Leftrightarrow -3 < x < 1$

Checking Convergence at $x = -3$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}(-3+1)^n = \sum_{n=0}^{\infty} \frac{n^2}{2^n}(-2)^n = \sum_{n=0}^{\infty} (-1)^n n^2$$ it is divergent.

Checking Convergence at $x = 1$

$$\sum_{n=0}^{\infty}\frac{n^2}{2^n}(1+1)^n = \sum_{n=0}^{\infty}\frac{n^2}{2^n}(2)^n = \sum_{n=0}^{\infty} n^2$$ it is also divergent.

Hence the **interval of convergence** is $-3 < x < 1$

and the **radius of convergence** $\rho = \dfrac{1+3}{2} = 2$.

Q. No: 2 (a) Find Maclaurin's series of $\ln(1+x)$ and hence deduce the Maclaurin series of $\ln(1+x^2)$. Use the first two non-zero terms of $\ln(1+x^2)$ to approximate $\ln(1.01)$ to four decimal places.

Solution: $f(x) = \ln(1+x) \Rightarrow f(o) = \ln(1) = 0$

$f^{/}(x) = \dfrac{1}{1+x} \Rightarrow f^{/}(o) = 1$. Write $f^{/}(x) = (1+x)^{-1}$

$f^{//}(x) = -(1+x)^{-2} \Rightarrow f^{//}(0) = -1$

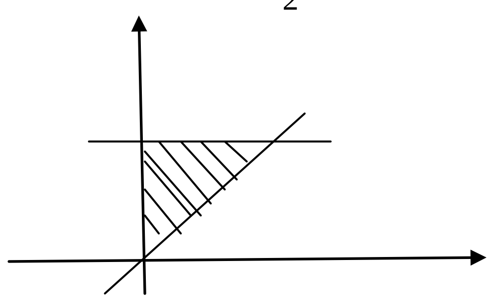
$f^{///}(x) = (-1)(-2)(1+x)^{-3} \Rightarrow f^{///}(0) = 2!$

$f^{////}(x) = (-1)(-2)(-3)(1+x)^{-4} \Rightarrow f^{////}(0) = -3!$ **.....** and so on

Therefore $f(x) = \ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ... + (-1)^n \dfrac{x^{n+1}}{n+1}....$ **if** $|x| < 1$.

(b) Evaluate the integral $\displaystyle\int_0^4 \int_{\frac{x}{2}}^{2} e^{y^2} dy dx$.

Solution: $\displaystyle\int_0^4 \int_{\frac{x}{2}}^{2} e^{y^2} dy dx \Rightarrow \quad 0 \le x \le 4$ **&** $\dfrac{x}{2} \le y \le 2.$



$\Rightarrow \quad 0 \le y \le 2$ **&** $0 \le x \le 2y \Rightarrow \displaystyle\int_0^4 \int_{\frac{x}{2}}^{2} e^{y^2} dy dx = \int_0^2 \int_0^{2y} e^{y^2} dx dy$

$$= \int_0^2 [x]_0^{2y} e^{y^2} dy = \int_0^2 2y e^{y^2} dy = \left. e^{y^2} \right]_0^2 = e^4 - 1.$$

(c) Find the surface area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$ in the first octant.

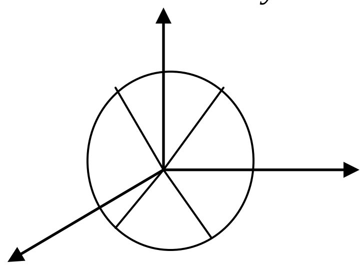
Solution: Here $z = xy = g(x, y) \Rightarrow g_x = y$ **&** $g_y = x$.

Therefore the surface area $A = \displaystyle\iint_R \sqrt{1+x^2+y^2} dA = \int_0^{\pi/2} \int_0^1 \sqrt{1+r^2} r dr d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left. \frac{(1+r^2)^{3/2}}{3/2} \right]_0^1 d\theta = \frac{1}{3}(\sqrt{8}-1)\int_0^{\pi/2} d\theta = \frac{1}{3}(\sqrt{8}-1)\frac{\pi}{2}.$$

Q. No: 3 (a) Sketch the graph of the solid region Q that lies inside the sphere $x^2+y^2+z^2=2$ and outside the cone $z^2=x^2+y^2$ and find its **volume.**

Solution:



$$\text{Volume} = \iiint_Q dV = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{\sqrt{2}} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{8\pi}{3}.$$

(b) Find the work done by the force $\vec{F}(x,y,z) = -\frac{1}{2}x\,\vec{i} - \frac{1}{2}y\,\vec{j} + \frac{1}{4}\vec{k}$ along the curve C given by : $x=\cos t, y=\sin t, z=t, 0\le t\le 2\pi$.

Solution: Work done $= \int_C -\frac{1}{2}xdx - \frac{1}{2}ydy + \frac{1}{4}dz$

$$= \int_0^{2\pi} -\frac{1}{2}\cos t(-\sin t)dt - \frac{1}{2}\sin t(\cos t)dt + \frac{1}{4}dt = \int_0^{2\pi}\frac{1}{4}dt$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

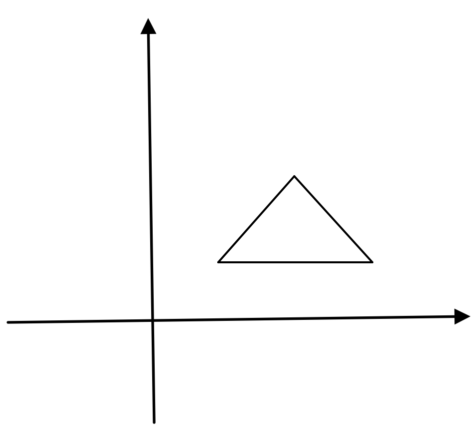
(c) Use the Green's theorem to evaluate $\oint_C xydx + \sin y dy$, where C is the triangle with vertices (1,1), (2,2), and (3,1).

Solution: Green's theorem is $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$.

Here $M = xy \Rightarrow \frac{\partial M}{\partial y} = x$ and $N = \sin y \Rightarrow \frac{\partial N}{\partial x} = 0$.

$$\oint_C xydx + \sin y dy = \iint_R -xdA = \int_1^2\int_y^{4-y} -xdxdy = \int_1^2 \left[-\frac{x^2}{2}\right]_y^{4-y} dy$$

$$= \int_1^2 -(8-4y)dy = \left[-8y + 4\frac{y^2}{2}\right]_1^2 = (-16+8)-(-8+2) = -2.$$



Q. No: 4 (a) Use the Divergence theorem to find the flux of

$\vec{F}(x,y,z) = e^x \sin y\ \vec{i} + e^x \cos y\ \vec{j} + y\ z^2\ \vec{k}$ across the surface S, where S is the the region bounded by the planes $x = 1, y = 1, z = 1,$ and the coordinate planes.

Solution: By the divergence theorem $\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV$

Here $M = e^x \sin y \Rightarrow \frac{\partial M}{\partial x} = e^x \sin y$

$N = e^x \cos y \Rightarrow \frac{\partial N}{\partial y} = -e^x \sin y$

and $P = yz^2 \Rightarrow \frac{\partial P}{\partial z} = 2yz$

Therefore $\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV = \iiint_Q (e^x \sin y - e^x \sin y + 2z)dV$

$$= \iiint_Q 2yzdV = 2\int_0^1 \int_0^1 \int_0^1 yzdzdydx = 2\int_0^1 \int_0^1 y\left(\frac{z^2}{2}\right]_0^1 dydx$$

$$= \int_0^1 \int_0^1 ydydx = \int_0^1 \frac{y^2}{2}\Bigg]_0^1 dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

(b) Use the Stoke's theorem to evaluate $\int_C \vec{F}.\vec{dr}$, where

$\vec{F}(x,y,z) = x^2z\ \vec{i} + xy^2\ \vec{j} + z^2\ \vec{k}$ where C is the curve of the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9,$ oriented counterclockwise as viewed from the above.

Solution: By Stoke's theorem $\int_C \vec{F}.\vec{dr} = \iint_S (curl\ \vec{F})\cdot \vec{n}\, dS$

$$curl\ \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & xy^2 & z^2 \end{vmatrix} = 0i + x^2 j + y^2 k\ . \Rightarrow M = 0, N = x^2, \textbf{ and } P = y^2$$

Here $z = -1 - x - y = g(x,y) \Rightarrow g_x = -1$ **and** $g_y = -1$

$$\iint_S (curl\ \vec{F})\cdot\vec{n}\, dS = \iint_R (-Mg_x - Ng_y + P)dA = \iint_R (x^2 + y^2)dA$$

Now using polar coordinates, we get

$$= \iint_R (x^2 + y^2)dA = \int_0^{2\pi} \int_0^3 r^2 rdrd\theta = \int_0^{2\pi} \int_0^3 r^3 drd\theta = \frac{81}{2}\pi.$$

Now $\ln(1+x^2) = x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \dots\dots + (-1)^n \frac{(x^2)^{n+1}}{n+1} \dots\dots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}.$$

Let $x = 0.1$. Then $\ln(1+x^2) = \ln(1+(0.1)^2) = \ln(1.001) \approx (0.1)^2 - \frac{(0.1)^2}{2} \approx 0.0099$.

(b) Evaluate the integral $\int_0^4 \int_{\frac{x}{2}}^{2} e^{y^2} dy dx$.

(c) Find the surface area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$ in the first octant.

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(b) Use the Stoke's theorem to evaluate $\int_C \vec{F} . \vec{dr}$, where $\vec{F}(x,y,z) = x^2 z\,\vec{i} + xy^2\,\vec{j} + z^2\,\vec{k}$ where C is the curve of the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9,$ oriented counterclockwise as viewed from the above.