King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second Mid-Term Examination
(II-Semester 1431/1432)

Max. Marks: 20   Time: 90 Minutes

Marking Scheme: **Q.1**(4),   **Q.2**:(3),   **Q.3**:(3),   **Q.4**:(3),   **Q.5**:(3),   **Q.6**:(4)

Q. No: 1 Reverse the order of integration, and evaluate the resulting integral

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx.$$

Solution: Here $0 \le x \le 4$ and $\sqrt{x} \le y \le 2$. .

$$\Rightarrow 0 \le y \le 2, \quad 0 \le x \le y^2.$$

$$\Rightarrow \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy = \int_0^2 \frac{1}{y^3+1}\left[x\right]_0^{y^2}\,dy$$

$$= \int_0^2 \frac{y^2}{y^3+1}\,dy = \frac{1}{3}\left[\ln(y^3+1)\right]_0^2 = \frac{1}{3}\left[\ln(9)-\ln(1)\right] = \frac{1}{3}\ln(9).$$

Q. No: 2 Use **polar coordinates** to evaluate the integral

$$\iint_R \left(x^2-y^2\right)dA \text{ , where R is the region}$$

bounded by the semi-circle $y = \sqrt{1-x^2}$ and the $x-axis$.

Solution: $\displaystyle\iint_R \left(x^2-y^2\right)dA = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2-y^2)\,dx\,dy = \int_0^\pi \int_0^1 (r^2\cos^2\theta - r^2\sin^2\theta)\,r\,dr\,d\theta$

$$= \int_0^\pi \int_0^1 (\cos^2\theta - \sin^2\theta)r^3\,dr\,d\theta = \int_0^\pi \cos(2\theta)\left[\frac{r^4}{4}\right]_0^1\,d\theta = \frac{1}{4}\int_0^\pi \cos(2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\frac{\sin(2\theta)}{2}\right]_0^\pi = \frac{1}{8}\left[\sin(2\pi)-\sin(0)\right] = \frac{1}{8}[0] = 0$$

Q. No: 3 Find the **surface area** of the surface S if S is the portion of the graph of
        $z = 2+xy$ that lies inside the **cylinder** $x^2+y^2 = 1$.

Solution: **Surface area=**

$$\iint_{R_{xy}} \sqrt{1+f_x^2+f_y^2}\,dA = \iint_{R_{xy}} \sqrt{1+y^2+x^2}\,dA = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left[\frac{\left(1+r^2\right)^{3/2}}{3/2} \right]_0^1 d\theta = \frac{1}{3}\left[(2)^{3/2} - 1 \right] 2\pi .$$

Q. No: 4 Set up integrals that can be used to find the centroid of the solid Q, where

Q is **bounded** by the **co-ordinate planes,** and graphs of the equations

$z = 9 - x^2$ and $2x + y = 6$.

Solution: Region is $0 \le x \le 3, 0 \le y \le 6 - 2x, 0 \le z \le 9 - x^2$.

$$m = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} dz\,dy\,dx ,$$

$$M_{xy} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} z\,dz\,dy\,dx , \qquad M_{yz} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} x\,dz\,dy\,dx ,$$

$$M_{xz} = \int_0^3 \int_0^{6-2x} \int_0^{9-x^2} y\,dz\,dy\,dx .$$

Q. No: 5 Find the mass of the solid bounded by $z = x^2 + y^2 - 4$ **and** $z = 0$
having density $\delta(x,y,z) = 1 + x^2 + y^2.$

Solution: Region is $x^2 + y^2 - 4 \le z \le 0, -\sqrt{4-x^2} \le y \le +\sqrt{4-x^2}, -2 \le x \le 2.$
In Cylindrical system region is $r^2 - 4 \le z \le 0, 0 \le r \le \sqrt{2}, 0 \le \theta \le 2\pi.$
Density $\delta(x,y,z) = 1 + x^2 + y^2 = 1 + r^2.$

$$Mass = \int_0^{2\pi} \int_0^2 \int_{r^2-4}^0 (1+r^2)r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 (r+r^3)\left[z \right]_{r^2-4}^0 dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 -(r+r^3)(r^2-4)dr\,d\theta = \int_0^{2\pi} \int_0^2 (+3r^3 + 4r - r^5)dr\,d\theta$$

$$= \int_0^{2\pi} \left[+3\frac{r^4}{4} + 4\frac{r^2}{2} - \frac{r^6}{6} \right]_0^2 d\theta = \int_0^{2\pi} \frac{28}{3}d\theta = \frac{56\pi}{3}.$$

Use spherical coordinates to evaluate the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z \; dzdydx$$

Solution: Here $-1 \le x \le 1$,

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$

and $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$.

In spherical coordinates $0 \le \rho \le 1, \; 0 \le \phi \le \dfrac{\pi}{4}$, $0 \le \theta \le 2\pi$

$$\Rightarrow \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z \; dzdydx = \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{1} (\rho\cos\phi)\rho^2 \sin\phi d\, \rho d \phi d\,\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{2} \cos\phi \sin\phi \rho^3 d\, \rho d \phi d\,\theta = \int_{0}^{2\pi} \int_{0}^{\pi/4} \cos\phi \sin\phi \left[\frac{\rho^4}{4}\right]_{0}^{1} d\, \phi d\,\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi/4} \frac{1}{4}\cos\phi \sin\phi d\, \phi d\,\theta = \frac{1}{8}\int_{0}^{2\pi} \int_{0}^{\pi/4} \sin 2\phi d\, \phi d\,\theta = \frac{1}{8}\int_{0}^{2\pi} \left[-\frac{\cos 2\phi}{2}\right]_{0}^{\pi/4} d\,\theta$$

$$= \frac{1}{16}\int_{0}^{2\pi} -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] d\,\theta = \frac{1}{16}\int_{0}^{2\pi} d\,\theta = \frac{\pi}{8}.$$