King Saud University
College of Sciences
Department of Mathematics

M-104
GENERAL MATHEMATICS -2-

CLASS NOTES
DRAFT - 2014

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CHAPTER ONE

CONIC SECTIONS

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1. Parabola

Definition: A **parabola** is the set of all points in the plane equidistant from a fixed point F (called the **focus**) and a fixed line D (called the **directrix**) in the same plane.

Notes:

1. The line passing through the focus F and perpendicular to the directrix D is called the **axis** of the parabola .

2. The point half-way from the focus F to the directrix D is called the **vertex** of the parabola and is denoted by V .
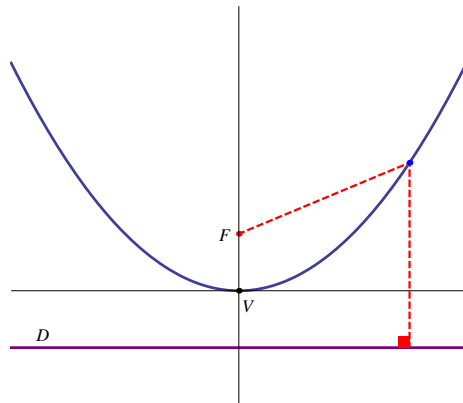
1.1 The vertex of the parabola is the origin :

This section discusses the special case where the vertex of the parabola is $(0, 0)$. There are four different cases :

1) $x^2 = 4ay$, where $a > 0$
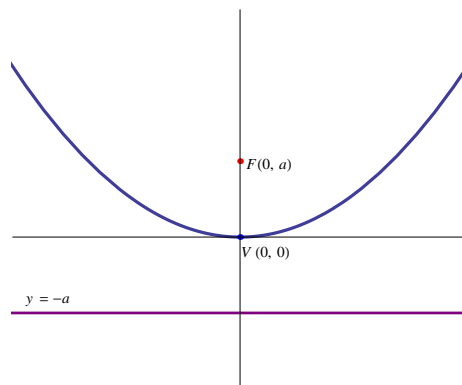
The parabola opens upwards .

The focus is $F(0, a)$.

The equation of the directrix is $y = -a$.

The axis of the parabola is the y-axis .

2) $x^2 = -4ay$, where $a > 0$

The parabola opens downwards (note the negative sign in the formula).

The focus is $F(0, -a)$.

The equation of the directrix is $y = a$.

The axis of the parabola is the y-axis .
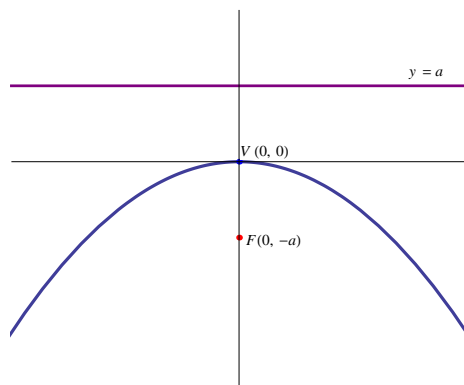
3) $y^2 = 4ax$, where $a > 0$
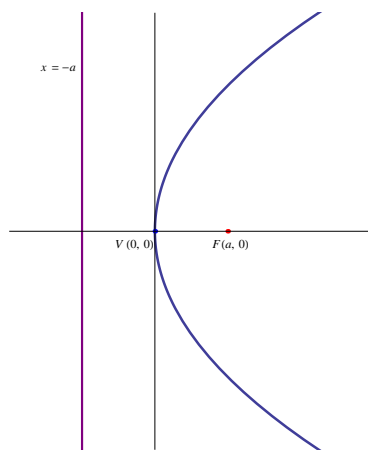
The parabola opens to the right.

The focus is $F(a, 0)$.

The equation of the directrix is $x = -a$.

The axis of the parabola is the x-axis .

4) $y^2 = -4ax$, where $a > 0$
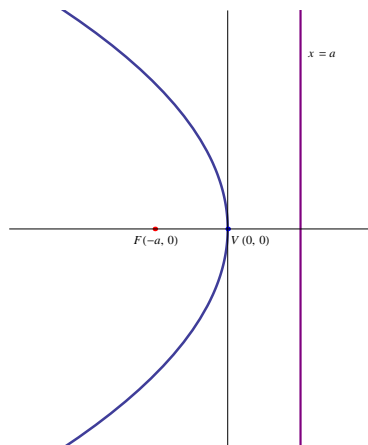
The parabola opens to the left (note the negative sign in the formula) .

The focus is $F(-a, 0)$.

The equation of the directrix is $x = a$.

The axis of the parabola is the x-axis .

Example 1: Find the focus and the directrix of the parabola $x^2 = 4y$, and sketch its graph.

Solution: Since the variable x is of degree 2 and the formula contains a positive sign then $x^2 = 4y$ is similar to case(1), where the parabola opens upwards .

$4a = 4 \implies a = 1$

The focus is $F(0,1)$, and the equation of the directrix is $y = -1$.

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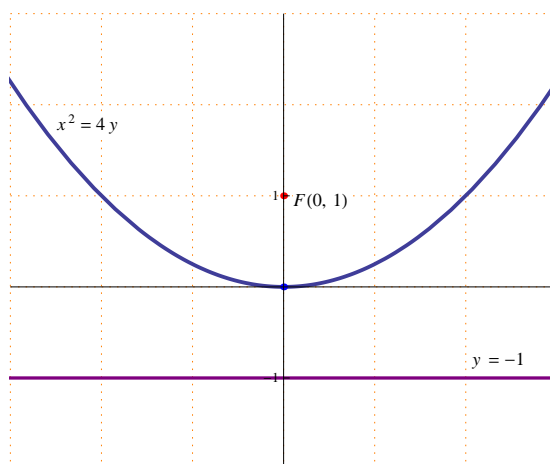
Example 2: Find the focus and the directrix of the parabola $y^2 = -8x$, and sketch its graph.

Solution: Since the variable y is of degree 2 and the formula contains a negative sign then $y^2 = -8x$ is similar to case(4), where the parabola opens to the left .

$-4a = -8 \implies a = 2$

The focus is F(-2,0) , and the equation of the directrix is $x = 2$.
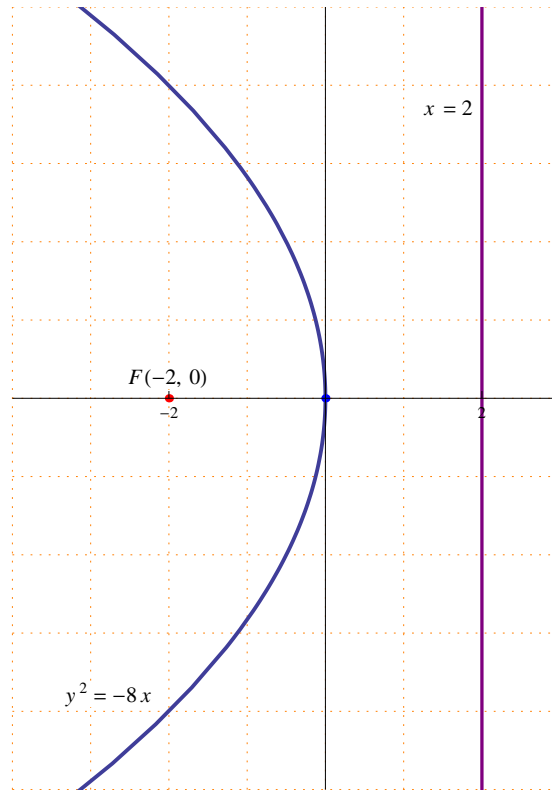
1.2 The general formula of a parabola :
This section discusses the general formula of a parabol where the vertex of the parabola is any point $V(h, k)$ in the plane.
There are four different cases :

No.	The general formula	Focus	Directrix	The parabola opens
1	$(x - h)^2 = 4a(y - k)$	$F(h, k + a)$	$y = k - a$	upwards
2	$(x - h)^2 = -4a(y - k)$	$F(h, k - a)$	$y = k + a$	downwards
3	$(y - k)^2 = 4a(x - h)$	$F(h + a, k)$	$x = h - a$	to the right
4	$(y - k)^2 = -4a(x - h)$	$F(h - a, k)$	$x = h + a$	to the left

Example 1: Find the focus and the directrix of the parabola $(x + 1)^2 = -4(y - 1)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (2).
$(x - h)^2 = (x + 1)^2 = (x - (-1))^2 \Rightarrow h = -1$.
$(y - k) = (y - 1) \Rightarrow k = 1$.
$-4a = -4 \Rightarrow a = 1$.
The vertex is $V(-1, 1)$
The focus is $F(-1, 0)$ and the equation of the directrix is $y = 2$.
The parabola opens downwards (note the negative sign in the formula).

Example 2: Find the focus and the directrix of the parabola $(y - 1)^2 = 8(x + 2)$, and sketch its graph.
Solution : The equation of the parabola is similar to case (3).
$(y - k)^2 = (y - 1)^2 \Rightarrow k = 1$.
$(x - h) = (x + 2) = (x - (-2)) \Rightarrow h = -2$.
$4a = 8 \Rightarrow a = 2$.
The vertex is $V(-2, 1)$
The focus is $F(0, 1)$ and the equation of the directrix is $x = -4$.
The parabola opens to the right .

$x = -4$

$V (-2, 1)$ $F(0, 1)$

$(y - 1)^2 = 8 (x + 2)$

Example 3: Find the focus and the directrix of the parabola $2y^2 - 4y + 8x + 10 = 0$, and sketch its graph.

Solution : By completing the square

$2y^2 - 4y + 8x + 10 = 0 \ \Rightarrow \ 2y^2 - 4y = -8x - 10 \ \Rightarrow \ 2(y^2 - 2y) = -8x - 10$
$\Rightarrow \ 2(y^2 - 2y + 1) = -8x - 10 + 2 \ \Rightarrow \ 2(y-1)^2 = -8x - 8 \Rightarrow \ 2(y-1)^2 = -8(x+1)$
$\Rightarrow \ (y - 1)^2 = -4(x + 1)$

The equation of the parabola is similar to case (4).

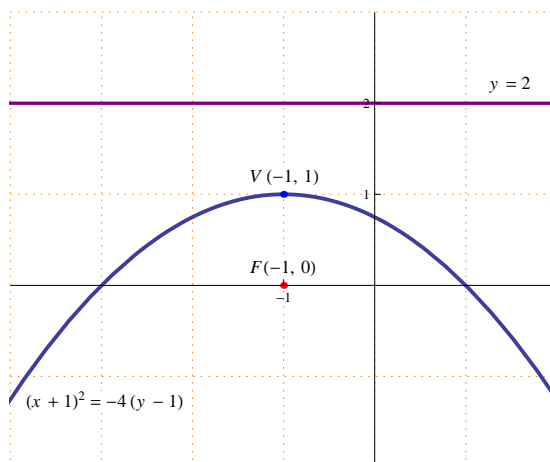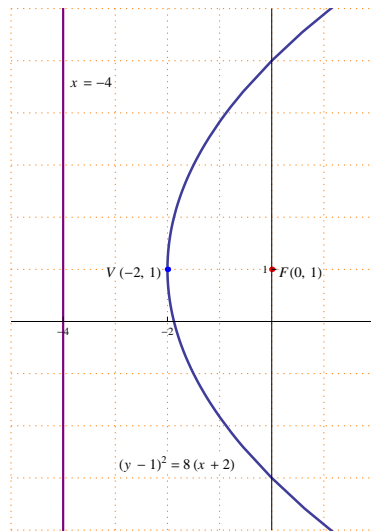$(y - k)^2 = (y - 1)^2 \ \Rightarrow \ k = 1$.

$(x - h) = (x + 1) = (x - (-1)) \ \Rightarrow \ h = -1$.

$-4a = -4 \ \Rightarrow \ a = 1$.

The vertex is $V(-1, 1)$.

The focus is $F(-2, 1)$ and the equation of the directrix is $x = 0$ (the y-axis).

The parabola opens to the left (note the negative sign in the formula)

$x = 0$

$F(-2, 1)$ $V (-1, 1)$

$(y - 1)^2 = -4 (x + 1)$

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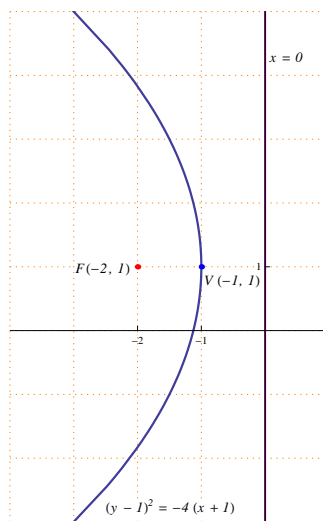
Example 4: Find the focus and the directrix of the parabola $x^2 - 6y - 2x = -7$
, and sketch its graph.

Solution : By completing the square

$x^2 - 6y - 2x = -7 \implies x^2 - 2x = 6y - 7 \implies x^2 - 2x + 1 = 6y - 7 + 1$

$\implies (x-1)^2 = 6y - 6 \implies (x-1)^2 = 6(y-1)$

The equation of the parabola is similar to case (1).

$(x-h)^2 = (x-1)^2 \implies h = 1$.

$(y-k) = (y-1)) \implies k = 1$.

$4a = 6 \implies a = \dfrac{6}{4} = \dfrac{3}{2}$.

The vertex is $V(1,1)$

The focus is $F\left(1, \dfrac{5}{2}\right)$ and the equation of the directrix is $y = -\dfrac{1}{2}$.

The parabola opens upwards.
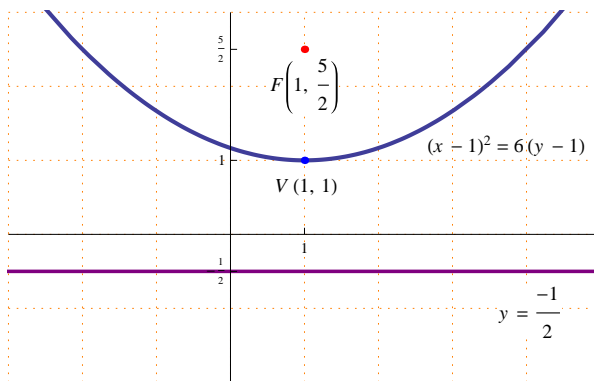
Example 5: Find the equation of the parabola with vertex $V(2,1)$ and focus
$F(2,3)$ and sketch its graph.

Solution : Since the focus is located upper than the vertex then the parabola
opens upwards.

Hence its equation is $(x-h)^2 = 4a(y-k)$.

Since the vertex is $V(2,1)$ then $h = 2$ and $k = 1$

a equals the distance between $V(2,1)$ and $F(2,3)$ which equals 2 .

The equation of the parabola with $V(2,1)$ and $F(2,3)$ is $(x-2)^2 = 8(y-1)$
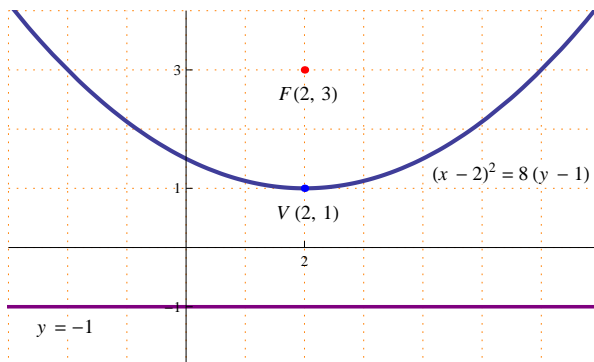
Example 6: Find the equation of the parabola with focus $F(-1, 1)$ and directrix $x = 1$ and sketch its graph.

Solution : Since the focus is located to the left of the directrix then the parabola opens to the left.

Hence its equation is $(y - k)^2 = -4a(x - h)$.

The vertex is half-way beween the focus and the directrix , hence $V(0, 1)$

a equals the distance between $V(0, 1)$ and $F(-1, 1)$ which equals 1 .

The equation of the parabola with $F(-1, 1)$ and directrix $x = 1$ is $(y-1)^2 = -4x$
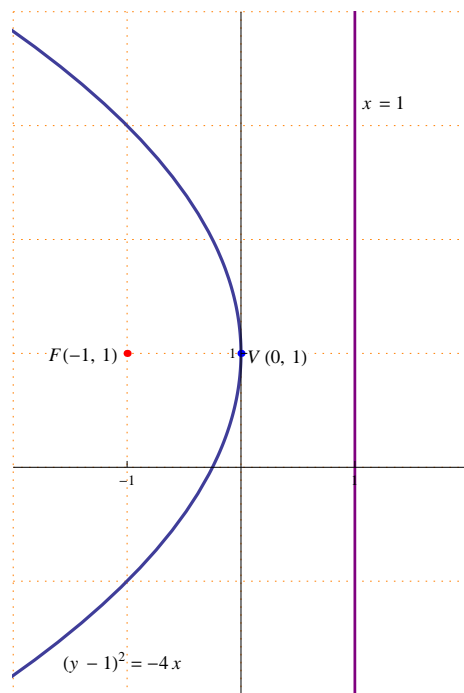
2. Ellipse

Definition: An **ellipse** is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the ellipse and are denoted by F_1 and F_2.

2. The midpoint between F_1 and F_2 is called the **center** of the ellipse and is denoted by P.

3. The endpoints of the **major axis** are called the vertices of the ellipse and are denoted by V_1 and V_2.

4. The endpoints of the **minor axis** are denoted by W_1 and W_2.
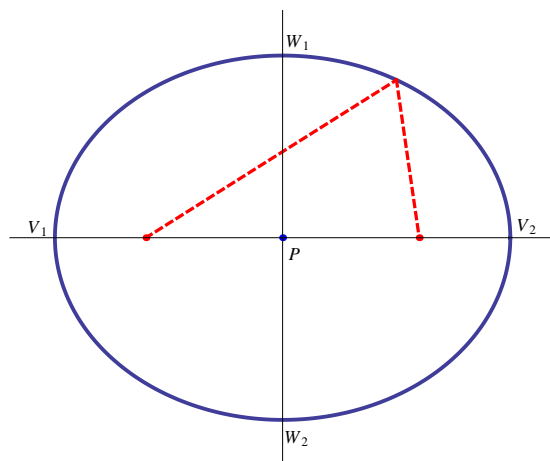
2.1 The center of the ellipse is the origin :
This section discusses the special case where the center of the ellipse is $(0,0)$.
There are two different cases :

1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$:

The foci of the ellipse are $F_1(-c,0)$ and $F_2(c,0)$, where $c = \sqrt{a^2 - b^2}$.

The vertices of the ellipse are $V_1(-a,0)$ and $V_2(a,0)$.

The endpoints of the minor axis are $W_1(0,b)$ and $W_2(0,-b)$.

The major axis lies on the x-axis , and its length is $2a$.

The minor axis lies on the y-axis , and its length is $2b$.
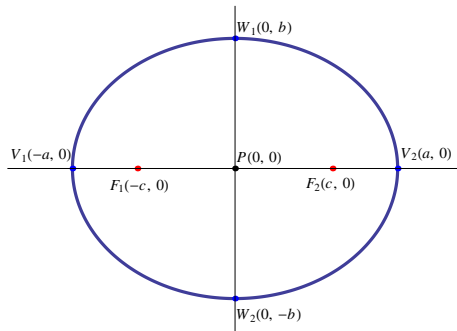
2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b > a$:

The foci of the ellipse are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{b^2 - a^2}$.

The vertices of the ellipse are $V_1(0, b)$ and $V_2(0, -b)$.

The endpoints of the minor axis are $W_1(-a, 0)$ and $W_2(a, 0)$.

The major axis lies on the y-axis , and its length is $2b$.

The minor axis lies on the x-axis , and its length is $2a$.
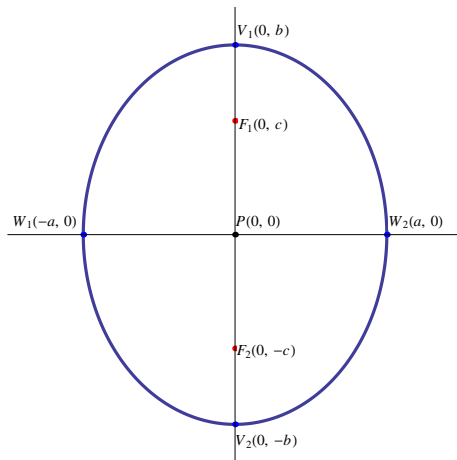
Example 1: Identify the features of the ellipse $9x^2 + 25y^2 = 225$, and sketch its graph.

Solution : $9x^2 + 25y^2 = 225 \Rightarrow \dfrac{9x^2}{225} + \dfrac{25y^2}{225} = 1 \Rightarrow \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

$a^2 = 25 \Rightarrow a = 5$ and $b^2 = 9 \Rightarrow b = 3$.

Since $a > b$ then $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ is similar to case (1).

$c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4$.

The foci are $F_1(-4, 0)$ and $F_2(4, 0)$.

The vertices are $V_1(-5, 0)$ and $V_2(5, 0)$.

The endpoints of the minor axis are $W_1(0, 3)$ and $W_2(0, -3)$.
The length of the major axis is $2a = 10$.
The length of the minor axis is $2b = 6$.
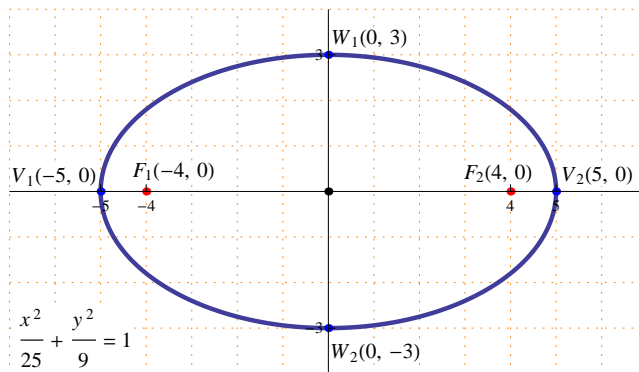
Example 2: Identify the features of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

Solution : $16x^2 + 9y^2 = 144 \;\Rightarrow\; \dfrac{16x^2}{144} + \dfrac{9y^2}{144} = 1 \;\Rightarrow\; \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

$a^2 = 9 \;\Rightarrow\; a = 3$ and $b^2 = 16 \;\Rightarrow\; b = 4$.

Since $b > a$ then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ is similar to case (2).

$c^2 = \sqrt{b^2 - a^2} = \sqrt{16 - 9} = \sqrt{7}$.

The foci are $F_1\left(0, \sqrt{7}\right)$ and $F_2\left(0, -\sqrt{7}\right)$.

The vertices are $V_1(0, 4)$ and $V_2(0, -4)$.

The endpoints of the minor axis are $W_1(-3, 0)$ and $W_2(3, 0)$.

The length of the major axis is $2b = 8$.

The length of the minor axis is $2a = 6$.

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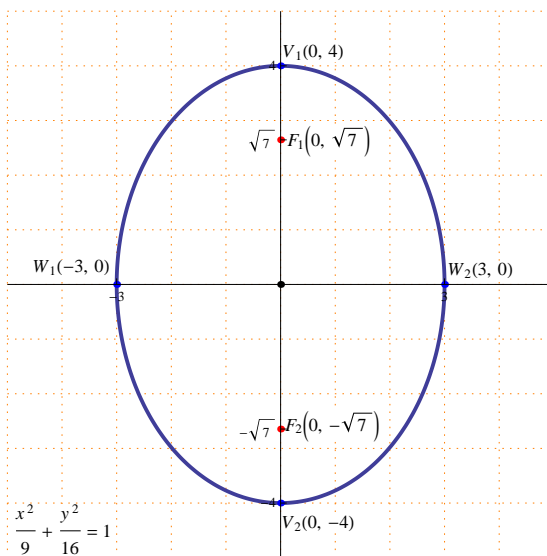
2.2 The general formula of an ellipse :

This section discusses the general formula of an ellipse where the center of the ellipse is any point $P(h,k)$ in the plane.

There are two different cases :

No.	The general Formula	The Foci	The Vertices	W_1 and W_2
1	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ ($a > b$) and $c = \sqrt{a^2 - b^2}$	$F_1(h-c,k)$ $F_2(h+c,k)$	$V_1(h-a,k)$ $V_2(h+a,k)$	$W_1(h,k-b)$ $W_2(h,k+b)$
2	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ ($b > a$) and $c = \sqrt{b^2 - a^2}$	$F_1(h,k-c)$ $F_2(h,k+c)$	$V_1(h,k-b)$ $V_2(h,k+b)$	$W_1(h-a,k)$ $W_2(h+a,k)$

Example 1: Find the equation of the ellipse with foci at $(-3,1)$, $(5,1)$, and one of its vertices is $(7,1)$, and sketch its graph.

Solution : The center of the ellipse $P(h,k)$ is located in the middle of the two foci, hence $(h,k) = \left(\dfrac{-3+5}{2}, \dfrac{1+1}{2} \right) = (1,1)$.

c is the distance between the center and one of the foci , and it equals to 4 (see the figure).

Since the major axis (where the two foci lie) is parallel to the x-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a > b$.

a is the distance between the center and one of the vertices, and it equals 6 (see the figure).

$c^2 = a^2 - b^2 \;\Rightarrow\; (4)^2 = (6)^2 - b^2 \;\Rightarrow\; b^2 = 36 - 16 = 20 \;\Rightarrow\; b = 2\sqrt{5}.$

The equation of the ellipse is $\dfrac{(x-1)^2}{36} + \dfrac{(y-1)^2}{20} = 1$.

The vertices of the ellipse are $V_1(-5,1)$ and $V_2(7,1)$.

The endpoints of the minor axis are $W_1\left(1, 1+2\sqrt{5}\right)$ and $W_2\left(1, 1-2\sqrt{5}\right)$.

Example 2: Find the equation of the ellipse with foci at $(2, 5)$, $(2, -3)$, and the length of its minor axis equals 6 , and sketch its graph.

Solution : The center of the ellipse $P(h, k)$ is located in the middle of the two foci, hence $(h, k) = \left(\dfrac{2+2}{2}, \dfrac{-3+5}{2} \right) = (2, 1)$.

c is the distance between the center and one of the foci , and it equals to 4 (see the figure).

Since the major axis (where the two foci lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.

The length of the minor axis is 6 means that $2a = 6 \Rightarrow a = 3$.

$c^2 = b^2 - a^2 \Rightarrow (4)^2 = b^2 - (3)^2 \Rightarrow b^2 = 16 + 9 = 25 \Rightarrow b = 5$.

The equation of the ellipse is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{25} = 1$.

The vertices of the ellipse are $V_1(2, 6)$ and $V_2(2, -4)$.

The endpoints of the minor axis are $W_1(-1, 1)$ and $W_2(5, 1)$.
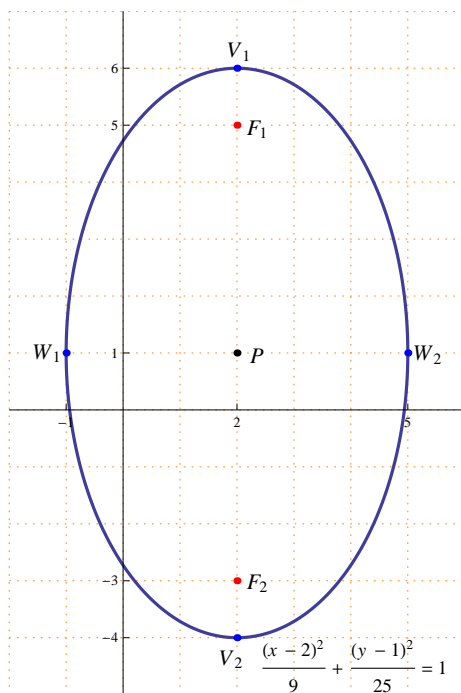
Example 3: Find the equation of the ellipse with vertices at $(-1, 4)$, $(-1, -2)$ and the distance between its two foci equals 4 , and sketch its graph.

Solution : The center of the ellipse $P(h, k)$ is located in the middle of the two vertices, hence $(h, k) = \left(\dfrac{-1-1}{2}, \dfrac{-2+4}{2} \right) = (-1, 1)$.

The distance between the two foci equals 4 means that $2c = 4 \Rightarrow c = 2$.

Since the major axis (where the two vertices lie) is parallel to the y-axis , then the general formula of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.
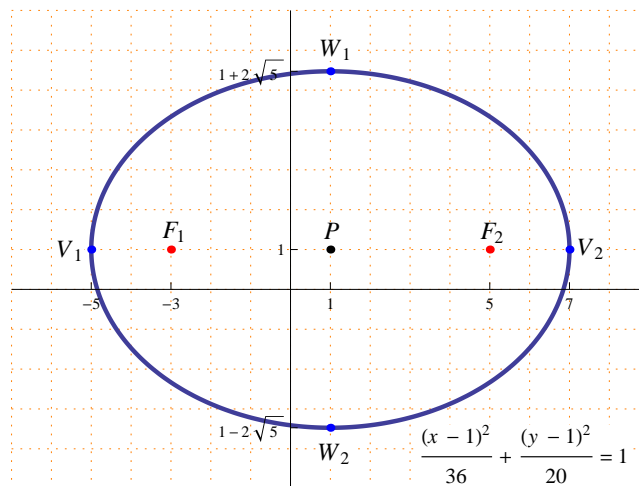
The length of the major axis (the distance between the two vertices) equals 6 ,

this means $2b = 6 \Rightarrow b = 3$.

$c^2 = b^2 - a^2 \Rightarrow (2)^2 = (3)^2 - a^2 \Rightarrow a^2 = 9 - 4 = 5 \Rightarrow a = \sqrt{5}$.

The equation of the ellipse is $\dfrac{(x+1)^2}{5} + \dfrac{(y-1)^2}{9} = 1$.

The foci of the ellipse are $F_1(-1, 3)$ and $F_2(-1, -1)$.

The endpoints of the minor axis are $W_1\left(-1 - \sqrt{5}, 1\right)$ and $W_2\left(-1 + \sqrt{5}, 1\right)$.

$$\frac{(x+1)^2}{5} + \frac{(y-1)^2}{9} = 1$$
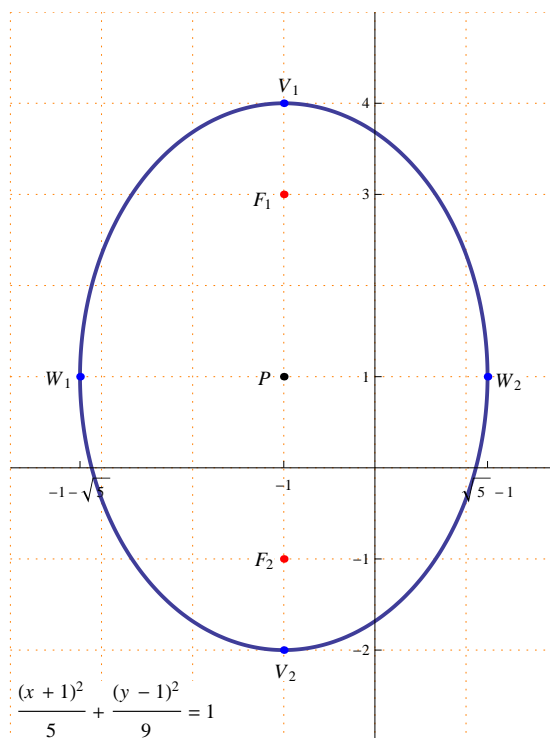
Example 4: Identify the features of the ellipse $4x^2 + 2y^2 - 8x - 8y - 20 = 0$, and sketch its graph.

Solution :

$4x^2 + 2y^2 - 8x - 8y - 20 = 0 \Rightarrow (4x^2 - 8x) + (2y^2 - 8y) = 20$

$\Rightarrow 4(x^2 - 2x) + 2(y^2 - 4y) = 20$

By completing the square

$4(x^2 - 2x) + 2(y^2 - 4y) = 20 \Rightarrow 4(x^2 - 2x + 1) + 2(y^2 - 4y + 4) = 20 + 12$

$\Rightarrow 4(x - 1)^2 + 2(y - 2)^2 = 32$

$\Rightarrow \dfrac{4(x-1)^2}{32} + \dfrac{2(y-2)^2}{32} = 1 \Rightarrow \dfrac{(x-1)^2}{8} + \dfrac{(y-2)^2}{16} = 1$

$b^2 = 16 \Rightarrow b = 4$ and $a^2 = 8 \Rightarrow b = \sqrt{8} = 2\sqrt{2}$.

$c^2 = b^2 - a^2 \Rightarrow c^2 = 16 - 8 = 8 \Rightarrow c = \sqrt{8} = 2\sqrt{2}$.

The center of the ellipse is $(1, 2)$.

The foci of the ellipse are $F_1\left(1, 2 + 2\sqrt{2}\right)$ and $F_2\left(1, 2 - 2\sqrt{2}\right)$.

The vertices of the ellipse are $V_1(1, 6)$ and $V_2(1, -2)$.

The endpoints of the minor axis are $W_1\left(1 - 2\sqrt{2}, 2\right)$ and $W_2\left(1 + 2\sqrt{2}, 2\right)$

The length of the major axis is 8 and the length of the minor axis is $2\sqrt{8} = 4\sqrt{2}$.

$$\frac{(x-1)^2}{8} + \frac{(y-2)^2}{16} = 1$$

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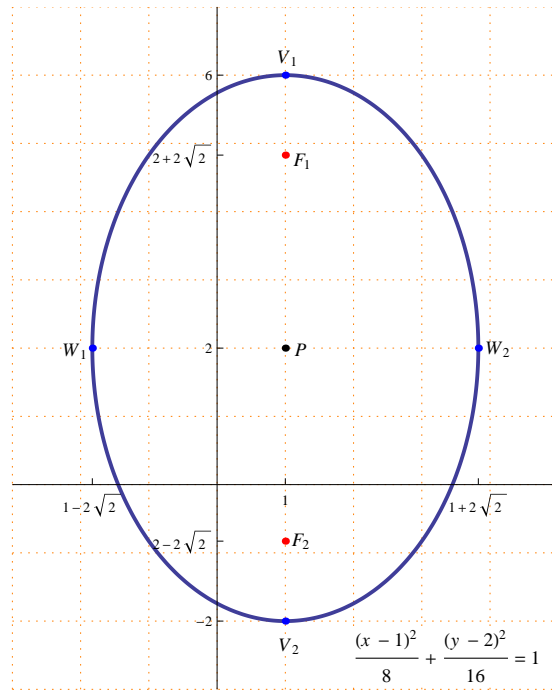
3. Hyperbola

Definition: A **hyperbola** is the set of all points in the plane for which the difference of the distances between two fixed points is constant.

Notes :

1. The two fixed points are called the **foci** of the hyperbola and are denoted by F_1 and F_2.

2. The midpoint between F_1 and F_2 is called the **center** of the hyperbola and is denoted by P.
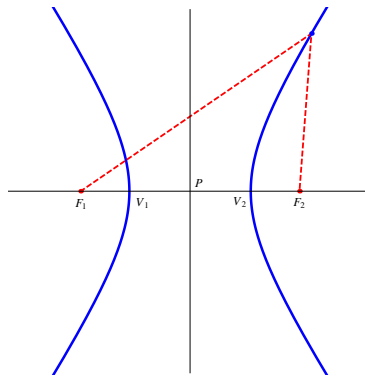
3.1 The center of the hyperbola is the origin :
This section discusses the special case where the center of the hyperbola is $(0, 0)$.
There are two different cases :

1) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$:

The foci of the hyperbola are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 + b^2}$.

The vertices of the hyperbola are $V_1(-a, 0)$ and $V_2(a, 0)$.

The line segment between V_1 and V_2 is the **transverse axis** , it lies on the x-axis and its length is $2a$.

The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.
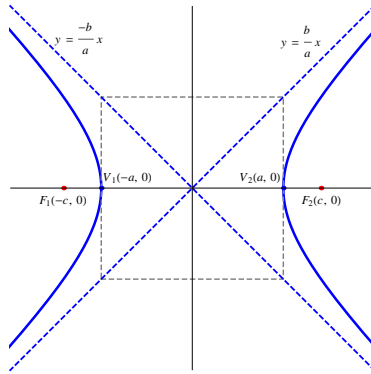
2) $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, where $a > 0$ and $b > 0$:

The foci of the hyperbola are $F_1(0, c)$ and $F_2(0, -c)$, where $c = \sqrt{a^2 + b^2}$.

The vertices of the hyperbola are $V_1(0, b)$ and $V_2(0, -b)$.

The line segment between V_1 and V_2 is the **transverse axis** , it lies on the y-axis and its length is $2b$.

The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.
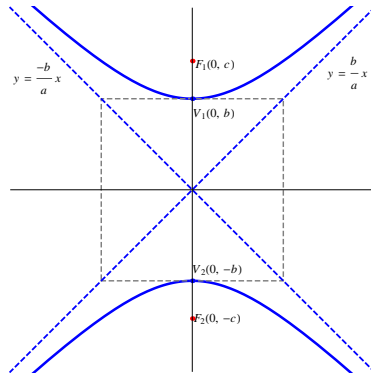
Example 1: Identify the features of the hyperbola $4x^2 - 16y^2 = 64$, and sketch its graph.

Solution :

$4x^2 - 16y^2 = 64 \ \Rightarrow \ \dfrac{4x^2}{64} - \dfrac{16y^2}{64} = 1 \ \Rightarrow \ \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

This form is similar to case (1).

$a^2 = 16 \ \Rightarrow \ a = 4$ and $b^2 = 4 \ \Rightarrow \ b = 2$

$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

The foci of the hyperbola are $F_1\left(-2\sqrt{5}, 0\right)$ and $F_2\left(2\sqrt{5}, 0\right)$.

The vertices are $V_1(-4, 0)$ and $V_2(4, 0)$.

The transverse axis lies on the x-axis and its length is $2a = 8$.

The equations of the asymptotes are $y = \dfrac{2}{4}x = \dfrac{1}{2}x$ and $y = -\dfrac{2}{4}x = -\dfrac{1}{2}x$
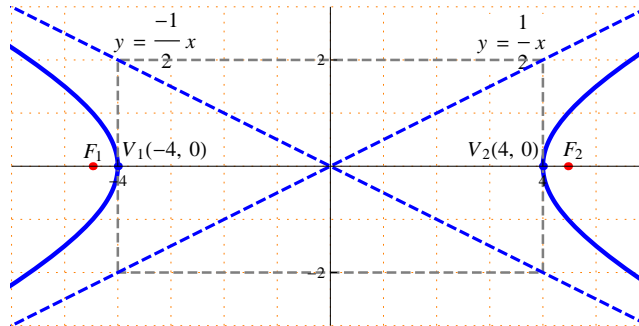
Example 2: Identify the features of the hyperbola $4y^2 - 9x^2 = 36$, and sketch its graph.

Solution :

$4y^2 - 9x^2 = 36 \;\Rightarrow\; \dfrac{4y^2}{36} - \dfrac{9x^2}{36} = 1 \;\Rightarrow\; \dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

This form is similar to case (2).

$a^2 = 4 \;\Rightarrow\; a = 2$ and $b^2 = 9 \;\Rightarrow\; b = 3$

$c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

The foci of the hyperbola are $F_1\left(0, \sqrt{13}\right)$ and $F_2\left(0, -\sqrt{13},\right)$.

The vertices are $V_1(0, 3)$ and $V_2(0, -3)$.

The transverse axis lies on the y-axis and its length is $2b = 6$.

The equations of the asymptotes are $y = \dfrac{3}{2}x$ and $y = -\dfrac{3}{2}x$
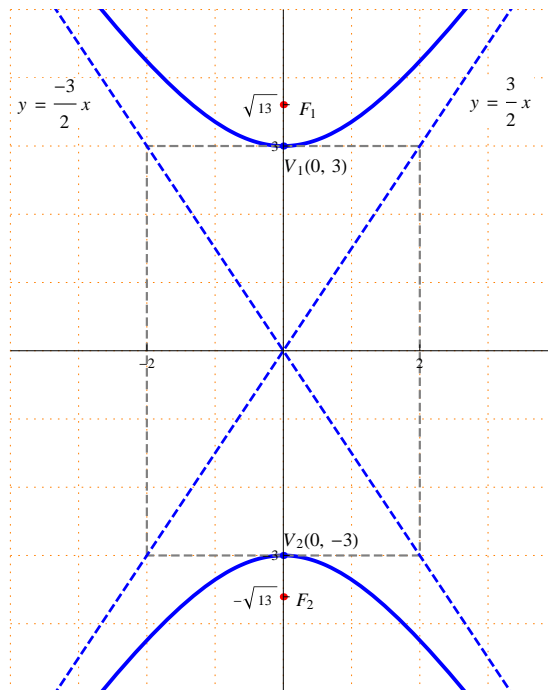
3.2 The general formula of a hyperbola :

This section discusses the general formula of a hyperbola where the center of the hyperbola is any point $P(h, k)$ in the plane.

There are two different cases :

No.	The general Formula	The Foci	The Vertices	Transverse axis
1	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ $(\ c^2 = a^2 + b^2\ )$	$F_1(h-c, k)$ $F_2(h+c, k)$	$V_1(h-a, k)$ $V_2(h+a, k)$	parallel to the x-axis
2	$\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$ $(\ c^2 = a^2 + b^2\ )$	$F_1(h, k+c)$ $F_2(h, k-c))$	$V_1(h, k+b)$ $V_2(h, k-b)$	parallel to the y-axis

The equations of the asymptotes are $y = \dfrac{b}{a}(x-h) + k$ and $y = -\dfrac{b}{a}(x-h) + k$

**Example 1:** Find the equation of the hyperbola with foci at $(-2, 2)$, $(6, 2)$ and one of its vertices is $(5, 2)$, and sketch its graph.

Solution :

The center of the hyperbola $P(h, k)$ is located in the middle of the two foci ,

hence $(h, k) = \left(\dfrac{-2+6}{2}, \dfrac{2+2}{2}\right) = (2, 2)$

Note that the two foci lie on a line parallel to the x-axis , hence the general

formula of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

$2c$ is the distance between the two foci , hence $2c = 8 \Rightarrow c = 4$.

a is the distance between the center $(2, 2)$ and the vertex $(5, 2)$, hence $a = 3$,

and the other vertex is $(-1, 2)$.

$c^2 = a^2 + b^2 \Rightarrow 4^2 = 3^2 + b^2 \Rightarrow b^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$.

The equation of the hyperbola is $\dfrac{(x-2)^2}{9} - \dfrac{(y-2)^2}{7} = 1$

The equations of the asymptotes are $L_1 : y = \dfrac{\sqrt{7}}{3}(x-2) + 2$ and

$L_2 : y = -\dfrac{\sqrt{7}}{3}(x-2) + 2$
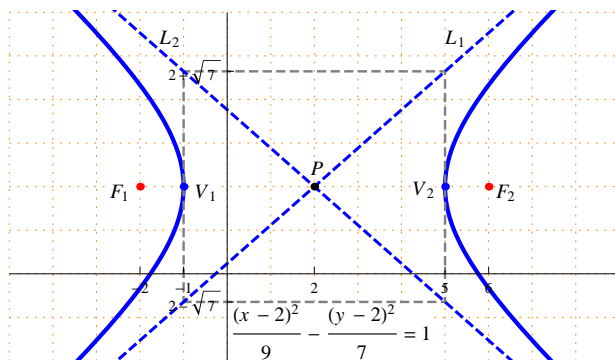
Example 2: Find the equation of the hyperbola with foci at $(-1, -6)$, $(-1, 4)$ and the length of its transverse axis is 8 , and sketch its graph.

Solution :

The center of the hyperbola $P(h, k)$ is located in the middle of the two foci ,

hence $(h, k) = \left(\dfrac{-1 - 1}{2}, \dfrac{-6 + 4}{2} \right) = (-1, -1)$

Note that the two foci lie on a line parallel to the y-axis , hence the general

formula of the hyperbola is $\dfrac{(y - k)^2}{b^2} - \dfrac{(x - h)^2}{a^2} = 1$.

$2c$ is the distance between the two foci , hence $2c = 10 \ \Rightarrow \ c = 5$.

The length of the transverse axis is 8 , this means $2b = 8 \ \Rightarrow \ b = 4$.

The vertices are $(-1, -5)$ and $(-1, 3)$.

$c^2 = a^2 + b^2 \ \Rightarrow \ 5^2 = a^2 + 4^2 \ \Rightarrow \ a^2 = 25 - 16 = 9 \ \Rightarrow \ a = 3$.

The equation of the hyperbola is $\dfrac{(y + 1)^2}{16} - \dfrac{(x + 1)^2}{9} = 1$.

The equations of the asymptotes are $L_1 : \ y = \dfrac{4}{3}(x + 1) - 1$ and

$L_2 : \ y = -\dfrac{4}{3}(x + 1) - 1$
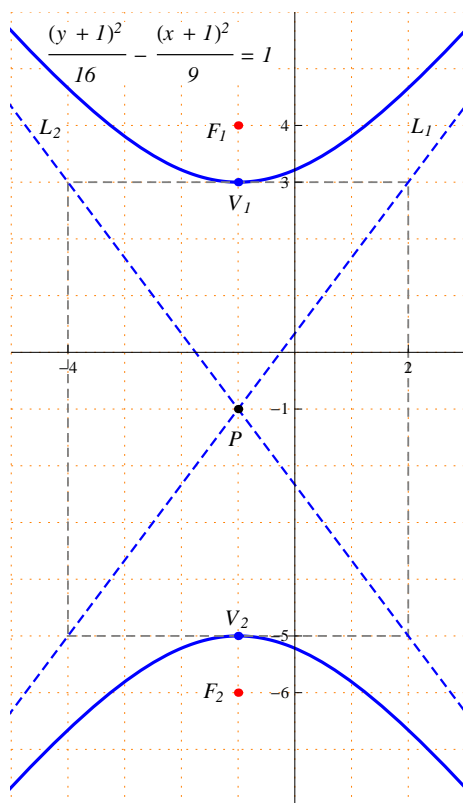
Example 3: Find the equation of the hyperbola with center at $(1, 1)$, one of its foci is $(5, 1)$ and one of its vertices is $(-1, 1)$, and sketch its graph.

Solution :

Since the center and the focus lie on a line parallel to the x-axis , then the

general formula of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

c is the distance between the center $(1,1)$ and the focus $(5,1)$, hence $c = 4$, the other foci is $(-3,1)$.

a is the distance between the center $(1,1)$ and the vertex $(-1,1)$, hence $a = 2$, the other vertex is $(3,1)$.

$c^2 = a^2 + b^2 \;\Rightarrow\; 4^2 = 2^2 + b^2 \;\Rightarrow\; b^2 = 16 - 4 = 12 \;\Rightarrow\; b = \sqrt{12} = 2\sqrt{3}$

The equation of the hyperbola is $\dfrac{(x-1)^2}{4} - \dfrac{(y-1)^2}{12} = 1$.

The equations of the asymptotes are

$L_1: \; y = \dfrac{2\sqrt{3}}{2}(x-1) + 1 = \sqrt{3}(x-1) + 1$ and $L_2: \; y = -\sqrt{3}(x-1) + 1$
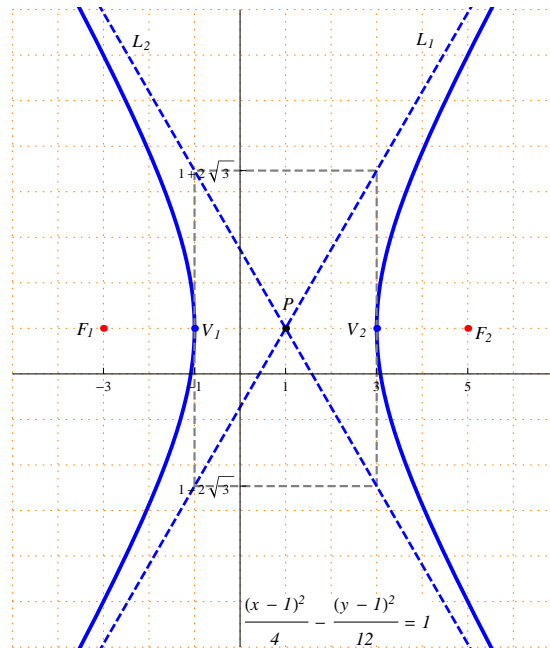
Example 4: Identify the features of the hyperbola $2y^2 - 4x^2 - 4y - 8x - 34 = 0$, and sketch its graph.

Solution :

$2y^2 - 4x^2 - 4y - 8x - 34 = 0 \;\Rightarrow\; (2y^2 - 4y) - (4x^2 + 8x) = 34$

$\Rightarrow\; 2(y^2 - 2y) - 4(x^2 + 2x) = 34$

$\Rightarrow\; 2(y^2 - 2y + 1) - 4(x^2 + 2x + 1) = 34 + 2 - 4 \;\Rightarrow\; 2(y-1)^2 - 4(x+1)^2 = 32$

$\Rightarrow\; \dfrac{2(y-1)^2}{32} - \dfrac{4(x+1)^2}{32} = 1 \;\Rightarrow\; \dfrac{(y-1)^2}{16} - \dfrac{(x+1)^2}{8} = 1$

$b^2 = 16 \;\Rightarrow\; b = 4$ and $a^2 = 8 \;\Rightarrow\; a = \sqrt{8} = 2\sqrt{2}.$

$c^2 = a^2 + b^2 \;\Rightarrow\; c^2 = 16 + 8 = 24 \;\Rightarrow\; c = \sqrt{24} = 2\sqrt{6}.$

The center of the hyperbola is $P(-1,1)$.

The foci of the hyperbola are $F_1\left(-1, 1 + 2\sqrt{6}\right)$ and $F_2\left(-1, 1 - 2\sqrt{6}\right)$.

The vertices of the hyperbola are $V_1(-1,5)$ and $V_2(-1,-3)$.

The transverse axis is parallel to the y-axis and its length is $2b = 8$.

The equations of the asymptotes are

$$L_1 : \ y = \frac{4}{2\sqrt{2}}(x+1) + 1 = \sqrt{2}(x+1) + 1 \text{ and } L_2 : \ y = -\sqrt{2}(x+1) + 1$$

L_2 L_1

F_1

V_1

5

P

1

$-1 - 2\sqrt{2}$ -1 $2\sqrt{2} - 1$

V_2

3

F_2

$$\frac{(y-1)^2}{16} - \frac{(x+1)^2}{8} = 1$$
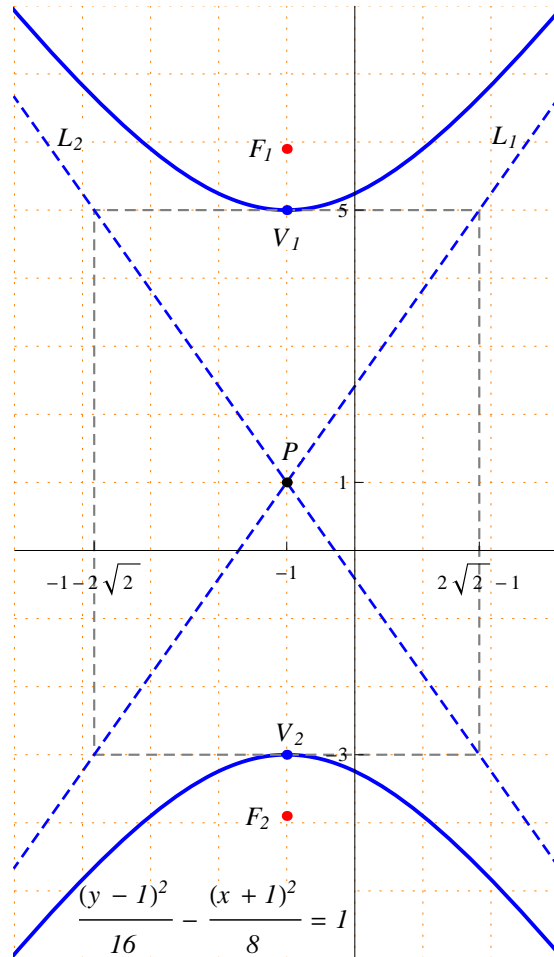
CHAPTER TWO

MATRICES AND DETERMINANTS

1. Matrices

2. Determinants

1. Matrices

Definition : A **matrix A** of order $m \times n$ is a set of real numbers arranged in a rectangular array of m rows and n columns. It is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

Notes :

1. a_{ij} represents the element of the matrix **A** that lies in row i and column j.

2. The matrix **A** can also be written as $\mathbf{A} = (a_{ij})_{m \times n}$.

3. If the number of rows equals the number of columns $(m = n)$ then **A** is called a **square** matrix of order n.

4. In a square matrix $\mathbf{A} = (a_{ij})$, the set of elements of the form a_{ii} is called the diagonal of the matrix.

Examples :

1. $\begin{pmatrix} -1 & 4 & 0 \\ 2 & -3 & 7 \end{pmatrix}$ is a matrix of order 2×3.

   $a_{11} = -1$, $a_{12} = 4$, $a_{13} = 0$, $a_{21} = 2$, $a_{22} = -3$ and $a_{23} = 7$.

2. $\begin{pmatrix} 5 & -3 & 2 \\ 0 & 1 & 7 \\ 0 & 8 & 13 \end{pmatrix}$ is a square matrix of order 3.

   The diagonal is the set $\{a_{11}, a_{22}, a_{33}\} = \{5, 1, 13\}$

1.1 Special types of matrices :

1. Row vector : A row vector of order n is a matrix of order $1 \times n$, and it is written as $\begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$

Example : $\begin{pmatrix} 2 & 7 & 0 & -1 \end{pmatrix}$ is a row vector of order 4.

2. Column vector : A column vector of order n is a matrix of order $n \times 1$, and it is written as $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Example : $\begin{pmatrix} 8 \\ -1 \\ 2 \end{pmatrix}$ is a column vector of order 3.

3. Null matrix : The matrix $(a_{ij})_{m \times n}$ of order $m \times n$ is called a **null matrix** if $a_{ij} = 0$ for all i and j, and it is denoted by $\mathbf{0}$.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

Example : $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a null matrix of order 3×4.

4. Upper triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called an **upper triangular matrix** if $a_{ij} = 0$ for all $i > j$, and it is written as $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{pmatrix}$

Example : $\begin{pmatrix} 8 & 5 & -2 & 1 \\ 0 & 3 & 1 & -6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is an upper triangular matrix of order 4.

5. Lower triangular matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **lower triangular matrix** if $a_{ij} = 0$ for all $i < j$, and it is written as

$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ a_{31} & a_{32} & a_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix}$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & -5 & 7 \end{pmatrix}$ is a lower triangular matrix of order 3.

6. Diagonal matrix : The square matrix $\mathbf{A} = (a_{ij})$ of order n is called a **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$, and it is written as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ 0 & 0 & a_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{pmatrix}$$

Example : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a diagonal matrix of order 3.

7. Identity matrix : The square matrix $I_n = (a_{ij})$ of order n is called an **identity matrix** if $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$, and it is

written as $I_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$

Example : $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix of order 3.

1.2 Elementary matrix operations :
1. Addition and subtraction of matrices :
Addition or subtraction of two matrices is defined if the two matricest have the same order.
If $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{m \times n}$ any two matrices of order $m \times n$ then

1. $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m \times n}$.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \ldots & a_{mn} + b_{mn} \end{pmatrix}$$

2. $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{m \times n}$.

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \ldots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \ldots & a_{2n} - b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \ldots & a_{mn} - b_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 1 & -4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 2 & 1 \\ -3 & 7 & -2 \end{pmatrix}$ then

$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+5 & -3+2 & 0+1 \\ 1+(-3) & -4+7 & 6+(-2) \end{pmatrix} = \begin{pmatrix} 7 & -1 & 1 \\ -2 & 3 & 4 \end{pmatrix}$

$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2-5 & -3-2 & 0-1 \\ 1-(-3) & -4-7 & 6-(-2) \end{pmatrix} = \begin{pmatrix} -3 & -5 & -1 \\ 4 & -11 & 8 \end{pmatrix}$

Notes:

1. The addition of matrices is commutative : if $\mathbf{A}$ and $\mathbf{B}$ any two matrices of the same order then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

2. The null matrix is the identity element of addition : if $\mathbf{A}$ is any matrix then $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

2. Multiplying a matrix by a scalar :
If $\mathbf{A} = (a_{ij})$ is a matrix of order $m \times n$ and $c \in \mathbb{R}$ then $c\mathbf{A} = (ca_{ij})$.

$$c\mathbf{A} = \begin{pmatrix} ca_{11} & ca_{12} & \ldots & ca_{1n} \\ ca_{21} & ca_{22} & \ldots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \ldots & ca_{mn} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -2 & 0 \end{pmatrix}$ then $3\mathbf{A} = \begin{pmatrix} 9 & -3 & 12 \\ 6 & -6 & 0 \end{pmatrix}$

3. Multiplying a row vector by a column vector :

If $\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}$ is a row vector of order n and

$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a column vector of order n then

$$\mathbf{AB} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Example : If $\mathbf{A} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} -1 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 1 \\ -1 \end{pmatrix} = -4 - 4 + 0 - 5 = -13$$

4. Multiplication of matrices :

1. If $\mathbf{A}$ and $\mathbf{B}$ any two matrices then $\mathbf{AB}$ is defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

2. If $\mathbf{A} = (a_{ij})_{m \times n}$ and $\mathbf{B} = (b_{ij})_{n \times p}$ then $\mathbf{AB} = (c_{ij})_{m \times p}$.

   c_{ij} is calculated by multiplying the ith row of $\mathbf{A}$ by the jth column of $\mathbf{B}$.

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

Example 1 :

1. $\begin{pmatrix} -1 & 3 & 4 \\ -2 & 0 & 5 \end{pmatrix}_{2 \times 3} \begin{pmatrix} 1 & 3 \\ -1 & -2 \\ 4 & 0 \end{pmatrix}_{3 \times 2}$

$$= \begin{pmatrix} (-1 \times 1) + (3 \times -1) + (4 \times 4) & (-1 \times 3) + (3 \times -2) + (4 \times 0) \\ (-2 \times 1) + (0 \times -1) + (5 \times 4) & (-2 \times 3) + (0 \times -2) + (5 \times 0) \end{pmatrix}_{2 \times 2}$$

$$= \begin{pmatrix} -1 - 3 + 16 & -3 - 6 + 0 \\ -2 + 0 + 20 & -6 + 0 + 0 \end{pmatrix}_{2 \times 2} = \begin{pmatrix} 12 & -9 \\ 18 & -6 \end{pmatrix}_{2 \times 2}$$

2. $\begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}_{2 \times 2} \begin{pmatrix} 0 & -3 & 4 \\ -2 & 0 & 1 \end{pmatrix}_{2 \times 3}$

$$= \begin{pmatrix} (3 \times 0) + (-1 \times -2) & (3 \times -3) + (-1 \times 0) & (3 \times 4) + (-1 \times 1) \\ (-2 \times 0) + (5 \times -2) & (-2 \times -3) + (5 \times 0) & (-2 \times 4) + (5 \times 1) \end{pmatrix}_{2 \times 3}$$

$$\begin{pmatrix} 0 + 2 & -9 + 0 & 12 - 1 \\ 0 - 10 & 6 + 0 & -8 + 5 \end{pmatrix}_{2 \times 3} = \begin{pmatrix} 2 & -9 & 11 \\ -10 & 6 & -3 \end{pmatrix}_{2 \times 3}$$

Example 2: Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}$

Compute (if possible) : $2\mathbf{BA}$ and $\mathbf{AB}$

Solution : $\mathbf{A}$ is of order 3×3 and $\mathbf{B}$ is of order 3×2

$2\mathbf{BA}$ is not possible because the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$.

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{pmatrix}_{3 \times 3} \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{pmatrix}_{3 \times 2} = \begin{pmatrix} (1 - 4 + 0) & (-1 - 6 + 12) \\ (4 + 10 + 0) & (-4 + 15 + 24) \\ (2 + 0 + 0) & (-2 + 0 + 4) \end{pmatrix}_{3 \times 2}$$

$$\mathbf{AB} = \begin{pmatrix} -3 & 5 \\ 14 & 35 \\ 2 & 2 \end{pmatrix}_{3 \times 2}$$

Notes :

1. The identity matrix is the identity element in matrix multiplication :

   If A is a matrix of order $m \times n$ and $\mathbf{I}_n$ is the identity matrix of order n then $\mathbf{A}\,\mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$.

2. Matrix multiplication is not commutative :

   If $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

   $$\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 8 & 5 \end{pmatrix}$$

   $$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

   $\mathbf{AB} \neq \mathbf{BA}$.

3. $\mathbf{AB} = \mathbf{0}$ does not imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

   For example, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \mathbf{0}$

   But $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

1.3 Transpose of a matrix :

If $\mathbf{A} = (a_{ij})_{m \times n}$ then the transpose of $\mathbf{A}$ is $\mathbf{A}^t = (a_{ji})_{n \times m}$.

Example : If $\mathbf{A} = \begin{pmatrix} 4 & 0 & -2 \\ -3 & 5 & 1 \end{pmatrix}$ then $\mathbf{A}^t = \begin{pmatrix} 4 & -3 \\ 0 & 5 \\ -2 & 1 \end{pmatrix}$

Note : The transpose of a lower triangular matrix is an upper triangular matrix , and the transpose of an upper triangular matrix is a lower triangular matrix .

Theorem :

If $\mathbf{A}$ and $\mathbf{B}$ any two matrices and $\lambda \in \mathbb{R}$ then

1. $\left(\mathbf{A}^t\right)^t = \mathbf{A}$.

2. $(\mathbf{A} + \mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.

3. $(\lambda\mathbf{A})^t = \lambda\,\mathbf{A}^t$.

4. $(\mathbf{A}\mathbf{B})^t = \mathbf{B}^t\mathbf{A}^t$.

1.4 Properties of operations on matrices :

1. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ any three matrices of the same order then

   $$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{C}) + \mathbf{B}$$

2. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m \times n$ and $\mathbf{C}$ a matrix of order $n \times p$ then $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$

3. If $\mathbf{A}$, $\mathbf{B}$ any two matrices of order $m \times n$ and $\mathbf{C}$ a matrix of order $p \times m$ then $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B}$

4. If $\mathbf{A}$ a matrix of order $m \times n$, $\mathbf{B}$ a matrix of order $n \times p$ and $\mathbf{C}$ a matrix of order $p \times q$ then $\mathbf{A}\mathbf{B}\mathbf{C} = (\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C})$

2. Determinants

If $\mathbf{A}$ is a square matrix then the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$.

2.1 The determinant of a 2×2 matrix :
If $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example :
If $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$ then $\det(\mathbf{A}) = (5 \times 3) - (2 \times -1) = 15 + 2 = 17$

2.2 The determinant of a 3×3 matrix :
Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix of order 3.

1). The determinant of $\mathbf{A}$ is defined as :
$$\det(\mathbf{A}) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det(\mathbf{A}) = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

2). Sarrus Method for calculating the deteminant of a 3×3 matrix :

Write the first two columns to the right of the matrix to get a 3×5 matrix
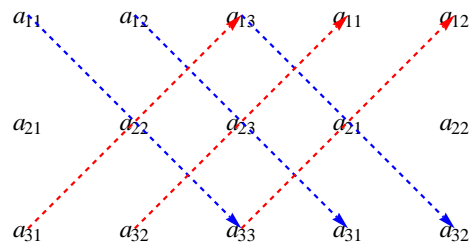
$$\det(\mathbf{A}) = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Example : If $\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 4 \\ -5 & 0 & 1 \end{pmatrix}$

1) Using the definition of the determinant of a 3×3 matrix
$$\det(\mathbf{A}) = 2 \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 4 \\ -5 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 \\ -5 & 0 \end{vmatrix}$$

$$\det(\mathbf{A}) = 2\left(2 \times 1 - 4 \times 0\right) - 3\left(1 \times 1 - 4 \times -5\right) - 1\left(1 \times 0 - 2 \times -5\right)$$

$$\det(\mathbf{A}) = 2(2 - 0) - 3(1 + 20) - 1(0 + 10) = 4 - 63 - 10 = -69$$

2) Using Sarrus Method

$$\begin{array}{ccccc}
2 & 3 & -1 & 2 & 3 \\
1 & 2 & 4 & 1 & 2 \\
-5 & 0 & 1 & -5 & 0
\end{array}$$

$\det(\mathbf{A}) = (2 \times 2 \times 1 + 3 \times 4 \times -5 + (-1) \times 1 \times 0) - (-5 \times 2 \times -1 + 0 \times 4 \times 2 + 1 \times 1 \times 3)$

$\det(\mathbf{A}) = (4 - 60 + 0) - (10 + 0 + 3) = -56 - 13 = -69$.

2.3 The determinant of a 4×4 matrix :

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ be a 4×4 matrix , then

$\det(\mathbf{A}) = a_{11} \det(\mathbf{A}_1) - a_{12} \det(\mathbf{A}_2) + a_{13} \det(\mathbf{A}_3) - a_{14} \det(\mathbf{A}_4)$

where

$$\mathbf{A}_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} \quad , \quad \mathbf{A}_2 = \begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} \quad , \quad \mathbf{A}_4 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

Example : If $\mathbf{A} = \begin{pmatrix} 3 & 1 & -2 & 1 \\ 0 & 4 & -1 & 5 \\ 2 & 1 & -3 & 0 \\ 1 & -2 & -1 & 3 \end{pmatrix}$

$\det(\mathbf{A}) = (3) \det(\mathbf{A}_1) - (1) \det(\mathbf{A}_2) + (-2) \det(\mathbf{A}_3) - (1) \det(\mathbf{A}_4)$

where

$$\mathbf{A}_1 = \begin{pmatrix} 4 & -1 & 5 \\ 1 & -3 & 0 \\ -2 & -1 & 3 \end{pmatrix} \quad , \quad \mathbf{A}_2 = \begin{pmatrix} 0 & -1 & 5 \\ 2 & -3 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 4 & 5 \\ 2 & 1 & 0 \\ 1 & -2 & 3 \end{pmatrix} \quad , \quad \mathbf{A}_4 = \begin{pmatrix} 0 & 4 & -1 \\ 2 & 1 & -3 \\ 1 & -2 & -1 \end{pmatrix}$$

- To calculate $\det(\mathbf{A}_1)$

$$\begin{array}{ccccc}
4 & -1 & 5 & 4 & -1 \\
1 & -3 & 0 & 1 & -3 \\
-2 & -1 & 3 & -2 & -1
\end{array}$$

$\det(\mathbf{A}_1) = (-36 + 0 - 5) - (30 + 0 - 3) = -36 - 5 - 30 + 3 = -68$

- To calculate $\det(\mathbf{A}_2)$

$$\begin{array}{ccccc}
0 & -1 & 5 & 0 & -1 \\
2 & -3 & 0 & 2 & -3 \\
1 & -1 & 3 & 1 & -1
\end{array}$$

$\det(\mathbf{A}_2) = (0 + 0 - 10) - (-15 + 0 - 6) = -10 + 21 = 11$

- To calculate $\det(\mathbf{A}_3)$

$$\begin{array}{ccccc} 0 & 4 & 5 & 0 & 4 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & -2 & 3 & 1 & -2 \end{array}$$

$\det(\mathbf{A}_3) = (0 + 0 - 20) - (5 + 0 + 24) = -20 - 29 = -49$

- To calculate $\det(\mathbf{A}_4)$

$$\begin{array}{ccccc} 0 & 4 & -1 & 0 & 4 \\ 2 & 1 & -3 & 2 & 1 \\ 1 & -2 & -1 & 1 & -2 \end{array}$$

$\det(\mathbf{A}_4) = (0 - 12 + 4) - (-1 + 0 - 8) = -8 + 9 = 1$

$\det(\mathbf{A}) = (3)\det(\mathbf{A}_1) - (1)\det(\mathbf{A}_2) + (-2)\det(\mathbf{A}_3) - (1)\det(\mathbf{A}_4)$
$\det(\mathbf{A}) = (3 \times -68) - (1 \times 11) + (-2 \times -49) - (1 \times 1)$
$\det(\mathbf{A}) = -204 - 11 + 98 - 1 = -216 + 98 = -118$.

2.4 Properties of determinants :

1. If $\mathbf{A}$ is a square matrix that contains a zero row (or a zero column) then $\det(\mathbf{A}) = 0$.

2. If $\mathbf{A}$ is a square matrix that contains two equal rows (or two equal columns) then $\det(\mathbf{A}) = 0$.

3. If $\mathbf{A}$ is a square matrix that contains a row which is a multiple of another row (or a column which is a multiple of another column) then $\det(\mathbf{A}) = 0$.

4. If $\mathbf{A}$ is a diagonal matrix or an upper triangular matrix or a lower triangular matrix the $\det(\mathbf{A})$ is the the product of the elements of the main diagonal.

5. The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.

6. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by multiplying one of the rows (or columns) of $\mathbf{A}$ by a non-zero constant λ then $\det(\mathbf{B}) = \lambda \det(\mathbf{A})$.

7. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by interchanging two rows (or two columns) of $\mathbf{A}$ then $\det(\mathbf{B}) = -\det(\mathbf{A})$.

8. If $\mathbf{A}$ is a square matrix and $\mathbf{B}$ is the matrix formed by mutliplying a row by a non-zero constant and adding the result to another row (or mutliplying a column by a non-zero constant and adding the result to another column) then $\det(\mathbf{B}) = \det(\mathbf{A})$.

Examples : Use properties of determinants to calculate the derminants of the following matrices

1. $\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{pmatrix}$

   Since the second row (R_2) is a zero row then $\det(\mathbf{A}) = 0$

2. $\mathbf{A} = \begin{pmatrix} 4 & -5 & 4 \\ 0 & 2 & 0 \\ -3 & 1 & -3 \end{pmatrix}$

   Since the first column equals to the third column $(C_1 = C_3)$ then

   $\det(\mathbf{A}) = 0$.

3. $\mathbf{A} = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 5 & 1 \\ 6 & 3 & -9 \end{pmatrix}$

   Since the third row is a multiple of the first row $(R_3 = 3R_1)$ then

   $\det(\mathbf{A}) = 0$

4. $\mathbf{A} = \begin{pmatrix} 3 & 0 & 4 \\ 6 & -1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$

   Interchange the the first column and the second column $(C_1 \longleftrightarrow C_2)$, then interchange the first row and the second row $(R_1 \longleftrightarrow R_2)$

   $$\det(\mathbf{A}) = \begin{vmatrix} 3 & 0 & 4 \\ 6 & -1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 3 & 4 \\ -1 & 6 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

   $$= (-1)^2 \begin{vmatrix} -1 & 6 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix} = -1 \times 3 \times 5 = -15$$

5. $\mathbf{A} = \begin{pmatrix} 5 & 2 & 3 \\ 15 & 8 & 1 \\ 10 & 6 & 2 \end{pmatrix}$

   $$\det(\mathbf{A}) = \begin{vmatrix} 5 & 2 & 3 \\ 15 & 8 & 1 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-3R_1 + R_2} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 10 & 6 & 2 \end{vmatrix} \xrightarrow{-2R_1 + R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 2 & -4 \end{vmatrix}$$

   $$\xrightarrow{-R_2 + R_3} \begin{vmatrix} 5 & 2 & 3 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{vmatrix} = 5 \times 2 \times 4 = 40$$

6. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & -4 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 1 & 0 \end{vmatrix}$

$\xrightarrow{-R_1+R_3} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} = 0 \ (\ R_2 = -8R_3 \ )$

7. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 \end{vmatrix}$

$\xrightarrow{-R_3+R_4} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 \end{vmatrix} = 0 \ (\ R_2 = R_4 \ )$

8. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{pmatrix}$

Multiply C_2 by 3 and multiply C_3 by 2

$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 3 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \begin{vmatrix} 1 & 6 & 6 & 4 \\ 0 & 6 & 6 & 0 \\ 1 & 6 & 6 & 5 \\ 3 & 0 & 0 & 0 \end{vmatrix} = 0 \ (\text{since } C_2 = C_3)$

CHAPTER THREE

SYSTEMS OF LINEAR EQUATIONS

1. Cramer's Rule

2. Gauss Elimination Method

3. Gauss-Jordan Method

Systems of Linear Equations

Consider the system of linear equations in n different variables

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{array} \qquad (*)
$$

Using multiplication of matrices , the above system of linear equations can be written as : $\mathbf{A}\,\mathbf{X} = \mathbf{B}$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$\mathbf{A}$ is called the coefficients matrix
$\mathbf{X}$ is called the column vector of variables (or column vector of the unknowns)
$\mathbf{B}$ is called the column vector of constants (or column vector of the resultants)

Theorem : The system of linear equations (*) has a solution if $\det(\mathbf{A}) \neq 0$.

This chapter presents three metods of solving the system of linear equations (*), the first method is Cramer's rule , the second is Gauss elimination method , and the third is Gauss-Jordan method .

1. Cramer's rule

Consider the system of linear equations in n different variables

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{array}
\qquad (*)
$$

$$\mathbf{A}\,\mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

If $\det(\mathbf{A}) \neq 0$ then the solution of the system (*) is given by

$x_i = \dfrac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$ for every $i = 1, 2, \cdots, n$.

Where $\mathbf{A}_i$ is the matrix formed by replacing the ith column of $\mathbf{A}$ by the column vector of constants.

$$
\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \ldots & a_{nn} \end{pmatrix} , \quad \mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & \ldots & a_{1n} \\ a_{21} & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \ldots & a_{nn} \end{pmatrix}
$$

$$
\mathbf{A}_n = \begin{pmatrix} a_{11} & a_{12} & \ldots & b_1 \\ a_{21} & a_{22} & \ldots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & b_n \end{pmatrix}
$$

Example 1: Use Cramer's rule to solve the system of linear equations

$$
\begin{array}{ccccc}
2x & + & 3y & = & 7 \\
-x & + & y & = & 4
\end{array}
$$

Solution : In this system of linear equations

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = (2 \times 1) - (-1 \times 3) = 2 - (-3) = 2 + 3 = 5$

$\mathbf{A}_1 = \begin{pmatrix} 7 & 3 \\ 4 & 1 \end{pmatrix} \implies \det(\mathbf{A}_1) = 7 - 12 = -5$

$\mathbf{A}_2 = \begin{pmatrix} 2 & 7 \\ -1 & 4 \end{pmatrix} \implies \det(\mathbf{A}_2) = 8 - (-7) = 15$

$x = \dfrac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \dfrac{-5}{5} = -1$ and $y = \dfrac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \dfrac{15}{5} = 3$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Example 2: Use Cramer's rule to solve the system of linear equations
$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 3 \\ 4x & + & y & - & z & = & -2 \\ 2x & - & 2y & + & z & = & 6 \end{array}$$

Solution : In this system of linear equations

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

To calculate $\det(\mathbf{A})$:

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 4 & 1 & -1 & 4 & 1 \\ 2 & -2 & 1 & 2 & -2 \end{array}$$

$\det(\mathbf{A}) = (2 - 2 - 8) - (2 + 4 + 4) = -8 - 10 = -18$

$$\mathbf{A}_1 = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 1 & -1 \\ 6 & -2 & 1 \end{pmatrix}$$

To calculate $\det(\mathbf{A}_1)$:

$$\begin{array}{ccccc} 3 & 1 & 1 & 3 & 1 \\ -2 & 1 & -1 & -2 & 1 \\ 6 & -2 & 1 & 6 & -2 \end{array}$$

$\det(\mathbf{A}_1) = (3 - 6 + 4) - (6 + 6 - 2) = 1 - 10 = -9$

$$\mathbf{A}_2 = \begin{pmatrix} 2 & 3 & 1 \\ 4 & -2 & -1 \\ 2 & 6 & 1 \end{pmatrix}$$

To calculate $\det(\mathbf{A}_2)$:

$$\begin{array}{ccccc} 2 & 3 & 1 & 2 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 2 & 6 & 1 & 2 & 6 \end{array}$$

$\det(\mathbf{A}_2) = (-4 - 6 + 24) - (-4 - 12 + 12) = 14 + 4 = 18$

$$\mathbf{A}_3 = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \\ 2 & -2 & 6 \end{pmatrix}$$

To calculate $\det(\mathbf{A}_3)$:

$$\begin{array}{ccccc} 2 & 1 & 3 & 2 & 1 \\ 4 & 1 & -2 & 4 & 1 \\ 2 & -2 & 6 & 2 & -2 \end{array}$$

$\det(\mathbf{A}_3) = (12 - 4 - 24) - (6 + 8 + 24) = 1 - 10 = -16 - 38 = -54$

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-9}{-18} = \frac{1}{2}$$

$$y = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{18}{-18} = -1$$

$$z = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})} = \frac{-54}{-18} = 3$$

The solution of the system of linear equations is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 3 \end{pmatrix}$

2. Gauss elimination method

Consider the system of linear equations in n different variables

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{array}
\quad (*)
$$

$$\mathbf{A}\,\mathbf{X} = \mathbf{B}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss elimination method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$
\left(
\begin{array}{cccc|c}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn} & b_n
\end{array}
\right)
$$

2. Use **elementary row operations** on the augmented matrix to transform the matrix $\mathbf{A}$ to an upper triangular matrix with leading coeficient of each row equals 1.

   (Note: the leading coefficient of a row is the leftmost non-zero element of that row).

$$
\left(
\begin{array}{cccccc|c}
1 & c_{12} & c_{13} & c_{14} & \ldots & c_{1n} & d_1 \\
0 & 1 & c_{23} & c_{24} & \ldots & a_{2n} & d_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & c_{(n-1)n} & d_{n-1} \\
0 & 0 & 0 & \ldots & 0 & 1 & d_n
\end{array}
\right)
$$

3. From the last augmented matrix , $x_n = d_n$ and the rest of the unknowns can be calculated by backward substitution.

Example 1: Use Gauss elimination method to solve the system

$$
\begin{array}{ccccccc}
x & - & 2y & + & z & = & 4 \\
-x & + & 2y & + & z & = & -2 \\
4x & - & 3y & - & z & = & -4
\end{array}
$$

Solution : The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 4 & -3 & -1 & -4 \end{array} \right) \xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 4 & -3 & -1 & -4 \end{array} \right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 4 & -3 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array} \right) \xrightarrow{-4R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -5 & -20 \\ 0 & 0 & 2 & 2 \end{array} \right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 2 & 2 \end{array} \right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

Therefore, $z = 1$.

$y - z = -4 \Rightarrow y - 1 = -4 \Rightarrow y = -4 + 1 = -3$

$x - 2y + z = 4 \Rightarrow x - 2(-3) + 1 = 4 \Rightarrow x + 6 + 1 = 4 \Rightarrow x = 4 - 7 = -3$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

Example 2: Use Gauss elimination method to solve the system

$$\begin{array}{rcrcrcrcr} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : The augmented matrix is

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & 3 & 8 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 1 & 1 & 1 & -1 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 3 & 2 & -1 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array} \right)$$

$$\xrightarrow{-R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 3 & 1 & -1 & -2 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array} \right) \xrightarrow{-3R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 2 & -1 & 1 & 3 & 8 \end{array} \right)$$

$$\xrightarrow{-2R_1+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & -2 & -4 & 1 & 3 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -3 & -1 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefor, $w = 1$

$z - \dfrac{1}{3}w = -\dfrac{1}{3} \ \Rightarrow \ z - \dfrac{1}{3} = -\dfrac{1}{3} \ \Rightarrow \ z = 0$

$y + \dfrac{1}{2}z = -1 \ \Rightarrow \ y + \dfrac{1}{2}(0) = -1 \ \Rightarrow \ y = -1$

$x + y + z - w = 0 \ \Rightarrow \ x - 1 + 0 - 1 = 0 \ \Rightarrow \ x = 2$

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

3. Gauss-Jordan method

Consider the system of linear equations in n different variables

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{array}
\qquad (*)
$$

$$
\mathbf{A}\,\mathbf{X} = \mathbf{B}
$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

To solve the system of linear equations (*) by Gauss-Jordan method :

1. Construct the augmented matrix $[\mathbf{A}|\mathbf{B}]$

$$
\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn} & b_n
\end{array} \right)
$$

2. Use elementary row operations on the augmented matrix to transform the matrix $\mathbf{A}$ to the identity matrix .

$$
\left(\begin{array}{ccccc|c}
1 & 0 & \ldots & 0 & 0 & d_1 \\
0 & 1 & \ldots & 0 & 0 & d_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & d_{n-1} \\
0 & 0 & \ldots & 0 & 1 & d_n
\end{array} \right)
$$

3. From the last augmented matrix , $x_i = d_i$ for every $i = 1, 2, \cdots, n$

Example 1: Use Gauss-Jordan method to solve the system

$$
\begin{array}{ccccccc}
x & + & y & + & z & = & 2 \\
x & - & y & + & 2z & = & 0 \\
2x & & & + & z & = & 2
\end{array}
$$

Solution : The augmented matrix is

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 2 \\
1 & -1 & 2 & 0 \\
2 & 0 & 1 & 2
\end{array} \right)
$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 1 & -1 & 2 & | & 0 \\ 2 & 0 & 1 & | & 2 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -2 & 1 & | & -2 \\ 2 & 0 & 1 & | & 2 \end{pmatrix}$$

$$\xrightarrow{-2R_1+R_3} \begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -2 & 1 & | & -2 \\ 0 & -2 & -1 & | & -2 \end{pmatrix} \xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -2 & 1 & | & -2 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -2 & 1 & | & -2 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{-R_3+R_2} \begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -2 & 0 & | & -2 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-R_3+R_1} \begin{pmatrix} 1 & 1 & 0 & | & 2 \\ 0 & -2 & 0 & | & -2 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-R_2+R_1} \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

Therefore , $x = 1$, $y = 1$ and $z = 0$.

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Example 2: Use Gauss-Jordan method to solve the system

$$\begin{array}{rcrcrcrcrcr} 2x & - & y & + & z & + & 3w & = & 8 \\ x & + & 3y & + & 2z & - & w & = & -2 \\ 3x & + & y & - & z & - & 2w & = & 3 \\ x & + & y & + & z & - & w & = & 0 \end{array}$$

Solution : (Note : This is example 2 in Gauss elimination method)
The augmented matrix is

$$\begin{pmatrix} 2 & -1 & 1 & 3 & | & 8 \\ 1 & 3 & 2 & -1 & | & -2 \\ 3 & 1 & -1 & -2 & | & 3 \\ 1 & 1 & 1 & -1 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 1 & 3 & | & 8 \\ 1 & 3 & 2 & -1 & | & -2 \\ 3 & 1 & -1 & -2 & | & 3 \\ 1 & 1 & 1 & -1 & | & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 1 & -1 & | & 0 \\ 1 & 3 & 2 & -1 & | & -2 \\ 3 & 1 & -1 & -2 & | & 3 \\ 2 & -1 & 1 & 3 & | & 8 \end{pmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{pmatrix} 1 & 1 & 1 & -1 & | & 0 \\ 0 & 2 & 1 & 0 & | & -2 \\ 3 & 1 & -1 & -2 & | & 3 \\ 2 & -1 & 1 & 3 & | & 8 \end{pmatrix} \xrightarrow{-3R_1+R_3} \begin{pmatrix} 1 & 1 & 1 & -1 & | & 0 \\ 0 & 2 & 1 & 0 & | & -2 \\ 0 & -2 & -4 & 1 & | & 3 \\ 2 & -1 & 1 & 3 & | & 8 \end{pmatrix}$$

$$\xrightarrow{-2R_1+R_4} \begin{pmatrix} 1 & 1 & 1 & -1 & | & 0 \\ 0 & 2 & 1 & 0 & | & -2 \\ 0 & -2 & -4 & 1 & | & 3 \\ 0 & -3 & -1 & 5 & | & 8 \end{pmatrix} \xrightarrow{R_2+R_3} \begin{pmatrix} 1 & 1 & 1 & -1 & | & 0 \\ 0 & 2 & 1 & 0 & | & -2 \\ 0 & 0 & -3 & 1 & | & 1 \\ 0 & -3 & -1 & 5 & | & 8 \end{pmatrix}$$

$$\xrightarrow{2R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & -6 & -2 & 10 & 16 \end{array}\right) \xrightarrow{3R_2+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 10 & 10 \end{array}\right)$$

$$\xrightarrow{3R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 3 & 30 & 30 \end{array}\right) \xrightarrow{R_3+R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 31 & 31 \end{array}\right)$$

$$\xrightarrow{\frac{1}{31}R_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_4+R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_4+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3+R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_3+R_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, $x = 2$, $y = -1$, $z = 0$ and $w = 1$.

The solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

CHAPTER FOUR

INTEGRATION

1. Indefinite integral

2. Integration by substitution

3. Integration by parts

4. Integration of rational functions
(Method of partial fractions)

1. Indefinite integral

Definition (Antiderivative): A function G is called an antiderivative of the function f on the interval $[a, b]$ if $G'(x) = f(x)$ for all $x \in [a, b]$.

Examples : What is the antiderivative of the following functions

1. $f(x) = 2x$.

2. $f(x) = \cos x$.

3. $f(x) = \sec^2 x$

4. $f(x) = \dfrac{1}{x}$

5. $f(x) = e^x$

Solution :

1. $G(x) = x^2 + c$

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx}\left(x^2 + c\right) = 2x + 0 = 2x$$

2. $G(x) = \sin x + c$

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx}\left(\sin x + c\right) = \cos x$$

3. $G(x) = \tan x + c$

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx}\left(\tan x + c\right) = \sec^2 x$$

4. $G(x) = \ln|x| + c$

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx}\left(\ln|x| + c\right) = \frac{1}{x}$$

5. $G(x) = e^x + c$

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx}\left(e^x + c\right) = e^x$$

Note: If $G_1(x)$ and $G_2(x)$ are both antiderivatives of the function $f(x)$ then $G_1(x) - G_2(x) = \text{constant}$.

Definition (indefinite integral): If $G(x)$ is the antiderivative of $f(x)$ then $\int f(x)\, dx = G(x) + c$, $\int f(x)\, dx$ is called the indefinite integral of the function $f(x)$.

Basic Rules of integration :

1. $\displaystyle\int 1\ dx = x + c$

2. $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$

3. $\displaystyle\int \cos x\ dx = \sin x + c$

4. $\displaystyle\int \sin x\ dx = -\cos x + c$

5. $\displaystyle\int \sec^2 x\ dx = \tan x + c$

6. $\displaystyle\int \csc^2 x\ dx = -\cot x + c$

7. $\displaystyle\int \sec x\ \tan x\ dx = \sec x + c$

8. $\displaystyle\int \csc x\ \cot x = -\csc x + c$

9. $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + c$

10. $\displaystyle\int e^x\ dx = e^x + c$

11. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + c$, where $|x| < 1$

12. $\displaystyle\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + c$

13. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + c$, where $|x| > 1$

Properties of indefinite integral :

1. $\displaystyle\int k\ f(x)\ dx = k\ \int f(x)\ dx$, where $k \in \mathbb{R}$

2. $\displaystyle\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

Examples : Evaluate the following integrals

1. $\int \left(4x^2 - \dfrac{5}{x^3} \right) \ dx$

   Solution : $\int \left(4x^2 - \dfrac{5}{x^3} \right) \ dx = \int 4x^2 \ dx - \int \dfrac{5}{x^3} \ dx$

   $= 4 \int x^2 \ dx - 5 \int x^{-3} dx = 4 \ \dfrac{x^3}{3} - 5 \ \dfrac{x^{-2}}{-2} + c = \dfrac{4}{3}x^3 + \dfrac{5}{2x^2} + c$

2. $\int \left(3x^{\frac{1}{3}} + \dfrac{1}{\sqrt{x}} \right) \ dx$

   Solution : $\int \left(3x^{\frac{1}{3}} + \dfrac{1}{\sqrt{x}} \right) \ dx = 3 \int x^{\frac{1}{3}} \ dx + \int x^{-\frac{1}{2}} \ dx$

   $= 3 \left(\dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} \right) + \left(\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} \right) + c = \dfrac{9}{4}x^{\frac{4}{3}} + 2x^{\frac{1}{2}} + c$

3. $\int (2 \cos x - 3 \sec^2 x) \ dx$

   Solution : $\int (2 \cos x - 3 \sec^2 x) \ dx = 2 \int \cos x \ dx - 3 \int \sec^2 x \ dx$

   $= 2 \sin x - 3 \tan x + c$

4. $\int (7 \sec x \ \tan x + 5 \csc^2 x) \ dx$

   Solution : $\int (7 \sec x \ \tan x + 5 \csc^2 x) \ dx = 7 \int \sec x \tan x \ dx + 5 \int \csc^2 x \ dx$

   $= 7 \sec x + 5(- \cot x) + c = 7 \sec x - 5 \cot x + c$

5. $\int \left(\dfrac{2}{x} - \dfrac{3}{x^2} \right) \ dx$

   Solution : $\int \left(\dfrac{2}{x} - \dfrac{3}{x^2} \right) \ dx = 2 \int \dfrac{1}{x} \ dx - 3 \int x^{-2} \ dx$

   $= 2 \ln |x| - 3 \left(\dfrac{x^{-1}}{-1} \right) + c = 2 \ln |x| + \dfrac{3}{x} + c$

6. $\int \left(9e^x - \dfrac{3}{1+x^2}\right) \, dx$

Solution : $\int \left(9e^x - \dfrac{3}{1+x^2}\right) \, dx = 9\int e^x \, dx - 3\int \dfrac{1}{1+x^2} \, dx$

$= 9e^x - 3\tan^{-1} x + c$

7. $\int \left(\dfrac{4}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt[3]{x}}\right) \, dx$

Solution : $\int \left(\dfrac{4}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt[3]{x}}\right) \, dx = 4\int \dfrac{1}{\sqrt{1-x^2}} \, dx + \int x^{-\frac{1}{3}} \, dx$

$= 4\sin^{-1} x + \left(\dfrac{x^{\frac{2}{3}}}{\frac{2}{3}}\right) + c = 4\sin^{-1} x + \dfrac{3}{2}x^{\frac{2}{3}} + c$

The definite integral :
If f is a continuous function on the interval $[a, b]$ and G is the antiderivative of f on $[a, b]$ then the definite integral of f on $[a, b]$ is

$$\int_a^b f(x) \, dx = [G(x)]_a^b = G(b) - G(a)$$

Examples : Evaluate the following integrals :

1. $\displaystyle\int_1^3 (3x^2 + 5)dx$

Solution : $\displaystyle\int_1^3 (3x^2 + 5)dx = \left[x^3 + 5x\right]_1^3$

$= \left(3^3 + 5 \times 3\right) - \left(1^3 + 5 \times 1\right) = (27 + 15) - (1 + 5) = 36$

2. $\displaystyle\int_0^1 (2x + e^x) \, dx$

Solution : $\displaystyle\int_0^1 (2x + e^x) \, dx = \left[x^2 + e^x\right]_0^1$

$= \left(1^2 + e^1\right) - \left(0^2 + e^0\right) = 1 + e - 1 = e$

2. Integration by substitution

The main idea of integration by substitution is to use a suitable substitution to transform the given integral to an easier integral that can be solved by one of the basic rules of integration.

Example : Evaluate the integral $\int x(x^2 + 3)^6 \ dx$

Solution : Use the substitution $u = x^2 + 3$

Then $du = 2x \ dx \ \Rightarrow \ \dfrac{1}{2} \ du = x \ dx$

$$\int x(x^2 + 3)^6 \ dx = \int u^6 \dfrac{1}{2} \ du = \dfrac{1}{2} \int u^6 \ du$$

$$= \dfrac{1}{2} \dfrac{u^7}{7} + c = \dfrac{(x^2 + 3)^7}{14} + c$$

By the chain rule $\dfrac{d}{dx} \ [f(x)]^{n+1} = (n+1) \ [f(x)]^n \ f'(x)$, where $n \neq -1$

Hence $\int [f(x)]^n \ f'(x) \ dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

So, the above integral can be solved as follows

$$\int x(x^2 + 3)^6 \ dx = \dfrac{1}{2} \int (x^2 + 3)^6 \ (2x) \ dx = \dfrac{1}{2} \ \dfrac{(x^2 + 3)^7}{7} + c$$

Basic rules of integrations and their general forms :

1. $\displaystyle\int x^n \ dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \neq -1$

   $\displaystyle\int [f(x)]^n \ f'(x) \ dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$

2. $\displaystyle\int \dfrac{1}{x} \ dx = \ln |x| + c$

   $\displaystyle\int \dfrac{f'(x)}{f(x)} \ dx = \ln |f(x)| + c$

3. $\displaystyle\int e^x \ dx = e^x + c$

   $\displaystyle\int e^{f(x)} f'(x) \ dx = e^{f(x)} + c$

4. $\displaystyle\int \cos x \ dx = \sin x + c$

   $\displaystyle\int \cos (f(x)) f'(x) \ dx = \sin (f(x)) + c$

5. $\int \sin x \; dx = -\cos x + c$

$\int \sin \left(f(x) \right) f'(x) \; dx = -\cos \left(f(x) \right) + c$

6. $\int \sec^2 x \; dx = \tan x + c$

$\int \sec^2 \left(f(x) \right) f'(x) \; dx = \tan \left(f(x) \right) + c$

7. $\int \csc^2 x \; dx = -\cot x + c$

$\int \csc^2 \left(f(x) \right) f'(x) \; dx = -\cot \left(f(x) \right) + c$

8. $\int \sec x \tan x \; dx = \sec x + c$

$\int \sec \left(f(x) \right) \tan \left(f(x) \right) f'(x) \; dx = \sec \left(f(x) \right) + c$

9. $\int \csc x \cot x \; dx = -\csc x + c$

$\int \csc \left(f(x) \right) \cot \left(f(x) \right) f'(x) \; dx = -\csc \left(f(x) \right) + c$

10. $\int \dfrac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1}\left(\dfrac{x}{a} \right) + c$, where $a > 0$ and $|x| < a$

$\int \dfrac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} \; dx = \sin^{-1}\left(\dfrac{f(x)}{a} \right) + c$, where $a > 0$ and $|f(x)| < a$

11. $\int \dfrac{1}{a^2 + x^2} \; dx = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a} \right) + c$, where $a > 0$

$\int \dfrac{f'(x)}{a^2 + [f(x)]^2} \; dx = \dfrac{1}{a} \tan^{-1}\left(\dfrac{f(x)}{a} \right) + c$, where $a > 0$

12. $\int \dfrac{1}{x\sqrt{x^2 - a^2}} \; dx = \dfrac{1}{a} \sec^{-1}\left(\dfrac{x}{a} \right) + c$, where $a > 0$ and $|x| > a$

$\int \dfrac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} \; dx = \dfrac{1}{a} \sec^{-1}\left(\dfrac{f(x)}{a} \right) + c$, where $|f(x)| > a$

Examples : Evaluate the following integrals

1. $\int (x^2 + 2x)(x^3 + 3x^2 + 5)^{10} \ dx$

   Solution :

   $$\int (x^2 + 2x)(x^3 + 3x^2 + 5)^{10} \ dx = \frac{1}{3} \int (x^3 + 3x^2 + 5)^{10} \left[3(x^2 + 2x)\right] \ dx$$

   $$= \frac{1}{3} \int (x^3 + 3x^2 + 5)^{10}(3x^2 + 6x) \ dx = \frac{1}{3} \frac{(x^3 + 3x^2 + 5)^{11}}{11} + c$$

2. $\int \dfrac{x + 1}{(x^2 + 2x + 6)^5} \ dx$

   Solution :

   $$\int \frac{x + 1}{(x^2 + 2x + 6)^5} \ dx = \int (x^2 + 2x + 6)^{-5}(x + 1) \ dx$$

   $$= \frac{1}{2} \int (x^2 + 2x + 6)^{-5}(2x + 2) \ dx = \frac{1}{2} \frac{(x^2 + 2x + 6)^{-4}}{-4} + c$$

3. $\int \dfrac{x^3 + x}{\sqrt{x^4 + 2x^2 + 5}} \ dx$

   Solution :

   $$\int \frac{x^3 + x}{\sqrt{x^4 + 2x^2 + 5}} \ dx = \int (x^4 + 2x^2 + 5)^{-\frac{1}{2}}(x^3 + x) \ dx$$

   $$= \frac{1}{4} \int (x^4 + 2x^2 + 5)^{-\frac{1}{2}}(4x^3 + 4x) \ dx = \frac{1}{4} \frac{(x^4 + 2x^2 + 5)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

4. $\int \dfrac{x^2 + 1}{x^3 + 3x + 8} \ dx$

   Solution :

   $$\int \frac{x^2 + 1}{x^3 + 3x + 8} \ dx = \frac{1}{3} \int \frac{3(x^2 + 1)}{x^3 + 3x + 8} \ dx$$

   $$= \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x + 8} \ dx = \frac{1}{3} \ln|x^3 + 3x + 8| + c$$

5. $\int \dfrac{\sin x}{1 + \cos x} \ dx$

   Solution :

   $$\int \frac{\sin x}{1 + \cos x} \ dx = -\int \frac{-\sin x}{1 + \cos x} \ dx = -\ln|1 + \cos x| + c$$

6. $\displaystyle\int \frac{e^{5x}}{e^{5x}-2}\ dx$

Solution :

$$\int \frac{e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\int \frac{5e^{5x}}{e^{5x}-2}\ dx = \frac{1}{5}\ln\left|e^{5x}-2\right|+c$$

7. $\displaystyle\int \left(3x^2+1\right)\sin(x^3+x+1)\ dx$

Solution :

$$\int \left(3x^2+1\right)\sin(x^3+x+1)\ dx = \int \sin(x^3+x+1)\ \left(3x^2+1\right)\ dx$$

$$= -\cos(x^3+x+1)+c$$

8. $\displaystyle\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx$

Solution :

$$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx = \int \sec^2\sqrt{x}\ \frac{1}{\sqrt{x}}\ dx$$

$$= 2\int \sec^2\sqrt{x}\ \frac{1}{2\sqrt{x}}\ dx = 2\tan\sqrt{x}+c$$

9. $\displaystyle\int x\ \csc(x^2+2)\cot(x^2+2)\ dx$

Solution :

$$\int x\ \csc(x^2+2)\cot(x^2+2)\ dx = \int \csc(x^2+2)\cot(x^2+2)\ x\ dx$$

$$\frac{1}{2}\int \csc(x^2+2)\cot(x^2+2)\ (2x)\ dx = -\frac{1}{2}\csc(x^2+2)+c$$

10. $\displaystyle\int e^{7\sin x}\cos x\ dx$

Solution :

$$\int e^{7\sin x}\cos x\ dx = \frac{1}{7}\int e^{7\sin x}(7\cos x)\ dx = \frac{1}{7}e^{7\sin x}+c$$

11. $\displaystyle\int \frac{e^{\frac{3}{x}}}{x^2}\ dx$

Solution :

$$\int \frac{e^{\frac{3}{x}}}{x^2}\ dx = \int e^{\frac{3}{x}}\ \frac{1}{x^2}\ dx$$

$$= -\frac{1}{3} \int e^{\frac{3}{x}} \frac{-3}{x^2} \, dx = -\frac{1}{3} e^{\frac{3}{x}} + c$$

12. $\displaystyle\int \frac{x}{\sqrt{9 - x^4}} \, dx$

Solution :

$$\int \frac{x}{\sqrt{9 - x^4}} \, dx = \int \frac{x}{\sqrt{3^2 - (x^2)^2}} \, dx$$

$$= \frac{1}{2} \int \frac{2x}{\sqrt{3^2 - (x^2)^2}} \, dx = \frac{1}{2} \sin^{-1}\left(\frac{x^2}{3}\right) + c$$

13. $\displaystyle\int \frac{1}{x^2 - 6x + 10} \, dx$

Solution

$$\int \frac{1}{x^2 - 6x + 10} \, dx = \int \frac{1}{(x^2 - 6x + 9) + (10 - 9)} \, dx$$

$$= \int \frac{1}{(x - 3)^2 + 1} \, dx = \tan^{-1}(x - 3) + c$$

14. $\displaystyle\int \frac{3}{x^2 + 2x + 5} \, dx$

Solution

$$\int \frac{3}{x^2 + 2x + 5} \, dx = \int \frac{3}{(x^2 + 2x + 1) + (5 - 1)} \, dx$$

$$= 3 \int \frac{1}{(x + 1)^2 + 2^2} \, dx = 3 \, \frac{1}{2} \tan^{-1}\left(\frac{x + 1}{2}\right) + c$$

15. $\displaystyle\int \frac{1}{x \ln |x|} \, dx$

Solution

$$\int \frac{1}{x \ln |x|} \, dx = \int \frac{\frac{1}{x}}{\ln |x|} \, dx = \ln |\ln |x|| + c$$

16. $\displaystyle\int \frac{2x - 1}{x^2 + 1} \, dx$

Solution :

$$\int \frac{2x - 1}{x^2 + 1} \, dx = \int \frac{2x}{x^2 + 1} \, dx - \int \frac{1}{x^2 + 1} \, dx$$

$$= \ln(x^2 + 1) - \tan^{-1} x + c$$

3. Integration by parts

It is used to solve an integral of a product of two functions using the formula

$$\int u \; dv = u \; v - \int v \; du$$

Examples : Evaluate the following integrals

1. $\int xe^x \; dx$

   Solution : Using integration by parts

   $u = x \qquad dv = e^x \; dx$
   $du = dx \qquad v = e^x$

   $$\int xe^x \; dx = xe^x - \int e^x \; dx = xe^x - e^x + c$$

2. $\int x^2 \sin x \; dx$

   Solution : Using integration by parts

   $u = x^2 \qquad dv = \sin x \; dx$
   $du = 2xdx \qquad v = -\cos x$

   $$\int x^2 \sin x \; dx = -x^2 \cos x - \int 2x(-\cos x) \; dx$$

   $$= -x^2 \cos x + 2 \int x \cos x \; dx$$

   Using integration by parts again

   $u = x \qquad dv = \cos x \; dx$
   $du = dx \qquad v = \sin x$

   $$\int x^2 \sin x \; dx = -x^2 \cos x + 2 \left(x \sin x - \int \sin x \; dx \right)$$

   $$= -x^2 \cos x + 2 \left(x \sin x - (-\cos x) \right) + c$$

   $$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

3. $\int x \ln |x| \; dx$

   Solution : Using integration by parts

   $u = \ln |x| \qquad dv = x \; dx$
   $du = \dfrac{1}{x} \; dx \qquad v = \dfrac{x^2}{2}$

$$\int x \ln|x| \ dx = \frac{x^2}{2} \ln|x| - \int \frac{1}{x} \frac{x^2}{2} \ dx$$

$$= \frac{x^2}{2} \ln|x| - \frac{1}{2} \int x \ dx = \frac{x^2}{2} \ln|x| - \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + c$$

4. $\int \ln|x| \ dx$

Solution : Using integration by parts

$$u = \ln|x| \qquad dv = dx$$
$$du = \frac{1}{x} \ dx \qquad v = x$$

$$\int \ln|x| \ dx = x \ln|x| - \int x \frac{1}{x} \ dx = x \ln|x| - \int 1 \ dx$$

$$= x \ln|x| - x + c$$

5. $\int \tan^{-1} x \ dx$

Solution : Using integration by parts

$$u = \tan^{-1} x \qquad dv = dx$$
$$du = \frac{1}{1+x^2} \ dx \qquad v = x$$

$$\int \tan^{-1} x \ dx = x \tan^{-1} x - \int x \frac{1}{1+x^2} \ dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \ dx = x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + c$$

6. $\int \sin^{-1} x \ dx$

Solution : Using integration by parts

$$u = \sin^{-1} x \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}} \ dx \qquad v = x$$

$$\int \sin^{-1} x \ dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \ dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}}(-2x) \ dx = x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

7. $\int e^x \sin x \ dx$

Solution : Using integration by parts

$u = \sin x$ $dv = e^x \ dx$
$du = \cos x \ dx$ $v = e^x$

$$\int e^x \sin x \ dx = e^x \sin x - \int e^x \cos x \ dx$$

Using integration by parts again

$u = \cos x$ $dv = e^x \ dx$
$du = -\sin x \ dx$ $v = e^x$

$$\int e^x \sin x \ dx = e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \ dx \right)$$

$$\int e^x \sin x \ dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \ dx \right)$$

$$\int e^x \sin x \ dx = e^x \sin x - e^x \cos x - \int e^x \sin x \ dx$$

$$2 \int e^x \sin x \ dx = e^x \sin x - e^x \cos x + c$$

$$\int e^x \sin x \ dx = \frac{1}{2} \left(e^x \sin x - e^x \cos x + c \right)$$

4. Integral of rational functions
(The metod of partial fractions)

Method of partial fractions is used to solve integrals of the form $\displaystyle\int \frac{P(x)}{Q(x)} \, dx$
where $P(x)$, $Q(x)$ are polynomials and degree $P(x) <$ degree $Q(x)$.
If degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials .

Definition (linear factor) :
A linear factor is a polynomial of degree 1.
It has the form $ax + b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

Examples :
x , $3x$, $2x - 7$ are examples of linear factors .

Definition (irreducible quadratic) :
An irreducible quadratic is a polynomial of degree 2.
It has the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $b^2 - 4ac < 0$.

Examples :

1. $x^2 + 9$ and $x^2 + x + 1$ are examples of irreducible quadratics.

2. $x^2 = x \, x$ and $x^2 - 1 = (x - 1)(x + 1)$ are reducible quadratics .

How to write $\dfrac{P(x)}{Q(x)}$ as partial fractions decomposition ?
Write $Q(x)$ as a product of linear factors and irreducible quadratics (if possible).
If $Q(x) = (a_1x + a_2)^m \ (b_1x^2 + b_2x + b_3)^n$ where $m, n \in \mathbb{N}$ then
$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \cdots + \frac{A_m}{(a_1x + a_2)^m}$$
$$+ \frac{B_1x + C_1}{b_1x^2 + b_2x + b_3} + \frac{B_2x + C_2}{(b_1x^2 + b_2x + b_3)^2} + \cdots + \frac{B_nx + C_n}{(b_1x^2 + b_2x + b_3)^n}$$
Where $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n, C_1, C_2, \cdots, C_n \in \mathbb{R}$.

Examples : Write the partial fractions decomposition of the follwoing

1. $\dfrac{2x + 6}{x^2 - 2x - 3}$

   Solution :

   $$\frac{2x + 6}{x^2 - 2x - 3} = \frac{2x + 6}{(x - 3)(x + 1)} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}$$

2. $\dfrac{x+5}{x^2+4x+4}$

   Solution :

   $$\frac{x+5}{x^2+4x+4} = \frac{x+5}{(x+2)^2} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2}$$

3. $\dfrac{x^2+1}{x^4+4x^2}$

   Solution :

   $$\frac{x^2+1}{x^4+4x^2} = \frac{x^2+1}{x^2(x^2+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1 x + C_1}{x^2+4}$$

4. $\dfrac{2x+7}{(x+1)(x^2+9)^2}$

   Solution :

   $$\frac{2x+7}{(x+1)(x^2+9)^2} = \frac{A_1}{x+1} + \frac{B_1 x + C_1}{x^2+9} + \frac{B_2 x + C_2}{(x^2+9)^2}$$

5. $\dfrac{x}{(x-1)(x^2-1)}$

   Solution :

   $$\frac{x}{(x-1)(x^2-1)} = \frac{x}{(x+1)(x-1)^2} = \frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}$$

6. $\dfrac{x^3+x}{x^2-1}$

   Solution : Using long division of polynomials

   $$\frac{x^3+x}{x^2-1} = \frac{(x^3-x)+2x}{x^2-1} = \frac{x(x^2-1)+2x}{x^2-1} = x + \frac{2x}{x^2-1}$$

   $$\frac{x^3+x}{x^2-1} = x + \frac{2x}{(x-1)(x+1)} = x + \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

Examples : Evaluate the following integrals

1. $\displaystyle\int \dfrac{x+3}{(x-3)(x-2)}\ dx$

   Solution : Using the method of partial fractions

   $$\frac{x+3}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

   $$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2)}{(x-3)(x-2)} + \frac{A_2(x-3)}{(x-2)(x-3)}$$

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$$\frac{x+3}{(x-3)(x-2)} = \frac{A_1(x-2) + A_2(x-3)}{(x-3)(x-2)}$$

$$x + 3 = A_1(x-2) + A_2(x-3) = A_1 x - 2A_1 + A_2 x - 3A_2$$

$$x + 3 = (A_1 + A_2)x + (-2A_1 - 3A_2)$$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 & + & A_2 & = & 1 & \longrightarrow (1) \\ -2A_1 & - & 3A_2 & = & 3 & \longrightarrow (2) \end{cases}$$

Muliplying equation (1) by 2 and adding it to equation (2) :

$$-A_2 = 5 \implies A_2 = -5$$

From Equation (1) : $A_1 - 5 = 1 \implies A_1 = 1 + 5 = 6$

$$\frac{x+3}{(x-3)(x-2)} = \frac{6}{x-3} + \frac{-5}{x-2}$$

$$\int \frac{x+3}{(x-3)(x-2)} \, dx = \int \left(\frac{6}{x-3} - \frac{5}{x-2} \right) dx$$

$$= 6 \int \frac{1}{x-3} \, dx - 5 \int \frac{1}{x-2} \, dx = 6 \ln|x-3| - 5 \ln|x-2| + c$$

2. $\int \dfrac{x+1}{x^2-1} \, dx$

Solution :

$$\int \frac{x+1}{x^2-1} \, dx = \int \frac{x+1}{(x-1)(x+1)} \, dx$$

$$= \int \frac{1}{x-1} \, dx = \ln|x-1| + c$$

3. $\int \dfrac{x-1}{(x+1)(x+2)^2} \, dx$

Solution : Using the method of partial fractions

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2}$$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{A_1(x+2)^2}{(x+1)(x+2)^2} + \frac{A_2(x+1)(x+2)}{(x+1)(x+2)^2} + \frac{A_3(x+1)}{(x+1)(x+2)^2}$$

$$x - 1 = A_1(x+2)^2 + A_2(x+1)(x+2) + A_3(x+1)$$

$$x - 1 = A_1(x^2 + 4x + 4) + A_2(x^2 + 3x + 2) + A_3(x+1)$$

$$x - 1 = A_1 x^2 + 4A_1 x + 4A_1 + A_2 x^2 + 3A_2 x + 2A_2 + A_3 x + A_3$$

$$x - 1 = (A_1 + A_2)x^2 + (4A_1 + 3A_2 + A_3)x + (4A_1 + 2A_2 + A_3)$$

By comparing the coefficients of the polynomials

$$\begin{cases} A_1 & + & A_2 & & & = & 0 & \longrightarrow (1) \\ 4A_1 & + & 3A_2 & + & A_3 & = & 1 & \longrightarrow (2) \\ 4A_1 & + & 2A_2 & + & A_3 & = & -1 & \longrightarrow (3) \end{cases}$$

Subtracting equation (3) from equation (2) : $A_2 = 2$

From equation (1) : $A_1 + 2 = 0 \implies A_1 = -2$

From equation (2) :

$(4 \times -2) + (3 \times 2) + A_3 = 1 \implies -8 + 6 + A_3 = 1 \implies A_3 = 3$

$$\frac{x-1}{(x+1)(x+2)^2} = \frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

$$\int \frac{x-1}{(x+1)(x+2)^2} \, dx = \int \left(\frac{-2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2} \right) \, dx$$

$$= -2 \int \frac{1}{x+1} \, dx + 2 \int \frac{1}{x+2} \, dx + 3 \int (x+2)^{-2} \, dx$$

$$= -2 \ln|x+1| + 2 \ln|x+2| + 3 \frac{(x+2)^{-1}}{-1} + c$$

$$= -2 \ln|x+1| + 2 \ln|x+2| - \frac{3}{x+2} + c$$

4. $\int \dfrac{2x^2 + 3x + 2}{x^3 + x} \, dx$

Solution : Using the method of partial functions

$$\frac{2x^2 + 3x + 2}{x^3 + x} = \frac{2x^2 + 3x + 2}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$\frac{2x^2 + 3x + 2}{x^3 + x} = \frac{A(x^2 + 1)}{x(x^2 + 1)} + \frac{x(Bx + C)}{x(x^2 + 1)}$$

$2x^2 + 3x + 2 = A(x^2 + 1) + x(Bx + C) = Ax^2 + A + Bx^2 + Cx$

$2x^2 + 3x + 2 = (A + B)x^2 + Cx + A$

By comparing the coefficients of the polynomials

$$\begin{cases} A + B & = & 2 & \longrightarrow (1) \\ C & = & 3 & \longrightarrow (2) \\ A & = & 2 & \longrightarrow (3) \end{cases}$$

From equation (1) : $2 + B = 2 \implies B = 0$

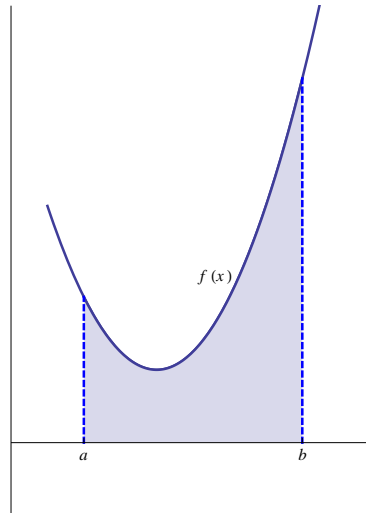$$\frac{2x^2 + 3x + 2}{x^3 + x} = \frac{2}{x} + \frac{3}{x^2 + 1}$$

$$\int \frac{2x^2 + 3x + 2}{x^3 + x} \, dx = \int \left(\frac{2}{x} + \frac{3}{x^2 + 1} \right) \, dx$$

$$= 2 \int \frac{1}{x} \, dx + 3 \int \frac{1}{x^2 + 1} \, dx$$

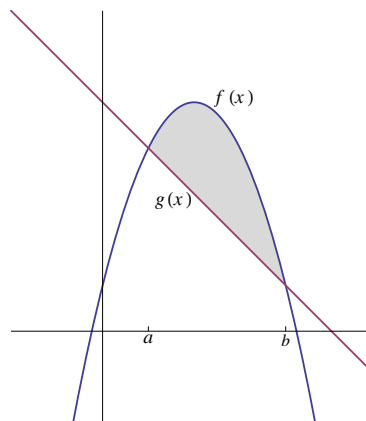$$= 2 \ln |x| + 3 \tan^{-1} x + c$$

CHAPTER FIVE

APPLICATIONS OF INTEGRATION

1. Area

2. Volume of a solid of revolution
(using disk or washer method)

3. Volume of a solid of revolution
(using cylindrical shells method)

1. Area

In the above figure the area under the graph of $f(x)$ on the interval $[a, b]$ is given by the definite integral $\displaystyle\int_a^b f(x) \, dx$

In the above figure the graphs of $f(x)$ and $g(x)$ intersect at the points $x = a$ and $x = b$.

The area bounded by the graphs of the curves of $f(x)$ and $g(x)$ equals

$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx$$

Examples :

1. Find the area of the region bounded by the graphs of $x = 0$, $y = 0$, $x = 2$ and $y = x^2 + 1$
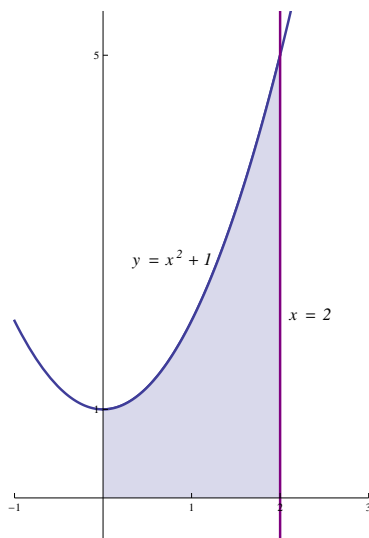
$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

$x = 0$ is the y-axis and $y = 0$ is the x-axis.

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$

$$\text{Area} = \int_0^2 (x^2 + 1)\ dx = \left[\frac{x^3}{3} + x\right]_0^2$$

$$\text{Area} = \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right) = \frac{8}{3} + 2 = \frac{14}{3}$$

2. Find the area of the region bounded by the graphs of $y = x$ and $y = x^2$
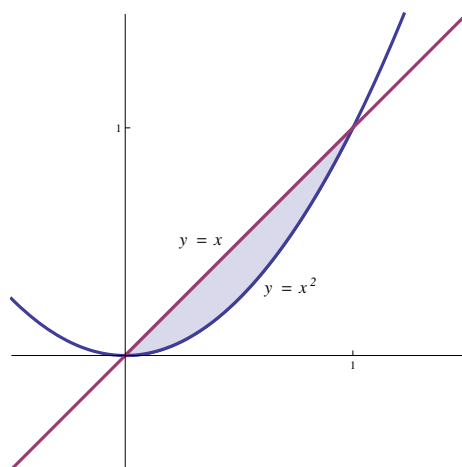
$y = x^2$ is a parabola with vertex $(0,0)$ and opens upwards.

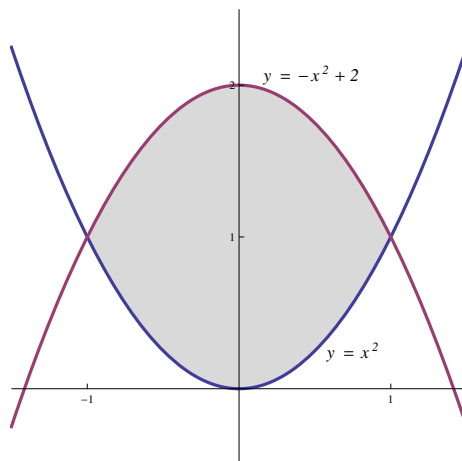$y = x$ is a straight line passing through the origin with slope equals 1.

Points of intersection of $y = x^2$ and $y = x$:

$x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0 , \ x = 1$

$\text{Area} = \int_0^1 (x - x^2) \ dx = \left[\dfrac{x^2}{2} - \dfrac{x^3}{3} \right]_0^1$

$\text{Area} = \left(\dfrac{1^2}{2} - \dfrac{1^3}{3} \right) - \left(\dfrac{0^2}{2} - \dfrac{0^3}{3} \right) = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

3. Find the area of the region bounded by the graphs of $y = x^2$ and $y = -x^2 + 2$

$y = -x^2 + 2$ is a parabola with vertex $(0,2)$ and opens downwards

$y = x^2$ is a parabola with vertex $(0,0)$ and opens upwards.
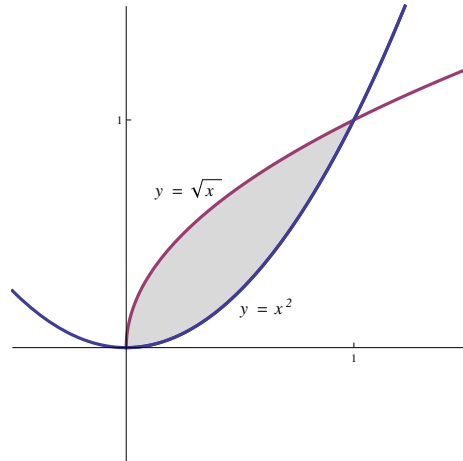
Points of intersection of $y = x^2$ and $y = -x^2 + 2$:

$x^2 = -x^2 + 2 \implies 2x^2 = 2 \implies x^2 = 1 \implies x = \pm 1$

$\text{Area} = \int_{-1}^1 \left[(-x^2 + 2) - x^2 \right] \ dx = \int_{-1}^1 (2 - 2x^2) \ dx$

$\text{Area} = \left[2x - \dfrac{2x^3}{3} \right]_{-1}^1 = \left[\left(2 - \dfrac{2}{3} \right) - \left(-2 + \dfrac{2}{3} \right) \right]$

$\text{Area} = 2 - \dfrac{2}{3} + 2 - \dfrac{2}{3} = 4 - \dfrac{4}{3} = \dfrac{12 - 4}{3} = \dfrac{8}{3}$

4. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$

$y = x^2$ is a parabola with vertex $(0,0)$ and opens upwards.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0,0)$ and opens to the right.

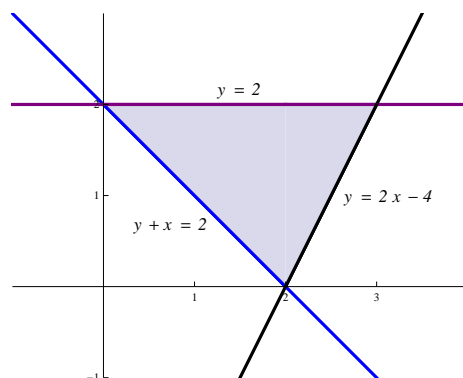Points of intersection of $y = x^2$ and $y = \sqrt{x}$:

$x^2 = \sqrt{x} \;\Rightarrow\; x^4 = x \;\Rightarrow\; x^4 - x = 0 \;\Rightarrow\; x(x^3 - 1) = 0$

$\Rightarrow\; x = 0 \;,\; x^3 = 1 \;\Rightarrow\; x = 0 \;,\; x = 1$

$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2 \right) \, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 = \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$

$\text{Area} = \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \frac{1}{3}$

5. Find the area of the region bounded by the graphs of $x + y = 2$, $y = 2$ and $y = 2x - 4$

$y = 2$, $y = 2x - 4$ and $y = -x + 2$ are three straight lines.

Point of intersection of $y = 2$ and $y = -x + 2$:

$-x + 2 = 2 \Rightarrow x = 0$

$y = 2$ and $y = -x + 2$ intersect at the point $(0, 2)$.

Point of intersection of $y = 2$ and $y = 2x - 4$:

$2x - 4 = 2 \Rightarrow x = 3$

$y = 2$ and $y = 2x - 4$ intersect at the point $(3, 2)$

Point of intersection of $y = -x + 2$ and $y = 2x - 4$:

$2x - 4 = -x + 2 \Rightarrow 3x = 6 \Rightarrow x = 2$

$y = -x + 2$ and $y = 2x - 4$ intersect at the point $(2, 0)$.

$\text{Area} = \int_0^2 [2 - (-x + 2)] \ dx + \int_2^3 [2 - (2x - 4)] \ dx$

$\text{Area} = \int_0^2 x \ dx + \int_2^3 (6 - 2x) \ dx = \left[\dfrac{x^2}{2}\right]_0^2 + \left[6x - x^2\right]_2^3$

$\text{Area} = \left[\dfrac{2^2}{2} - \dfrac{0^2}{2}\right] + \left[(6 \times 3 - 3^2) - (6 \times 2 - 2^2)\right]$

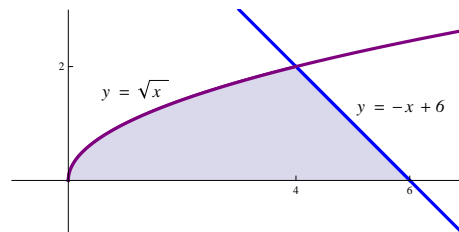$\text{Area} = (2 - 0) + [(18 - 9) - (12 - 4)] = 2 + (9 - 8) = 2 + 1 = 3$

Another solution :

$y + x = 2 \Rightarrow x = -y + 2$ and $y = 2x - 4 \Rightarrow 2x = y + 4 \Rightarrow x = \dfrac{1}{2}y + 2$

$\text{Area} = \int_0^2 \left[\left(\dfrac{1}{2}y + 2\right) - (-y + 2)\right] \ dy$

$\text{Area} = \int_0^2 \left(\dfrac{1}{2}y + y\right) \ dy = \int_0^2 \dfrac{3}{2}y \ dy$

$\text{Area} = \dfrac{3}{2}\left[\dfrac{y^2}{2}\right]_0^2 = \dfrac{3}{2}\left[\dfrac{2^2}{2} - \dfrac{0^2}{2}\right] = \dfrac{3}{2} \times 2 = 3$

6. Find the area of the region bounded by the graphs of $y = 0$, $y = -x + 6$ and $y = \sqrt{x}$

$y = -x + 6$ is a straight line passing through $(0, 6)$ with slope equals -1.

$y = \sqrt{x} \;\Rightarrow\; x = y^2$ is the upper half of the parabola with vertex $(0, 0)$ and opens to the right.

Points of intersection of $x = y^2$ and $x = -y + 6$:

$$y^2 = -y + 6 \;\Rightarrow\; y^2 + y - 6 = 0 \;\Rightarrow\; (y - 2)(y + 3) = 0 \;\Rightarrow\; y = 2 \;,\; y = -3$$

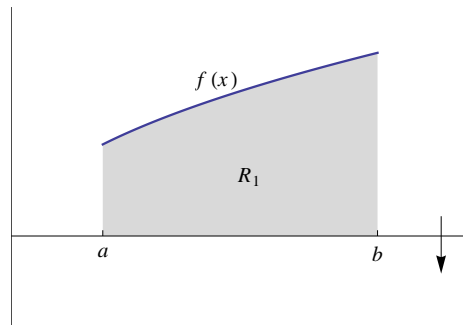(Note that $y = -3$ is not in the desired region).

$$\text{Area} = \int_0^2 \left[(-y + 6) - y^2 \right] \, dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2$$

$$\text{Area} = \left(12 - \frac{4}{2} - \frac{8}{3} \right) - (0 - 0 - 0) = 12 - 2 - \frac{8}{3} = 10 - \frac{8}{3} = \frac{30 - 8}{3} = \frac{22}{3}$$

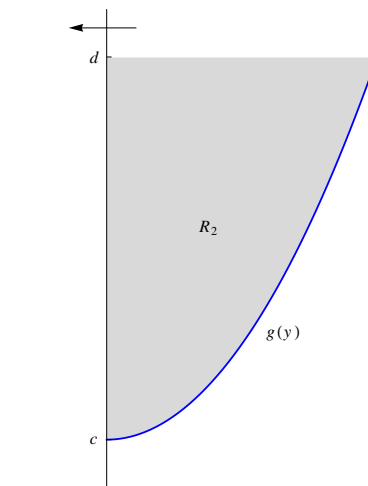# 2. Volume of a solid of revolution (using disk or washer method)

1. Disk Method

Recall that the volume of a right circular cylinder equals $\pi r^2 h$ where r is the radius of the base (which is a circle) and h is the height of the cylinder .

In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_1 around the x-axis is $V = \pi \displaystyle\int_a^b [f(x)]^2 \ dx$
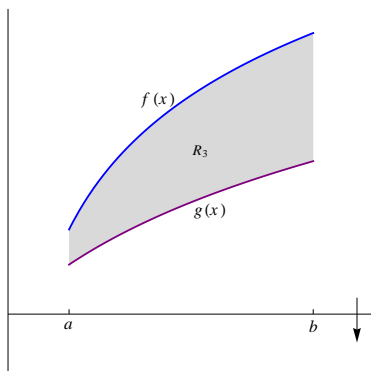

In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.

Using disk method , the volume of the solid of revolution generated by revolving the region R_2 around the y-axis is $V = \pi \displaystyle\int_c^d [g(y)]^2 \ dy$

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2. Washer Method

Volume of a washer $= \pi \left[(outer\ radius)^2 - (inner\ radius)^2\right]\ (thickness)$

In the above figure R_3 is the region bounded by the graphs of the curves of $f(x)$, $g(x)$, $x = a$ and $x = b$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_3 around the x-axis is $V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right]\ dx$
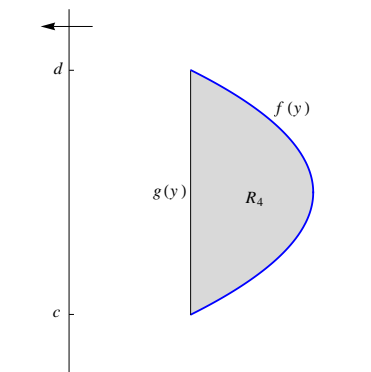

In the above figure R_4 is the region bounded by the graphs of the curves of $f(y)$ and $g(y)$, where $f(y)$ and $g(y)$ intersect at the points $y = c$ and $y = d$.

Using washer method , the volume of the solid of revolution generated by revolving the region R_4 around the y-axis is $V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2\right]\ dy$

Examples : Use Disk or washer method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

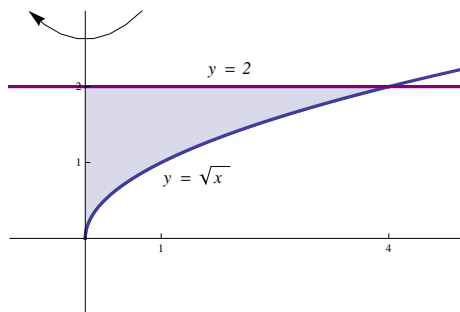1. $y = x^2 + 2$, $y = 0$, $x = 0$, $x = 1$, around the x-axis
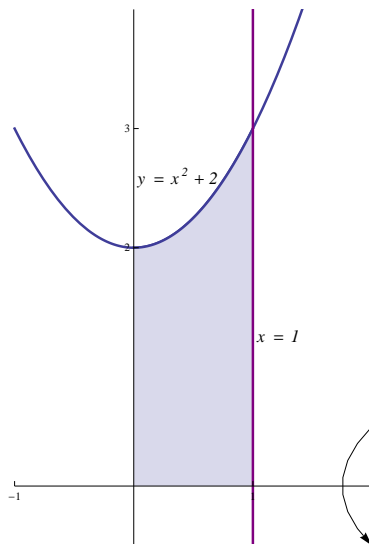
$y = x^2 + 2$ is a parabola with vertex $(0, 2)$ and opens upwards.

$x = 1$ is a straight line parallel to the y-axis and pasing through $(1, 0)$

Using Disk method :

$$\text{Volume} = \pi \int_0^1 (x^2 + 2)^2 \, dx = \pi \int_0^1 (x^4 + 4x^2 + 4) \, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_0^1 = \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 4 \right) - (0 + 0 + 0) \right] = \frac{83\pi}{15}$$

2. $y = \sqrt{x}$, $y = 2$ and $x = 0$, around the y-axis
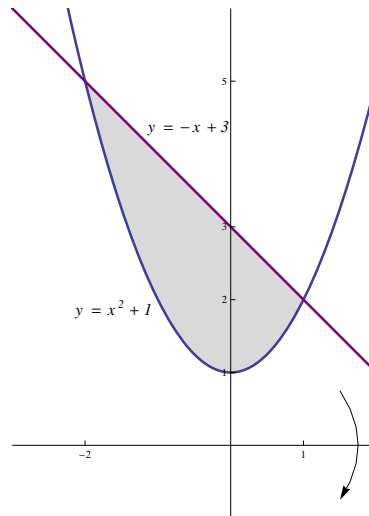

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

$y = 2$ is a straight line parallel to the x-axis and passing through $(0, 2)$

Using Disk method :

$$\text{Volume} = \pi \int_0^2 (y^2)^2 \ dy = \pi \int_0^2 y^4 \ dy$$

$$= \pi \left[\frac{y^5}{5}\right]_0^2 = \pi \left[\frac{2^5}{5} - 0\right] = \frac{32\pi}{5}$$

3. $y = x^2 + 1$ and $y = -x + 3$, around the x-axis

$y = x^2 + 1$ is a parabola with vertex $(0, 1)$ and opens upwards.

$y = -x + 3$ is a straight line with slope -1 and passing through $(0, 3)$.

Points of intersection of $y = x^2 + 1$ and $y = -x + 3$:

$x^2 + 1 = -x + 3 \ \Rightarrow \ x^2 + x - 2 = 0 \ \Rightarrow \ (x+2)(x-1) = 0 \ \Rightarrow \ x = -2 \ , \ x = 1$

Using Washer method :

$$\text{volume} = \pi \int_{-2}^1 \left[(-x + 3)^2 - (x^2 + 1)^2\right] \ dx$$

$$\text{Volume} = \pi \int_{-2}^1 \left[(x^2 - 6x + 9) - (x^4 + 2x^2 + 1)\right] \ dx$$
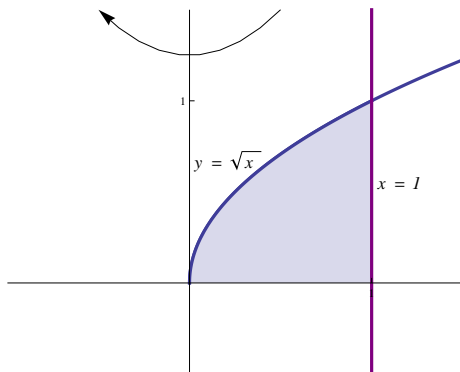
$$\text{Volume} = \pi \int_{-2}^1 \left(-x^4 - x^2 - 6x + 8\right) \ dx = \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x\right]_{-2}^1$$

$$= \pi \left[\left(-\frac{1}{5} - \frac{1}{3} - 3 + 8\right) - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16\right)\right]$$

$$= \pi \left(-\frac{1}{5} - \frac{1}{3} + 5 - \frac{32}{5} - \frac{8}{3} + 28\right)$$

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$$= \pi \left(33 - 3 - \frac{33}{5}\right) = \pi \left(30 - \frac{33}{5}\right) = \frac{150 - 33}{5}\pi = \frac{117\pi}{5}$$

4. $y = \sqrt{x}$, $y = 0$ and $x = 1$, around the y-axis

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0, 0)$ and opens to the right

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$

Note that $y = \sqrt{x}$ intersects $x = 1$ at the point $(1, 1)$.

Using Washer method :

$$\text{Volume} = \pi \int_0^1 \left[(1)^2 - (y^2)^2\right] \, dy = \pi \int_0^1 (1 - y^4) \, dy$$

$$= \pi \left[y - \frac{y^5}{5}\right]_0^1 = \pi \left[\left(1 - \frac{1}{5}\right) - (0 - 0)\right] = \pi \left(1 - \frac{1}{5}\right) = \frac{4\pi}{5}$$

3. Volume of a solid of revolution (using cylindrical shells method)

Volume of a shell $=$ 2π (average radius) (altitude) (thickness)

In the above figure R_1 is the region bounded by the graphs of the curves of $f(x)$, $x = a$, $x = b$ and the x-axis.

Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_1 around the y-axis is $V = 2\pi \int_{a}^{b} x \ f(x) \ dx$
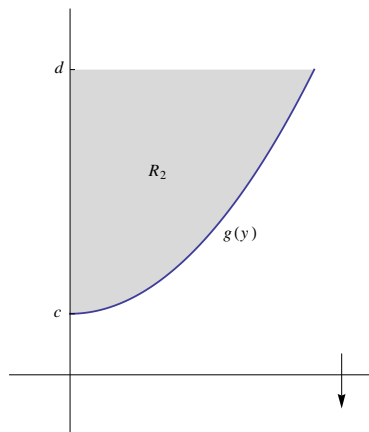
In the above figure R_2 is the region bounded by the graphs of the curves of $g(y)$, $y = d$ and the y-axis.

Using cylindrical shells method , the volume of the solid of revolution generated by revolving the region R_2 around the x-axis is $V = 2\pi \int_{c}^{d} y \ g(y) \ dy$

Examples : Use cylindrical shells method to calculate the volume of the solid of revolution generated by revolving the region bounded by the graphs of :

1. $y = \sqrt{x}$, $y = 0$ and $x = 4$, around the y-axis.
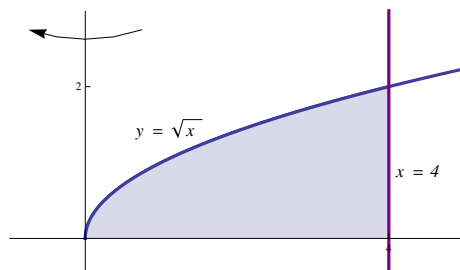
$y = 0$ is the x-axis

$y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ with vertex $(0,0)$ and opens to the right.

$x = 4$ is a straight line parallel to the y-axis and passing through $(4,0)$.
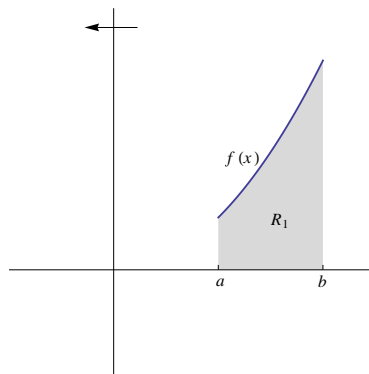
Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^4 x\sqrt{x} \; dx = 2\pi \int_0^4 x^{\frac{3}{2}} \; dx$$

$$\text{Volume} = 2\pi \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = 2\pi \; \frac{2}{5} \; (4)^{\frac{5}{2}} = 2\pi \; \frac{2}{5} \; (32) = \frac{128\pi}{5}$$

2. $x + y = 1$, $x = 1$ and $y = 2x + 1$, around the y-axis .
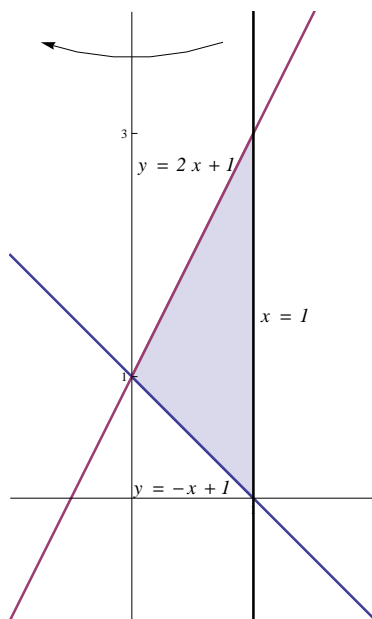
$y = -x + 1$ is a straight line with slope -1 and passing through $(0, 1)$.

$y = 2x + 1$ is a straight line with slope 2 and passing through $(0, 1)$.

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$.

Point of intersection of $x = 1$ and $y = -x + 1$ is $(1, 0)$.

Point of intersection of $x = 1$ and $y = 2x + 1$ is $(1, 3)$.

Point of intersection of $y = -x + 1$ and $y = 2x + 1$:

$2x + 1 = -x + 1 \implies 3x = 0 \implies x = 0$.

Using Cylindrical shells method

$$\text{Volume} = 2\pi \int_0^1 x[(2x + 1) - (-x + 1)] \ dx = 2\pi \int_0^1 x(3x) \ dx = 2\pi \int_0^1 3x^2 \ dx$$

$$\text{Volume} = 2\pi \left[x^3\right]_0^1 = 2\pi[1 - 0] = 2\pi$$

3. $y = x^2$ and $y = 1$, around the x-axis.
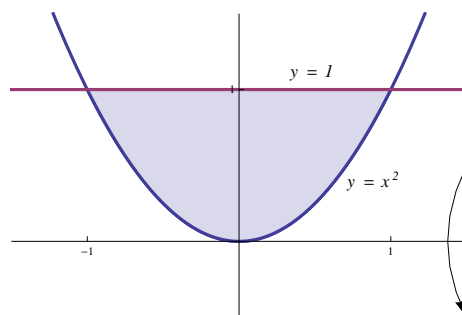
$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = 1$ is a straight line parallel to the x-axis and passing through $(0, 1)$.

Since the bounded region is symmetric with respect to the y-axis, consider the right half of the parabola $y = x^2$ which is $x = \sqrt{y}$.

Using Cylindrical shells method

$$\text{Volume} = 2\left(2\pi \int_0^1 y\sqrt{y} \ dy\right) = 4\pi \int_0^1 y^{\frac{3}{2}} \ dy$$

$$\text{Volume} = 4\pi \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = 4\pi \left(\frac{2}{5} - 0\right) = \frac{8\pi}{5}$$

4. $y = x^2$ and $y = x$, around the x-axis.

$y = x^2$ is a parabola with vertex $(0, 0)$ and opens upwards.

$y = x$ is a straigh line passing through the origin with solpe 1.
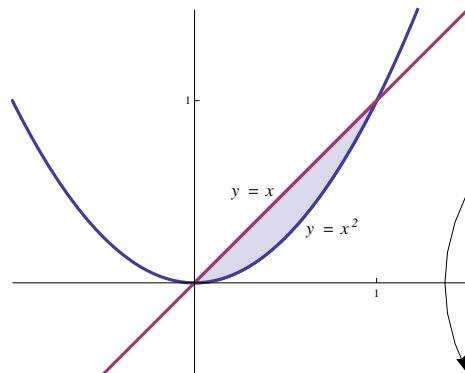
Consider $x = \sqrt{y}$ which is the right half of the parabola $y = x^2$.

Points of intersection of $x = \sqrt{y}$ and $x = y$:

$y = \sqrt{y} \ \Rightarrow \ y^2 = y \ \Rightarrow \ y^2 - y = 0 \ \Rightarrow \ y(y-1) = 0 \ \Rightarrow \ y = 0 \ , \ y = 1$

Using Cylindrical shells method

$\text{Volume} = 2\pi \int_0^1 y\left(\sqrt{y} - y\right) \ dy = 2\pi \int_0^1 \left(y^{\frac{3}{2}} - y^2\right) \ dy$

$\text{Volume} = 2\pi \left[\frac{2}{5} y^{\frac{5}{2}} - \frac{y^3}{3}\right]_0^1 = 2\pi \left[\left(\frac{2}{5} - \frac{1}{3}\right) - (0 - 0)\right]$

$\text{Volume} = 2\pi \left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi \left(\frac{6-5}{15}\right) = \frac{2\pi}{15}$

CHAPTER SIX

PARTIAL DERIVATIVES

1. Functions of several variables

2. Partial derivatives

3. Chain Rules

4. Implicit differentiation

1. Functions of several variables

1.1 Functions of two variables :

Definition: A function of two variables is a rule that assigns an ordered pair (x, y) (in the domain of the function) to a real number w.

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x, y) \longrightarrow w$$

Example :

$f(x, y) = \dfrac{y}{x^2 + y^2}$ is a function of two variables x and y

$f(3, 1) = \dfrac{1}{3^2 + 1^2} = \dfrac{1}{10}$.

Note that $f(x, y)$ takes $(3, 1) \in \mathbb{R}^2$ to $\dfrac{1}{10} \in \mathbb{R}$

1.2 Functions of three variables :

Definition: A function of three variables is a rule that assigns an ordered triple (x, y, z) (in the domain of the function) to a real number w .

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x, y, z) \longrightarrow w$$

Example :

$f(x, y, z) = \dfrac{z}{x + y^2 + 3}$ is a function of three variables x , y and z

$f(1, -2, 4) = \dfrac{4}{1 + (-2)^2 + 3} = \dfrac{4}{8} = \dfrac{1}{2}$.

Note that $f(x, y, z)$ takes $(1, -2, 4) \in \mathbb{R}^3$ to $\dfrac{1}{2} \in \mathbb{R}$

2. Partial derivatives

2.1 Partial derivatives of a function of two variables :
If $w = f(x, y)$ is a function of two variables, then :

1. The partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y as a constant .

2. The partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x as a constant .

Example 1: Calculate f_x and f_y of the functiuon $f(x, y) = x^2 y^3 + xy \ln(x + y)$
Solution:

1. $f_x = \dfrac{\partial}{\partial x} \left(x^2 y^3 + xy \ln(x + y) \right)$

   $f_x = (2x)y^3 + \left[(1)y \ln(x + y) + xy \dfrac{1}{x + y} \right] = 2xy^3 + y \ln(x + y) + \dfrac{xy}{x + y}$

2. $f_y = \dfrac{\partial}{\partial y} \left(x^2 y^3 + xy \ln(x + y) \right)$

   $f_y = x^2 (3y^2) + \left[x(1) \ln(x + y) + xy \dfrac{1}{x + y} \right] = 3x^2 y^2 + x \ln(x + y) + \dfrac{xy}{x + y}$

Example 2: Calculate f_x and f_y of the functiuon $f(x, y) = \dfrac{x + y^2}{x + y}$
Solution:

1. $f_x = \dfrac{\partial f}{\partial x} = \dfrac{(1 + 0)(x + y) - (x + y^2)(1 + 0)}{(x + y)^2} = \dfrac{x + y - (x + y^2)}{(x + y^2)}$

   $f_x = \dfrac{x + y - x - y^2}{(x + y)^2} = \dfrac{y - y^2}{(x + y)^2}$

2. $f_y = \dfrac{\partial f}{\partial y} = \dfrac{(0 + 2y)(x + y) - (x + y^2)(0 + 1)}{(x + y)^2} = \dfrac{2y(x + y) - (x + y^2)}{(x + y)^2}$

   $f_y = \dfrac{2xy + 2y^2 - x - y^2}{(x + y)^2} = \dfrac{2xy - x + y^2}{(x + y)^2}$

2.2 Partial derivatives of a function of three variables :

If $w = f(x, y, z)$ is a function of three variables, then :

1. The partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial w}{\partial x}$, f_x or w_x , and it is calculated by applying the rules of differentiation to x and regarding y and z as constants .

2. The partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial w}{\partial y}$, f_y or w_y , and it is calculated by applying the rules of differentiation to y and regarding x and z as constants .

3. The partial derivative of f with respect to z is denoted by $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial w}{\partial z}$, f_z or w_z , and it is calculated by applying the rules of differentiation to z and regarding x and y as constants .

Example : If $f(x, y, z) = 2z^3x - 4(x^2 + y^2)z$, then calculate f_x , f_y and f_z at $(0, 1, 2)$.

Solution :

1. $f_x = \dfrac{\partial}{\partial x}\left(2z^3x - 4(x^2 + y^2)z\right) = 2z^3 - 4(2x)z = 2z^3 - 8xz$

   $f_x(0, 1, 2) = 2\,(2^3) - 8(0)(2) = 16$

2. $f_y = \dfrac{\partial}{\partial y}\left(2z^3x - 4(x^2 + y^2)z\right) = 0 - 4(0 + 2y)z = -8yz$

   $f_y(0, 1, 2) = -8(1)(2) = -16$

3. $f_z = \dfrac{\partial}{\partial z}\left(2z^3x - 4(x^2 + y^2)z\right) = 6z^2x - 4(x^2 + y^2)$

   $f_z(0, 1, 2) = 6(2^2)(0) - 4(0^2 + 1^2) = -4$

2.3 Second partial derivatives :

If $w = f(x, y)$ is a function of two variables , then :

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}(f_x) = f_{xx}$.

2. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}(f_y) = f_{yy}$.

3. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}(f_y) = f_{yx}$.

4. $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}(f_x) = f_{xy}$.

Note : Second partial derivatives of a function of three variables are defined in a same manner.

Theorem : Let $f(x, y)$ be a function of two variables. If f , f_x , f_y , f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Note : If $f(x, y, z)$ is a function of three variables and f has continuous second partial derivatives, then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

Example 1: Let $f(x, y) = x^3 y + xy^2 \sin(x + y)$, calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$

Solution :

$$f_x = 3x^2 y + y^2 \sin(x + y) + xy^2 \cos(x + y)$$

$$f_y = x^3 + 2xy \sin(x + y) + xy^2 \cos(x + y)$$

$$f_{xy} = 3x^2 + 2y \sin(x + y) + y^2 \cos(x + y) + 2xy \cos(x + y) - xy^2 \sin(x + y)$$

$$f_{yx} = 3x^2 + 2y \sin(x + y) + 2xy \cos(x + y) + y^2 \cos(x + y) - xy^2 \sin(x + y)$$

Note : $f_{xy} = f_{yx}$ according to the theorem .

Example 2: Let $f(x, y, z) = x^3 y^2 z + xy \sin(y + z)$, calculate $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial z}$

Solution :

$$f_x = 3x^2 y^2 z + y \sin(y + z)$$

$$f_z = x^3 y^2 + xy \cos(y + z)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = 6x^2 yz + \sin(y + z) + y \cos(y + z)$$

$$\frac{\partial^2 f}{\partial x \partial z} = f_{zx} = 3x^2 y^2 + y \cos(y + z)$$

Example 3: Let $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$, Show that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$

Solution :

$$f_x = 0 - 3z(2x) = -6xz$$

$$f_y = 0 - 3z(2y) = -6yz$$

$$f_z = 6z^2 - 3(x^2 + y^2)$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = -6z$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = -6z$$

$$\frac{\partial^2 f}{\partial z^2} = f_{zz} = 12z$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$$

3. Chain Rules

Theorem (Chain Rules):

1. If $w = f(x, y)$ and $x = g(t)$, $y = h(t)$, such that f , g and h are differentiable then

$$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

2. If $w = f(x, y)$ and $x = g(t, s)$, $y = h(t, s)$, such that f , g and h are differentiable then

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

3. If $w = f(x, y, z)$ and $x = g(t, s)$, $y = h(t, s)$, $z = k(t, s)$ such that f , g , h and k are differentiable then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

Example 1 : Let $f(x, y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution :

1. $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x} \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \dfrac{\partial y}{\partial s}$

   $\dfrac{\partial f}{\partial x} = y$, $\dfrac{\partial x}{\partial s} = 2st$

   $\dfrac{\partial f}{\partial y} = x + 2y$, $\dfrac{\partial y}{\partial s} = 1$

   $\dfrac{\partial f}{\partial s} = y\,(2st) + (x + 2y)(1) = (s + t)2st + \left[s^2 t + 2(s + t)\right]$

   $= 2s^2 t + 2st^2 + s^2 t + 2s + 2t = 3s^2 t + 2st^2 + 2s + 2t$

2. $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x} \dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y} \dfrac{\partial y}{\partial t}$

   $\dfrac{\partial f}{\partial x} = y$, $\dfrac{\partial x}{\partial t} = s^2$

   $\dfrac{\partial f}{\partial y} = x + 2y$, $\dfrac{\partial y}{\partial t} = 1$

$$\frac{\partial f}{\partial t} = ys^2 + (x + 2y)(1) = (s+t)s^2 + s^2t + 2(s+t)$$

$$= s^3 + s^2t + s^2t + 2s + 2t = s^3 + 2s^2t + 2s + 2t$$

Example 2 : Let $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$ and $z = \dfrac{s}{t}$, calculate $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution :

1. $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial s}$

$\dfrac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)$, $\dfrac{\partial x}{\partial s} = t$

$\dfrac{\partial f}{\partial y} = x\cos(xy)$, $\dfrac{\partial y}{\partial s} = 1$

$\dfrac{\partial f}{\partial z} = -x\sin(xz)$, $\dfrac{\partial z}{\partial s} = \dfrac{1}{t}$

$\dfrac{\partial f}{\partial s} = t \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\dfrac{1}{t}\right)(-x\sin(xz))$

$\dfrac{\partial f}{\partial s} = t + ty\cos(xy) - tz\sin(xz) + x\cos(xy) - \dfrac{x\sin(xz)}{t}$

2. $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial t}$

$\dfrac{\partial f}{\partial x} = 1 + y\cos(xy) - z\sin(xz)$, $\dfrac{\partial x}{\partial t} = s$

$\dfrac{\partial f}{\partial y} = x\cos(xy)$, $\dfrac{\partial y}{\partial t} = 1$

$\dfrac{\partial f}{\partial z} = -x\sin(xz)$, $\dfrac{\partial z}{\partial t} = \dfrac{-s}{t^2}$

$\dfrac{\partial f}{\partial t} = s \ [1 + y\cos(xy) - z\sin(xz)] + x\cos(xy) + \left(\dfrac{-s}{t^2}\right)(-x\sin(xz))$

$\dfrac{\partial f}{\partial t} = s + sy\cos(xy) - sz\sin(xz) + x\cos(xy) + \dfrac{sx\sin(xz)}{t^2}$

4. Implicit differentiation

1. Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x say $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

2. Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable , then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 1 : Let $y^2 - xy + 3x^2 = 0$, find $\dfrac{dy}{dx}$.
Solution 1: Let $F(x, y) = x^2 - xy + 3x^2$ then $F(x, y) = 0$

$F_x = -y + 6x$ and $F_y = 2y - x$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y + 6x)}{2y - x} = \frac{y - 6x}{2y - x} \ .$$

Solution 2 : $y^2 - xy + 3x^2 = 0$

Differentiate both sides implicitly

$2yy' - (y + xy') + 6x = 0 \quad \Rightarrow \quad 2yy' - y - xy' + 6x = 0$

$\Rightarrow \quad 2yy' - xy' = y - 6x \quad \Rightarrow \quad (2y - x)y' = y - 6x$

$\Rightarrow \quad \dfrac{dy}{dx} = y' = \dfrac{y - 6x}{2y - x}$

Example 2 : Let $F(x, y, z) = x^2y + z^2 + \sin(xyz) = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution :

$F_x = 2xy + yz \cos(xyz)$

$F_y = x^2 + xz \cos(xyz)$

$F_z = 2z + xy \cos(xyz)$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xy + yz \cos(xyz)}{2z + xy \cos(xyz)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{x^2 + xz \cos(xyz)}{2z + xy \cos(xyz)}$

CHAPTER SEVEN

DIFFERENTIAL EQUATIONS

1. Definition of a differential equation

2. Separable Differential equations

3. First-order linear differential equations

1. Definition of a differential equation

Definition : An equation that involves x , y , y' , y'' , $\cdots$, $y^{(n)}$ for a function $y(x)$ with n^{th} derivative $y^{(n)}$ of y with respect to x is an ordinary differential Equation of order n.

Examples :

1. $y' = x^2 + 5$ is a differential equation of order 1.

2. $y'' + x\left(y'\right)^4 - y = x$ is a differential equation of order 2

3. $\left(y^{(4)}\right)^3 + x^2\ y'' = 2x$ is a differnetial equation of order 4

$y = y(x)$ is called a **solution** of a differential equation if $y = y(x)$ satisfies that differential equation.
Consider the differential equation $y' = 6x + 4$, then $y = 3x^2 + 4x + c$ is the **general solution** of that differential equation.

If an **initial condition** was added to the differential equation to assign a certain vallue for c then $y = y(x)$ is called the **particular solution** of the differential equation .
Consider the differential equation $y' = 6x + 4$ with the initial condition $y(0) = 2$, $y = 3x^2 + 4x + c$ is the general solution of the differential equation ,
$y(0) = 2\ \Rightarrow\ 3(0)^2 + 4 \times 0 + c = 2\ \Rightarrow\ c = 2$, hence $y = 3x^2 + 4x + 2$ is the particular solution of the differential equation.

2. Separable Differential equations

The separable differential equation has the form $M(x) + N(y)\, y' = 0$.
where $M(x)$ and $N(y)$ are continuous functions.

To solve the separable differential equation :

1. Write it as $M(x)\, dx + N(y)\, dy = 0 \implies N(y)\, dy = -M(x)\, dx$.

2. Integrate the left-hand side with respect to y and the right-hand side with respect to x

$$\int N(y)\, dy = -\int M(x)\, dx$$

Example 1 : Solve the differential equation $y' + y^3 e^x = 0$.
Solution :

$$y' + y^3 e^x = 0 \implies \frac{dy}{dx} = -y^3 e^x \ ,$$

$$\implies \quad -\frac{1}{y^3}\, dy = e^x\, dx \implies \quad -y^{-3}\, dy = e^x\, dx$$

$$\implies \quad -\int y^{-3}\, dy = \int e^x\, dx \implies \quad -\frac{y^{-2}}{-2} = e^x + c$$

$$\implies \quad \frac{1}{2y^2} = e^x + c \implies \quad \frac{1}{y^2} = 2\,(e^x + c)$$

$$\implies \quad y^2 = \frac{1}{2\,(e^x + c)} \implies \quad y = \sqrt{\frac{1}{2\,(e^x + c)}}$$

Example 2 : Solve the differential equation $\dfrac{dy}{dx} = y^2 e^x$, $y(0) = 1$.
Solution :

$$\frac{dy}{dx} = y^2 e^x \implies \frac{1}{y^2}\, dy = e^x\, dx$$

$$\implies \quad y^{-2}\, dy = e^x\, dx \implies \quad \int y^{-2}\, dy = \int e^x\, dx$$

$$\implies \quad \frac{y^{-1}}{-1} = e^x + c \implies \quad y = \frac{-1}{e^x + c}$$

Using the initial condition $y(0) = 1 \implies 1 = \dfrac{-1}{e^0 + c}$

$$\implies \quad 1 = \frac{-1}{1 + c} \implies 1 + c = -1 \implies c = -2$$

The particular solution is $y = \dfrac{-1}{e^x - 2}$

Example 3 : Solve the differential equation $dy - \sin x(1 + y^2)dx = 0$.

Solution :

$$dy - \sin x(1 + y^2)dx = 0 \implies dy = \sin x(1 + y^2)dx$$

$$\implies \frac{1}{1 + y^2} \, dy = \sin x \, dx \implies \int \frac{1}{1 + y^2} \, dy = \int \sin x \, dx$$

$$\implies \tan^{-1} y = -\cos x + c \implies y = \tan(-\cos x + c)$$

Example 4 : Solve the differential equation $e^{-y} \sin x - y' \cos^2 x = 0$.

Solution :

$$e^{-y} \sin x - y' \cos^2 x = 0 \implies -\cos^2 x \frac{dy}{dx} = -e^{-y} \sin x$$

$$\implies \frac{1}{e^{-y}} \, dy = \frac{-\sin x}{-\cos^2 x} \, dx \implies e^y \, dy = \frac{1}{\cos x} \frac{\sin x}{\cos x} \, dx$$

$$\implies e^y \, dy = \sec x \tan x \, dx \implies \int e^y \, dy = \int \sec x \tan x \, dx$$

$$\implies e^y = \sec x + c \implies y = \ln|\sec x + c|$$

Example 5 : Solve the differential equation $y' = 1 - y + x^2 - yx^2$.

Solution :

$$y' = 1 - y + x^2 - yx^2 \implies \frac{dy}{dx} = 1 - y + x^2(1 - y)$$

$$\implies \frac{dy}{dx} = (1 - y)(1 + x^2) \implies \frac{1}{1 - y} \, dy = (1 + x^2) \, dx$$

$$\implies \int \frac{1}{1 - y} \, dy = \int (1 + x^2) \, dx \implies -\int \frac{-1}{1 - y} \, dy = \int (1 + x^2) \, dx$$

$$\implies -\ln|1 - y| = x + \frac{x^3}{3} + c \implies \ln|1 - y| = -x - \frac{x^3}{3} - c$$

$$\implies 1 - y = e^{-x - \frac{x^3}{3} - c} \implies y = 1 - e^{-x - \frac{x^3}{3} - c}$$

3. First-order linear differential equations

The first-order linear differential equation has the form $y' + P(x)\,y = Q(x)$, where $P(x)$ and $Q(x)$ are continuous functions of x

To solve the first-order linear differential equation :

1. Compute the integrating factor $u(x) = e^{\int P(x)\ dx}$

2. The general solution of the first-order linear differential equation is

   $$y(x) = \frac{1}{u(x)} \int u(x)\,Q(x)\ dx$$

Example 1 : Solve the differential equation $x\dfrac{dy}{dx} + y = x^2 + 1$.

Solution :

$$x\frac{dy}{dx} + y = x^2 + 1 \implies y' + \left(\frac{1}{x}\right)y = \frac{x^2 + 1}{x}$$

$$\implies y' + \left(\frac{1}{x}\right)y = x + \frac{1}{x}$$

$$P(x) = \frac{1}{x} \text{ and } Q(x) = x + \frac{1}{x}$$

The integrating factor is $u(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

The general solution is $y = \dfrac{1}{x} \displaystyle\int x\left(x + \frac{1}{x}\right)\ dx$

$$y = \frac{1}{x} \int (x^2 + 1)\ dx = \frac{1}{x}\left(\frac{x^3}{3} + x + c\right) = \frac{x^2}{3} + 1 + \frac{c}{x}$$

Example 2 : Solve the differential equation $y' - \dfrac{2}{x}y = x^2 e^x$, $y(1) = e$.

Solution :

$$P(x) = -\frac{2}{x} \text{ and } Q(x) = x^2 e^x$$

The integrating factor is

$$u(x) = e^{\int -\frac{2}{x}dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

The general solution is $y = \dfrac{1}{x^{-2}} \displaystyle\int x^{-2}x^2 e^x dx$

$$y = x^2 \int e^x dx = x^2(e^x + c) = x^2 e^x + cx^2$$

Using the initial condition $y(1) = e$

$$y(1) = e \implies e = (1)^2 e^1 + c\,(1)^2 \implies e = e + c \implies c = 0$$

The particular solution is $y = x^2 e^x$

Example 3 : Solve the differential equation $y' + y = \cos(e^x)$
Solution :

$P(x) = 1$ and $Q(x) = \cos(e^x)$

The integrating factor is $u(x) = e^{\int 1\ dx} = e^x$

The general solution is $y = \dfrac{1}{e^x} \displaystyle\int e^x \cos(e^x)\ dx$

$$y = e^{-x} \int \cos(e^x) e^x\ dx = e^{-x}\left(\sin(e^x) + c\right) = e^{-x}\sin(e^x) + ce^{-x}$$

Example 4 : Solve the differential equation $xy' - 3y = x^2$
Solution :

$$xy' - 3y = x^2 \implies y' - \dfrac{3}{x}y = x$$

$$P(x) = -\dfrac{3}{x} \text{ and } Q(x) = x$$

The integrating factor is

$$u(x) = e^{\int -\frac{3}{x} dx} = e^{-3\int \frac{1}{x} dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

The general solution is $y = \dfrac{1}{x^{-3}} \displaystyle\int x^{-3} x\ dx$

$$y = x^3 \int x^{-2} dx = x^3 \left(\dfrac{x^{-1}}{-1} + c\right)$$

$$y = x^3 \left(-\dfrac{1}{x} + c\right) = -x^2 + cx^3$$