التكامل الثنائي

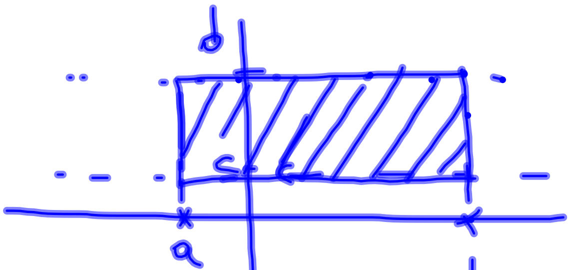
تعريف: ليكن $f(x,y) \geq 0$ لكل $(x,y) \in R$

$$I = \iint_R f(x,y)\, dA$$ هو حجم المنطقة

المحدودة بالدالة $f(x,y)$ فوق سطح R

تعريف
(1)

$$R = [a,b] \times [c,d]$$



$$I = \iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

مثال 1 ص 163

احسب التكامل $I = \int_1^2 \int_{-1}^1 (2y + 4y^2x)\,dx\,dy$

الحل: $R = [-1, 1] \times [1, 2]$

$$I = \int_1^2 \left(\int_{-1}^1 (2y + 4y^2x)\,dx \right) dy$$

$$= \int_1^2 2yx + 2y^2x^2 \Big]_{-1}^1 \, dy$$

$$= \int_1^2 \left((2y + 2y^2) - \left(2y(-1) + 2y^2(-1)^2 \right) \right) dy$$

$$= \int_1^2 4y\,dy = 2y^2 \Big]_1^2 = 2(2^2) - 2(1^2) = 6$$

مثال 2 ص 163

احسب التكامل

$$I = \int_{-1}^{1}\int_{1}^{2} (2y + 4y^2x)\,dy\,dx$$

$$I = \int_{-1}^{1}\left(\int_{1}^{2}(2y+4y^2x)\,dy\right)dx$$

$$= \int_{-1}^{1} y^2 + \frac{4}{3}y^3x \Big]_{1}^{2} dx$$

$$= \int_{-1}^{1}\left(\left(2^2+\frac{4}{3}2^3\cdot x\right)-\left(1^2+\frac{4}{3}1^3x\right)\right)dx$$

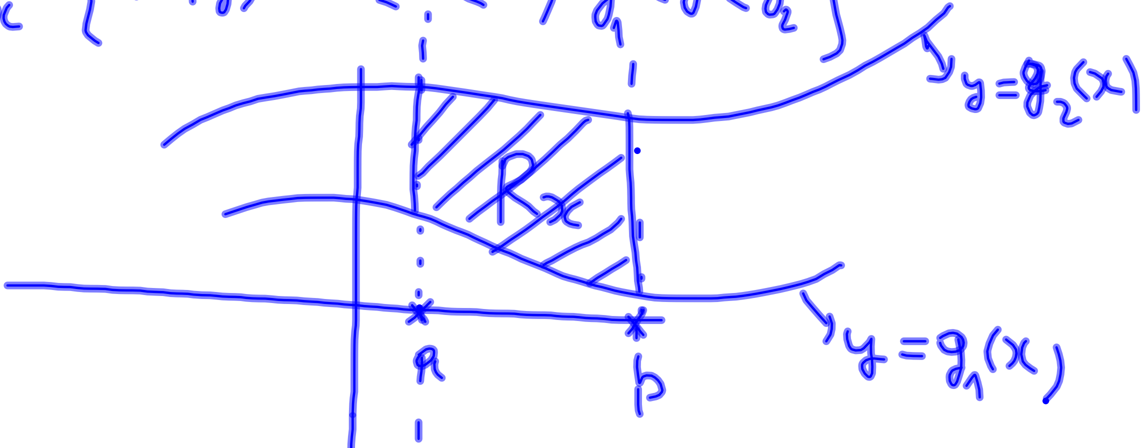
$$= \int_{-1}^{1}\left(3+\frac{4}{3}7x\right)dx$$

$$= 3x + \frac{28}{3}\frac{x^2}{2}\Big]_{-1}^{1} = \left(3+\frac{14}{3}\right)-\left(-3+\frac{14}{3}\right) = 6$$

$$\boxed{I = 6}$$

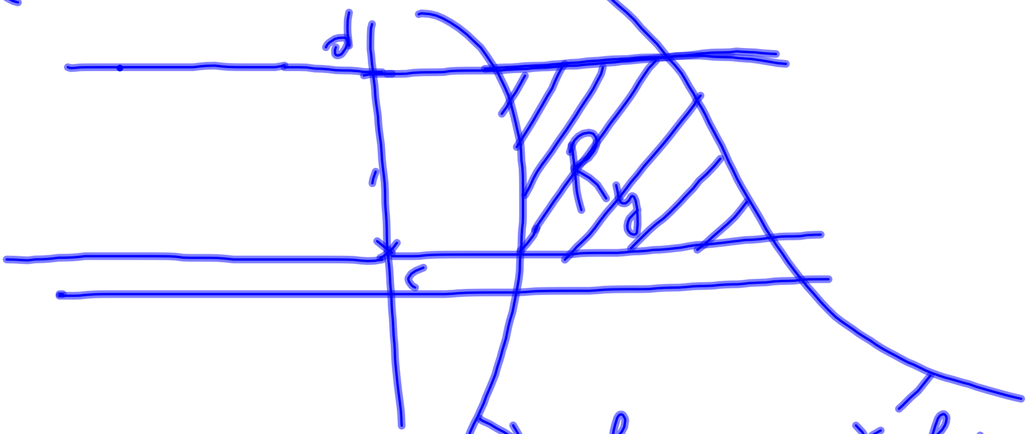
تعريف

$$R_x = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \quad (1)$$



$$\boxed{I = \iint_{R_x} f(x,y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx}$$

$$R_y = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$



$$\boxed{I = \iint_{R_y} f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy}$$

مبرهنة

إذا كان المجال R بكتب

$$R=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(y)\}$$

$$=\{(x,y): c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$$

فإن

$$I=\iint_R f(x,y)\,dA$$

$$=\int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$$

$$=\int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

مثال 3 ص 164

$$I=\int_0^1\int_x^{2x}(2x+3y^2)\,dy\,dx$$ احسب التكامل

الحل:

6

$$I=\int_0^1\left(\int_x^{2x}(2x+3y^2)\,dy\right)dx$$

$$=\int_0^1 \left[2xy+y^3\right]_x^{2x} dx$$

$$=\int_0^1\Big(\big(2x(2x)+(2x)^3\big)-\big(2x\,x+x^3\big)\Big)\,dx$$

$$=\int_0^1\left(4x^2+8x^3-2x^2-x^3\right)dx$$

$$=\int_0^1\left(2x^2+7x^3\right)dx$$

$$=\frac{2x^3}{3}+\frac{7}{4}x^4\Big]_0^1=\frac{2}{3}+\frac{7}{4}=\frac{29}{12}$$

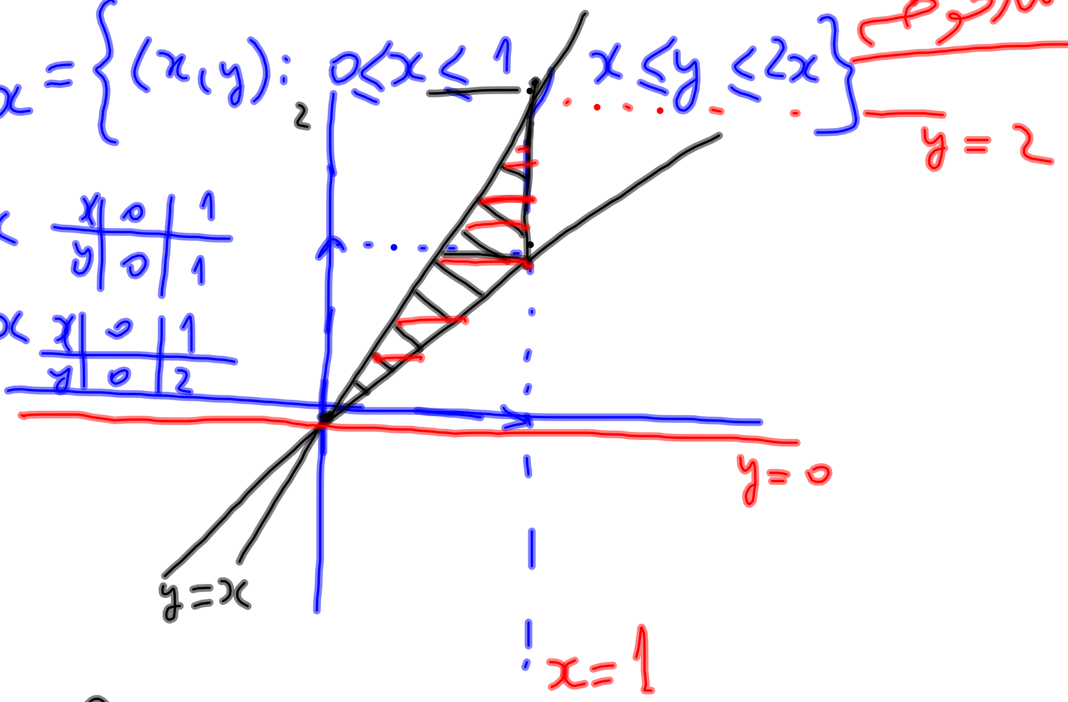
$$R_x = \left\{ (x,y) : 0 \le x \le 1 ,\ x \le y \le 2x \right\}$$ ملحوظة

$y = x$

x	0	1
y	0	1

$y = 2x$

x	0	1
y	0	2



$$R_x = R_1 \cup R_2$$

$$R_1 = \left\{ (x,y) : 0 \le y \le 1 ,\ \frac{y}{2} \le x \le y \right\}$$

$$R_2 = \left\{ (x,y) : 1 \le y \le 2 ,\ \frac{y}{2} \le x \le 1 \right\}$$

$$I = \iint_{R_1} + \iint_{R_2}$$

$$I = \int_0^1 \int_x^{2x} (2x+3y^2)\,dy\,dx$$

$$= \int_0^1 \left(\int_{\frac{y}{2}}^{y} (2x+3y^2)\,dx \right) dy + \int_1^2 \left(\int_{\frac{y}{2}}^{1} (2x+3y^2)\,dx \right) dy$$

$$= \int_0^1 x^2+3y^2x \Big]_{\frac{y}{2}}^{y} dy + \int_1^2 x^2+3y^2x \Big]_{\frac{y}{2}}^{1} dy$$

$$= \int_0^1 \left(\left(y^2+3y^2y\right) - \left(\left(\frac{y}{2}\right)^2 + 3y^2\left(\frac{y}{2}\right) \right) \right) dy + \int_1^2 \left(\left(1^2+3y^2\cdot 1\right) - \left(\left(\frac{y}{2}\right)^2 + 3y^2\left(\frac{y}{2}\right) \right) \right) dy$$

$$= \int_0^1 \left(\frac{3}{4}y^2 + \frac{3}{2}y^3 \right) dy + \int_1^2 \left(1 + \frac{11}{4}y^2 - \frac{3}{4}y^3 \right) dy$$

$$= \frac{1}{4}y^3 + \frac{3}{8}y^4 \Big]_0^1 + y + \frac{11}{12}y^3 - \frac{3}{16}y^4 \Big]_1^2$$

$$= \frac{1}{4} + \frac{3}{8} + 2 + \frac{11}{12}\cdot 2^3 - \frac{3}{16}\cdot 2^4 - 1 - \frac{11}{12} + \frac{3}{16}$$

$$= \frac{5}{4} + \frac{3}{8} + \frac{22}{3} - 3 - \frac{11}{12} + \frac{3}{16}$$

$$= \frac{15+88-11-36}{12} + \frac{9}{16} = \frac{56}{12} + \frac{9}{16} = \frac{14}{3} + \frac{9}{16}$$

$$= \frac{16\cdot 14 + 3\cdot 9}{48} = \frac{251}{48} =$$

مثال 4 ص 164

احسب التكامل:

$$I = \int_0^{\frac{\pi}{2}} \int_0^{y} 2x \sin y^3 \, dx \, dy$$

الحل:

$$I = \int_0^{\frac{\pi}{2}} \left(\int_0^{y} 2x \sin y^3 \, dx \right) dy$$

$$= \int_0^{\frac{\pi}{2}} x^2 \sin y^3 \Big]_0^{y} \, dy$$

$$= \frac{1}{3} \int_0^{\pi/2} 3y^2 \sin y^3 \, dy$$

$$(\cos u)' = -u' \sin u$$

$$= \frac{1}{3} \left(-\cos y^3 \right) \Big]_0^{\pi/2} = -\frac{1}{3} \cos \frac{\pi^3}{8} + \frac{1}{3}$$

مثال ٣ ص ١٦٥

$$I = \iint_R f(x,y)\,dA$$

حيث $R = \{(x,y) : -1 \le x \le 1,\ x^3 \le y \le x+1\}$, $f(x,y) = 3x + 2y$

الحل:

$$I = \int_{-1}^{1} \int_{x^3}^{x+1} (3x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} 3xy + y^2 \Big]_{x^3}^{x+1} dx$$

$$= \int_{-1}^{1} \left(\left[3x(x+1) + (x+1)^2\right] - \left[3x x^3 + (x^3)^2\right] \right) dx$$

$$= \int_{-1}^{1} \left(3x^2 + 3x + x^2 + 2x + 1 - 3x^4 - x^6\right) dx$$

$$= \int_{-1}^{1} \left(1 + 5x + 4x^2 - 3x^4 - x^6\right) dx = 2\int_0^1 (1 + 4x^2 - 3x^4 - x^6)\,dx$$

$$= x + \frac{5x^2}{2} + \frac{4}{3}x^3 - \frac{3x^5}{5} - \frac{x^7}{7} \Big]_{-1}^{1}$$

$$= \left(1 + \frac{5}{2}1 + \frac{4}{3}1 - \frac{3}{5}1 - \frac{1}{7}\right) - \left(-1 + \frac{5}{2}(-1)^2 + \frac{4}{3}(-1)^3 - \frac{3}{5}(-1)^5 - \frac{(-1)^7}{7}\right)$$

$$= 2 + \frac{8}{3} - \frac{6}{5} - \frac{2}{7} = \frac{210 + 8(35) - 6(21) - 30}{105}$$

$$= \frac{334}{105}$$

ملاحظة

دالة زوجية: $g(-x) = g(x)$

$$\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx \quad : (a > 0)$$

دالة فردية: $g(-x) = -g(x)$

$$\int_{-a}^{a} g(x)\,dx = 0$$

مثال 6 ص 165

$f(x,y)=x^3+4y$ ، R في المنطقة المستوية المحدودة بالمنحنين

$y=x^2$, $y=2x$

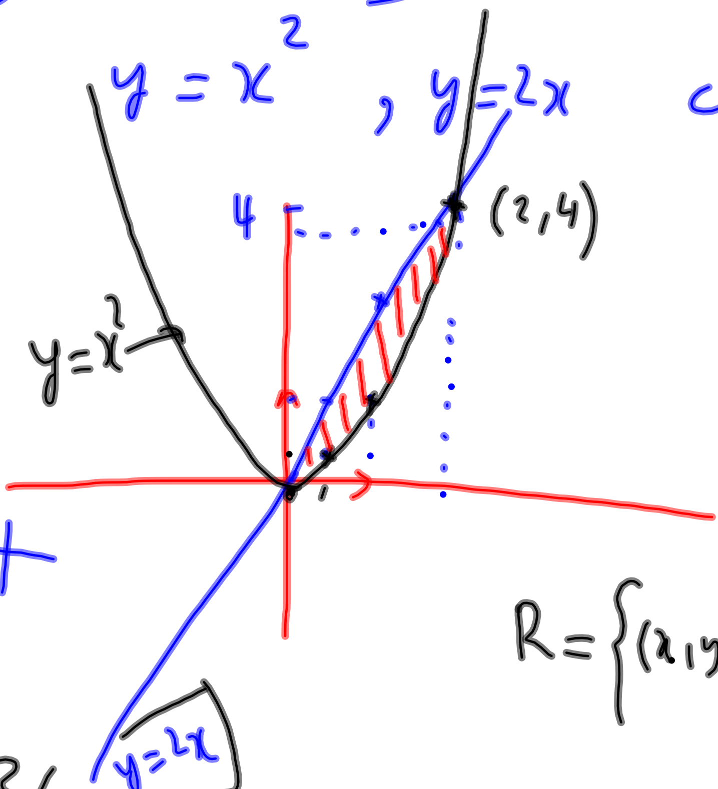
الحل:

$y=2x$

x	0	1
y	0	2

$y=x^2$

x	0	$\frac{1}{2}$	1	2
y	0	$\frac{1}{4}$	1	4



$$R=\{(x,y): 0\le x\le 2,\ x^2\le y\le 2x\}$$

$$I=\int_0^2\left(\int_{x^2}^{2x}(x^3+4y)\,dy\right)dx$$

$$=\int_0^2 x^3y+2y^2\Big]_{x^2}^{2x}dx=\int_0^2\left(\left(2x^3x+2(2x)^2\right)-\left(x^3x^2+2(x^2)^2\right)\right)dx$$

$$=\int_0^2\left(2x^4+8x^2-x^5-2x^4\right)dx=\int_0^2\left(8x^2-x^5\right)dx$$

$$=\frac{8}{3}x^3-\frac{x^6}{6}\Big]_0^2=\frac{8}{3}2^3-\frac{2^6}{6}=\frac{64-32}{3}=\frac{32}{3}$$

مثال 7 1668

طريقة ثانية

$R=\{(x,y): 0\le y\le 4,\ \frac{y}{2}\le x\le\sqrt{y}\}$

$I=\iint_R (x^3+4y)\,dA$

$=\int_0^4\left(\int_{\frac{y}{2}}^{\sqrt{y}}(x^3+4y)\,dx\right)dy$

$=\int_0^4 \frac{x^4}{4}+4xy\Big|_{\frac{y}{2}}^{\sqrt{y}}\,dy=\int_0^4\left(\frac{y^2}{4}+4\sqrt{y}\,y-\frac{(\frac{y}{2})^4}{4}-4\frac{y}{2}y\right)dy$

$=\int_0^4\left(-\frac{7}{4}y^2+4y^{\frac{3}{2}}-\frac{y^4}{64}\right)dy$

$=-\frac{7}{12}y^3+\frac{4}{\frac{5}{2}}y^{\frac{5}{2}}-\frac{y^5}{5\cdot 64}\Big|_0^4$

$=-\frac{7}{12}\cdot 4^3+\frac{8}{5}\,4^{\frac{5}{2}}-\frac{4^5}{5\cdot 64}$

$=-\frac{112}{3}+\frac{8}{5}\cdot 2^5-\frac{2^4}{5}=\frac{-112}{3}+\frac{\overset{3}{\cancel{15}}\cdot 2^4}{5}$

$=\frac{-112+3\times 48}{3}=\frac{32}{3}$

مثال 9 ص 169

احسب التكامل الثنائي: $I = \int_0^4 \int_{\sqrt{y}}^2 y\cos(x^5)\,dx\,dy$

الحل: ملاحظة:

$$I = \int_0^4 \left(\int_{\sqrt{y}}^2 y\cos(x^5)\,dx \right) dy$$

التكامل بهذا الترتيب لا يمكن

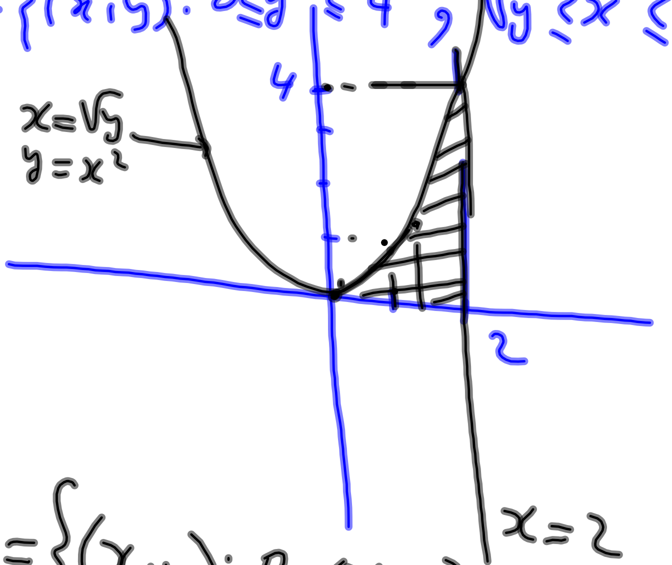
تذكير

$$\int 5x^4 \cos x^5\,dx = \sin x^5 + C$$

$R = \{(x,y) : 0 \le y \le 4 ,\ \sqrt{y} \le x \le 2\}$

$x = \sqrt{y} \rightarrow y = x^2$

$x = 2$



$R = \{(x,y) : 0 \le x \le 2 ,\ 0 \le y \le x^2\}$

$$I = \int_0^2 \left(\int_0^{x^2} y\cos x^5\,dy \right) dx$$

$$= \int_0^2 \frac{y^2}{2} \cos x^5 \Big|_0^{x^2} dx$$

$$= \frac{1}{5} \int_0^2 5\,\frac{x^4}{2} \cos x^5\,dx = \frac{1}{10} \sin x^5 \Big|_0^2 = \frac{1}{10}\left(\sin(32)\right)$$

مثال ١٠ ص 169

$$I = \int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy\, dx$$

الحل: ملحوظة:

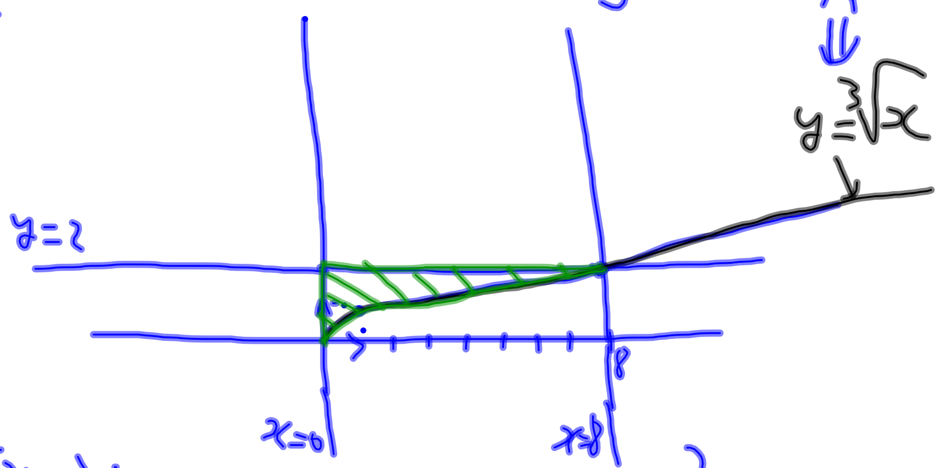
$$\int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy$$

صعب كثيراً وجود التكامل السابق.

$$R = \{(x,y) : 0 \le x \le 8 ,\ \sqrt[3]{x} \le y \le 2\}$$

$y = \sqrt[3]{x}$

$x = y^3 \Leftrightarrow y = \sqrt[3]{x}$



$$R = \{(x,y) : 0 \le y \le 2 ,\ 0 \le x \le y^3\}$$

$$I = \int_0^2 \left(\int_0^{y^3} \frac{1}{y^4+1}\, dx \right) dy = \int_0^2 \frac{1}{y^4+1}\, x \Big]_0^{y^3} dy$$

$$= \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\, dy = \frac{1}{4} \ln(y^4+1)\Big]_0^2$$

$$= \frac{1}{4}\left(\ln(17) - \ln(1)\right) = \frac{1}{4}\ln(17).$$

التكامل الثنائي باستعمال الاحداثيات القطبية

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\boxed{x^2+y^2=r^2} \Rightarrow r=\sqrt{x^2+y^2}$$

$$x \neq 0 \;,\; \frac{y}{x} = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right) = \arctan\left(\frac{y}{x}\right)}$$

مثال ١ ص ١٨١

أوجد المنحنى

إذا كانت الإحداثيات القطبية هي $r = 4\sin\theta$

الحل:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \Longleftrightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

!!

لدينا $r = 4\sin\theta$

ولدينا $\sin\theta = \frac{y}{r}$

فإن $r = 4\frac{y}{r} \Rightarrow r^2 = 4y$

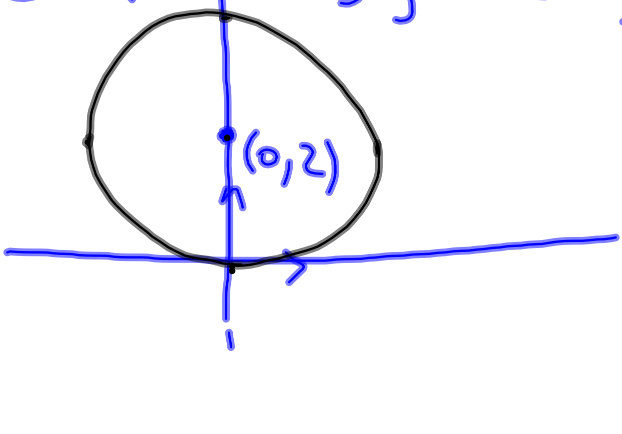
فإن بالإحداثيات الديكارتية

$$x^2+y^2 = 4y$$

وبالتالي: $x^2+y^2-4y+4 = 4$

$$C: \; x^2+(y-2)^2 = 2^2$$

الدائرة C مركزها $(0,2)$ ونصف قطرها 2



مثال ص ١٨١ , ١٨٢

أوجد الصيغة القطبية للمعادلة $x^2-y^2=16$

الحل:

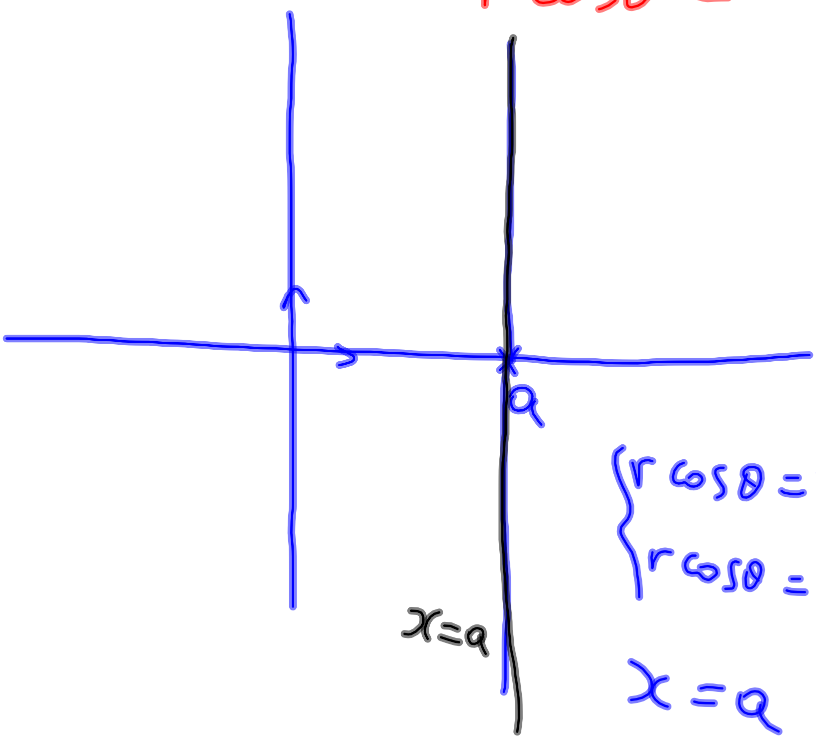
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{aligned} x^2-y^2 &= (r\cos\theta)^2-(r\sin\theta)^2 \\ &= r^2(\cos^2\theta-\sin^2\theta) \\ &= r^2\cos(2\theta) \end{aligned}$$

$$\boxed{r^2\cos(2\theta)=16}$$

إذن

$r\cos\theta = a$ (أ)



$$\begin{cases} r\cos\theta = x \\ r\cos\theta = a > 0 \end{cases}$$ لدينا

$x = a$ إذن

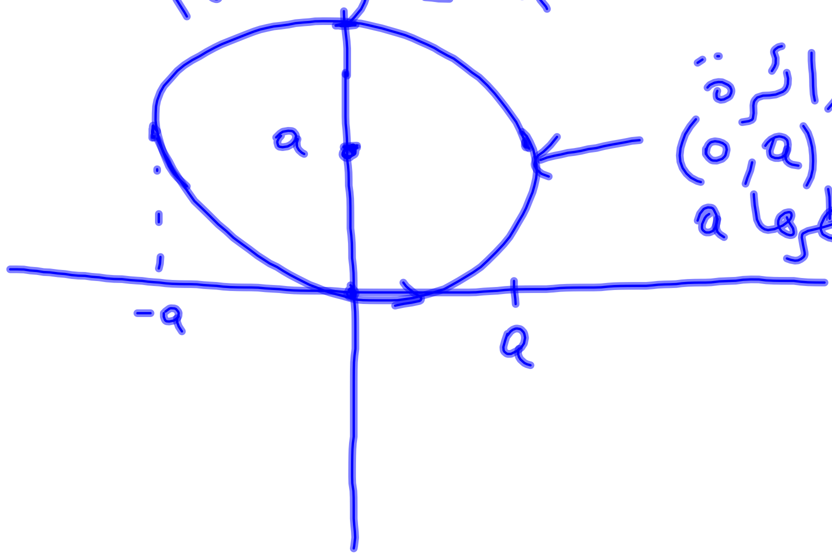
$r = -2a\sin\theta$ أو $r = 2a\sin\theta$ (2.)

$$r^2 = 2a\underbrace{r\sin\theta}$$

$$x^2 + y^2 = 2ay \; ; \quad a > 0$$

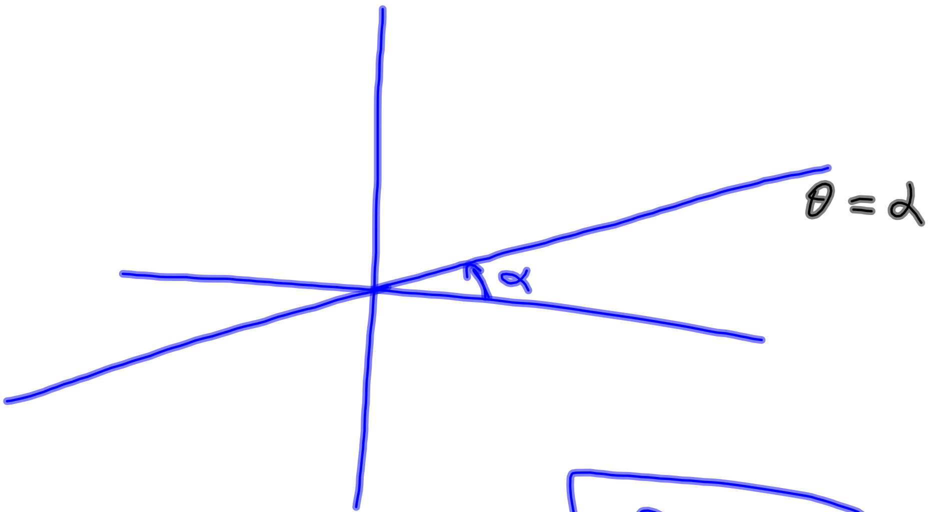
$$x^2 + \underbrace{y^2 - 2ay} = 0$$

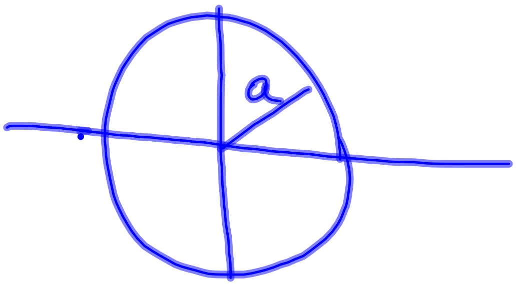
$$x^2 + (y - a)^2 = a^2$$



الدائرة
مركزها $(0,a)$
ونصف قطرها a

$\alpha \in \mathbb{R}_+$, $\theta = \alpha$ $(\theta$



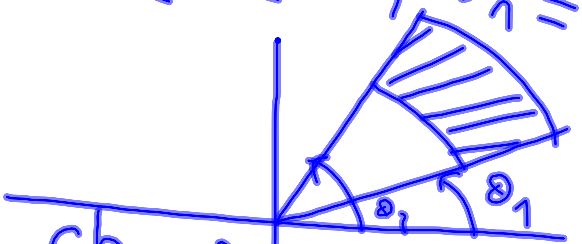


$\boxed{r = a}$ (θ)

التكامل الثنائي بالإحداثيات القطبية

مبرهنة

$$R = R_p = \{(r,\theta) ;\ a \le r \le b \ ;\ \theta_1 \le \theta \le \theta_2\}$$



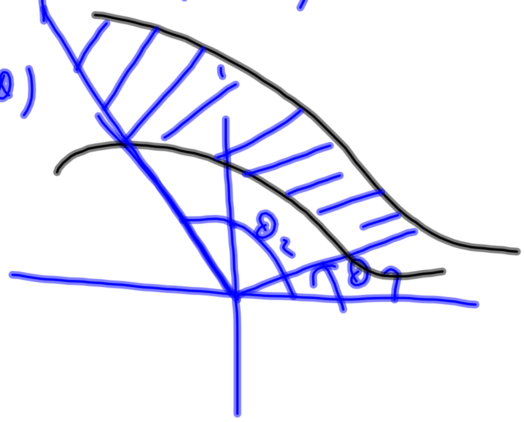
$$\iint_R f(x,y)\,dA = \int_a^b \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\,\boxed{r}\,d\theta\,dr$$

$$R = \{(r,\theta) : a \le r \le b,\ g_1(r) \le \theta \le g_2(r)\} \quad (2)$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$R = \{(r,\theta) : \theta_1 \le \theta \le \theta_2,\ h_1(\theta) \le r \le h_2(\theta)\} \quad (3)$$

$$\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



مثال 3 ص 186

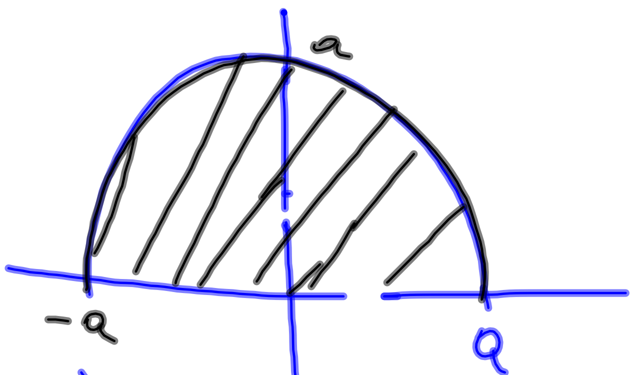
احسب التكامل

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

الحل:

$$R = \left\{ (x,y) : -a \le x \le a ,\ 0 \le y \le \sqrt{a^2-x^2} \right\}$$

$$y = \sqrt{a^2-x^2} \Rightarrow \begin{cases} y^2 + x^2 = a^2 \\ y \geq 0 \end{cases}$$



$$R = \left\{ (r,\theta) : 0 \le r \le a ,\ 0 \le \theta \le \pi \right\}$$

$$I = \int_0^a \int_0^{\pi} (r^2)^{\frac{3}{2}}\, r\, d\theta\, dr = \int_0^a \int_0^{\pi} r^4\, d\theta\, dr$$

$$= \int_0^a r^4\, \theta \Big]_0^{\pi} dr = \int_0^a r^4 \pi\, dr = \pi \frac{r^5}{5} \Big]_0^a$$

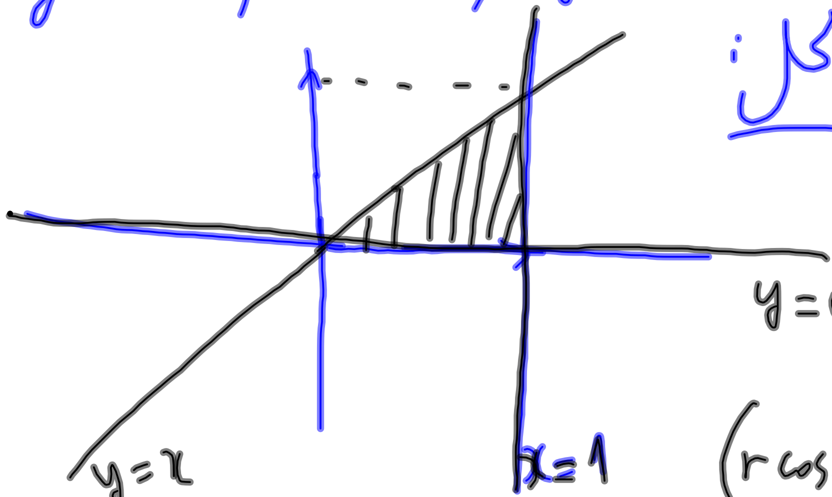
$$\boxed{I = \frac{\pi}{5} a^5}$$

مثال 4 ص 187

$$I = \iint_R \frac{dA}{(1+x^2+y^2)^{3/2}}$$

حيث R محدودة: $y = x$, $x = 1$, $y = 0$

الحل:



$$R = \left\{ (r,\theta) : 0 \le \theta \le \frac{\pi}{4} \; ; \; 0 \le r \le \frac{1}{\cos\theta} \right\}$$

$$\left(\begin{aligned} r\cos\theta &= 1 \\ r &= \frac{1}{\cos\theta} \end{aligned} \right.$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} \frac{1}{(1+r^2)^{3/2}} \, 2r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left. \frac{(1+r^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right|_0^{\frac{1}{\cos\theta}} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{\left(1+\left(\frac{1}{\cos\theta}\right)^2\right)^{-\frac{1}{2}}}{-\frac{1}{2}} + 2 \right) d\theta$$

$$= \int_0^{\pi/4} \left(1 - \frac{1}{\sqrt{\frac{\cos^2\theta + 1}{\cos^2\theta}}} \right) d\theta$$