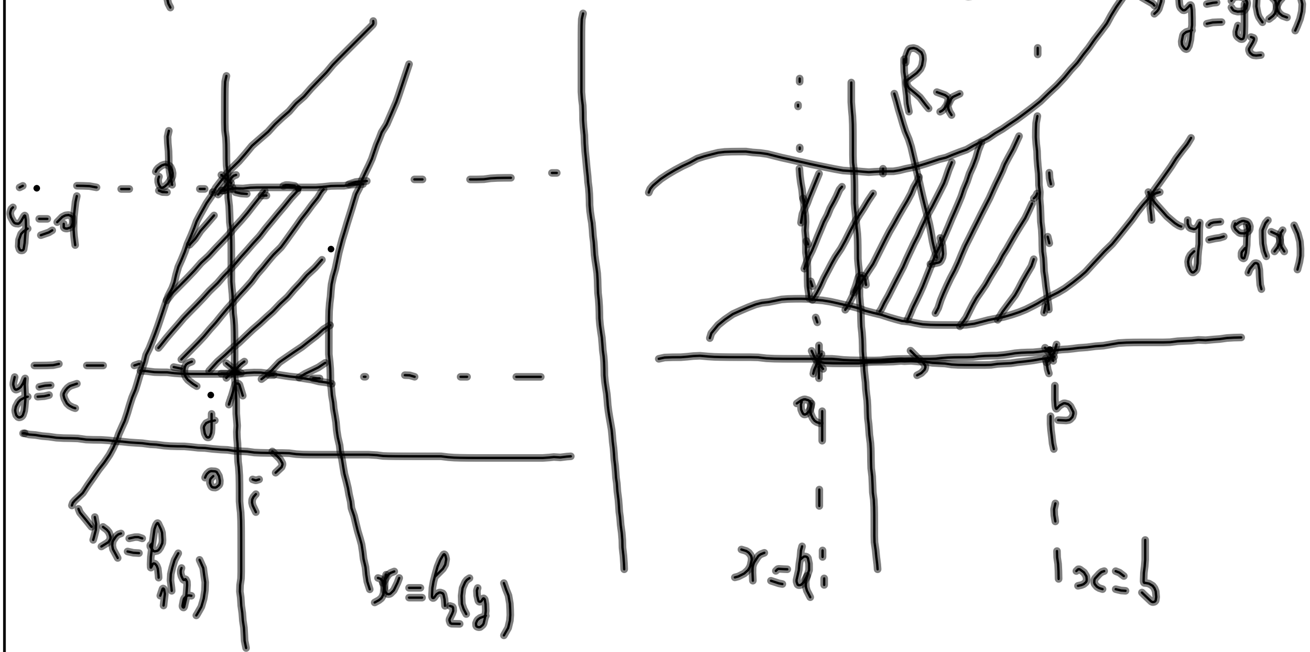
التكامل الثنائي

تعريف: ليكن $a,b \in \mathbb{R}$ ، $c,d \in \mathbb{R}$

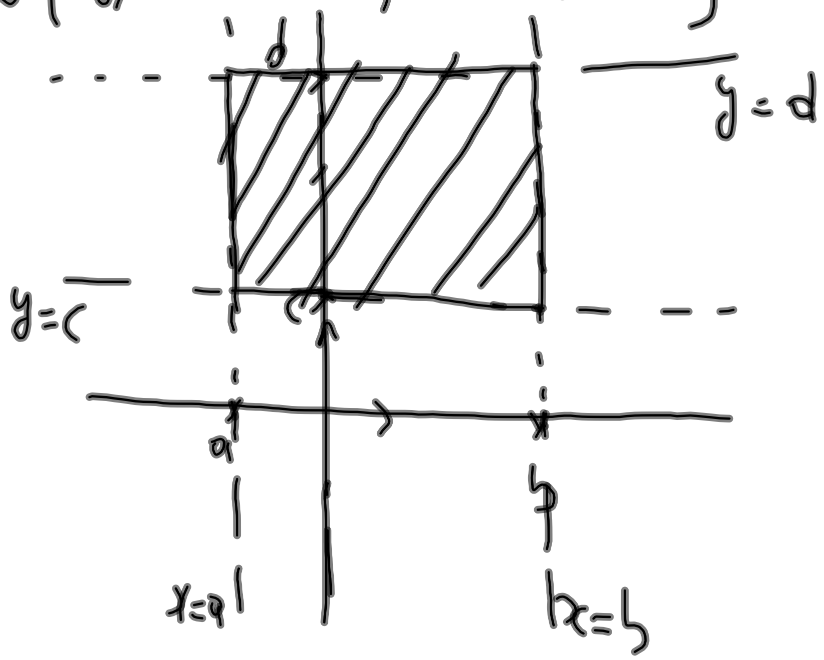
$$R_x = \{(x,y) : a \le x \le b \,,\, g_1(x) \le y \le g_2(x)\}$$

$$R_y = \{(x,y) : c \le y \le d \,;\, h_1(y) \le x \le h_2(y)\}$$



حالة خاصة:

$$[a,b]\times[c,d]=\{(x,y): a\le x\le b\ ,\ c\le y\le d\}$$



تعريف:

$$\int_a^b\int_c^d f(x,y)\,dA = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$\int_c^d\int_a^b f(x,y)\,dA = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

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مثال ١ ص 163 احسب التكامل:

$$I=\int_1^2\int_{-1}^1 (2y+4y^2x)\,dx\,dy$$

الحل:

$$I=\int_1^2\left[\int_{-1}^1 (2y+4y^2x)\,dx\right]dy$$

$$=\int_1^2 2yx+4y^2\frac{x^2}{2}\Big]_{-1}^1 dy=\int_1^2\left((2y+2y^2)-(2y(-1)+2y^2(-1)^2)\right)dy$$

$$=\int_1^2 4y\,dy=4\frac{y^2}{2}\Big]_1^2=2y^2\Big]_1^2=2(2^2)-2(1^2)=6$$

مثال ص 163 : احسب التكامل :

$$I=\int_{-1}^{1}\int_{1}^{2}(2y+4y^2x)\,dy\,dx$$

الحل :

$$I=\int_{-1}^{1}\left[\int_{1}^{2}(2y+4y^2x)\,dy\right]dx=\int_{-1}^{1}2\frac{y^2}{2}+4\frac{y^3}{3}x\Big]_{1}^{2}dx$$

$$=\int_{-1}^{1}\left(\left(2^2+\frac{4}{3}(2^3)x\right)-\left(1^2+\frac{4}{3}(1^3)x\right)\right)dx$$

$$=\int_{-1}^{1}\left(3+\frac{4}{3}\cdot 7\cdot x\right)dx=3x+\frac{4}{3}\cdot 7\frac{x^2}{2}\Big]_{-1}^{1}=3x+\frac{14}{3}x^2\Big]_{-1}^{1}$$

$$=\left(3(1)+\frac{14}{3}(1^2)\right)-\left(3(-1)+\frac{14}{3}(-1)^2\right)$$

$$=6$$

مبرهنة:

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$f(x,y)$: دالة في متغيرين متصلة بشكل النقاط: $[a,b]\times[c,d]$

تعريف:

$$\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx = \iint_{R_x} f(x,y)\,dA$$

$$\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy = \iint_{R_y} f(x,y)\,dA$$

تعريف: ليكن f دالة في متغيرين متصلة عند كل نقطة من R
$(R = R_x , R_y)$

$$\iint_{R_x} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

$$\iint_{R_y} f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

مثال 3 ص 164 : احسب التكامل:

$$I=\int_0^1\int_x^{2x}(2x+3y^2)\,dy\,dx$$

الحل:

$$I=\int_0^1\left[\int_x^{2x}(2x+3y^2)\,dy\right]dx=\int_0^1 \left.2xy+\cancel{3}\frac{y^3}{\cancel{3}}\right]_x^{2x}dx$$

$$=\int_0^1\Big(\big(2x(2x)+(2x)^3\big)-\big(2x.x+x^3\big)\Big)\,dx$$

$$=\int_0^1\left(4x^2+8x^3-2x^2-x^3\right)dx=\int_0^1\left(2x^2+7x^3\right)dx=2\frac{x^3}{3}+7\frac{x^4}{4}\Big|_0^1$$

$$=\left(\frac{2}{3}\,1^3+\frac{7}{4}\cdot1^4\right)-\left(\frac{2}{3}\cdot0^3+\frac{7}{4}\cdot0^4\right)=\frac{2}{3}+\frac{7}{4}=\frac{8+21}{12}=\frac{29}{12}$$

$0 \le x \le 1$

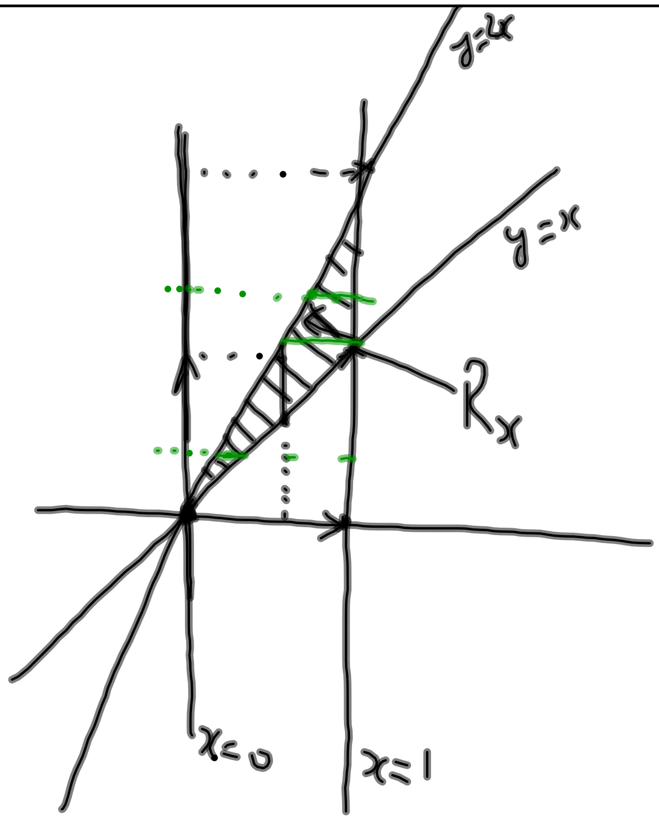
$x \le y \le 2x$

$y = x$

x	0	1
y	0	1

$y = 2x$

x	0	1
y	0	2



$$R_x = \{(x,y) : 0 \le x \le 1 ,\ x \le y \le 2x\}$$

مثال 4 ص 164

احسب التكامل:

$$I = \int_0^{\frac{\pi}{2}}\int_0^y 2x \sin y^3 \, dx \, dy$$

الحل:

$$I = \int_0^{\frac{\pi}{2}}\int_0^y 2x \sin y^3 \, dx \, dy = \int_0^{\frac{\pi}{2}}\left[\int_0^y 2x \sin y^3 \, dx\right] dy$$

$$= \int_0^{\frac{\pi}{2}} x^2 \sin y^3 \Big|_0^y \, dy = \int_0^{\frac{\pi}{2}} \left(y^2 \sin y^3 - \underbrace{0^2 \sin y^3}_{0}\right) dy$$

$$= -\frac{1}{3}\cos y^3 \Big|_0^{\frac{\pi}{2}} = -\frac{1}{3}\left(\cos\frac{\pi^3}{8} - 1\right)$$

$$\boxed{I = \frac{1}{3} - \frac{1}{3}\cos\frac{\pi^3}{8}}$$

تذكير

$$(\cos y^3)' = -3y^2 \sin y^3$$

<u>مبرهنة</u>: لتكن f دالة متصلة عند كل نقطة من R.

إذا كان $R = R_x = R_y$

فإن

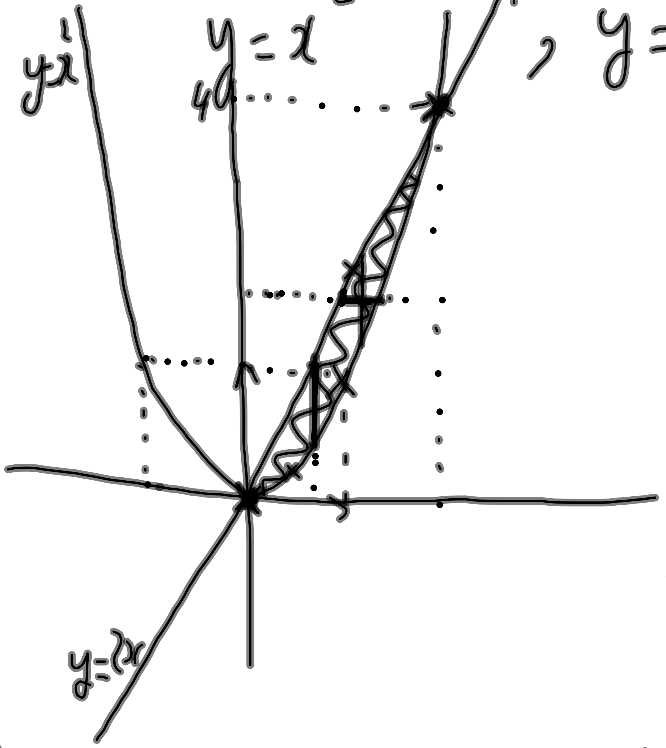
$$\iint_{R_x} f(x,y)\,dy\,dx = \iint_{R_y} f(x,y)\,dx\,dy$$

يعني:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

مثال ص 165 : احسب التكامل : $I = \iint_R f(x,y)\,dA$

حيث: $f(x,y) = x^3+4y$, R المنطقة المستوية المحدودة بالمنحنيين $y = x^2$, $y = 2x$



$y = x^2$

x	0	$\frac{1}{2}$	1	2
y	0	$\frac{1}{4}$	1	4

$y = 2x$

x	0	1	2
y	0	2	4

الحل :

$x^2 = 2x$

$\Leftrightarrow x^2 - 2x = 0$

$\Leftrightarrow x(x-2) = 0$

$\Leftrightarrow x = 0$, أ $x = 2$

$$R = R_x = \{(x,y) : 0 \le x \le 2 \ , \ x^2 \le y \le 2x\}$$

$$R = R_y = \{(x,y) : 0 \le y \le 4 \ , \ \frac{y}{2} \le x \le \sqrt{y}\}$$

الطريقة 1:

$$I = \iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\,dy\,dx$$

$$= \int_0^2 \left[\int_{x^2}^{2x} (x^3 + 4y)\,dy \right] dx$$

$$= \int_0^2 x^3 y + 2y^2 \Big]_{x^2}^{2x} dx = \int_0^2 \left[\left(x^3(2x) + 2(2x)^2\right) - \left(x^3 x^2 + 2(x^2)^2\right) \right] dx$$

$$I = \int_0^2 (8x^2 - x^5)\,dx = \frac{8x^3}{3} - \frac{x^6}{6} \Big]_0^2 = \frac{8}{3} 2^3 - \frac{2^6}{6}$$

$$= 64\left(\frac{1}{3} - \frac{1}{6}\right) = 64\left(\frac{2-1}{6}\right) = \frac{32}{3} \qquad \boxed{I = \frac{32}{3}}$$

الطريقة 2

$$I = \iint_R f(x,y)\,dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^3+4y)\,dx\,dy$$

$$= \int_0^4 \left[\int_{\frac{y}{2}}^{\sqrt{y}} (x^3+4y)\,dx \right] dy = \int_0^4 \frac{x^4}{4} + 4yx \Big]_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{\sqrt{y}^4}{4} + 4\cdot y \cdot \sqrt{y} - \frac{\left(\frac{y}{2}\right)^4}{4} - 4\cdot y\,\frac{y}{2} \right) dy$$

$$= \int_0^4 \left(\frac{y^2}{4} + 4y^{\frac{3}{2}} - \frac{y^4}{64} - 2y^2 \right) dy = \int_0^4 \left(4y^{\frac{3}{2}} - \frac{y^4}{64} - \frac{7}{4}y^2 \right) dy$$

$$= 4\cdot \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^5}{64\cdot 5} - \frac{7}{4}\cdot\frac{y^3}{3} \Big]_0^4 = \frac{8}{5}\,4^{\frac{5}{2}} - \frac{4^5}{64\cdot 5} - \frac{7\cdot 4^3}{4\cdot 3}$$

$$= \frac{8}{5}\cdot 2^5 - \frac{4^2}{5} - \frac{7\cdot 4^2}{3} = \frac{4^4 - 4^2}{5} - \frac{7\cdot 4^2}{3} = \frac{4^2(4^2-1)}{5} - \frac{7}{3}4^2$$

$$= 4^2\cdot 3 - \frac{7\cdot 4^2}{3} = \frac{4^2}{3}(9-7) = \boxed{\frac{32}{3} = I}$$

مثال ٢ ص ١٥٦: احسب التكامل $I = \iint_R f(x,y)\,dA$

حيث أن: $f(x,y) = 3x + 2y$: $R_x = R = \{(x,y) : -1 \le x \le 1,\ x^3 \le y \le x+1\}$

الحل:

$$I = \iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{x^3}^{x+1} (3x+2y)\,dy\,dx$$

$$= \int_{-1}^{1}\left[\int_{x^3}^{x+1}(3x+2y)\,dy\right]dx = \int_{-1}^{1} 3xy + y^2\Big]_{x^3}^{x+1} dx$$

$$= \int_{-1}^{1}\left(3x(x+1) + (x+1)^2 - 3x.x^3 - (x^3)^2\right)dx$$

$$= \int_{-1}^{1}\left(3x^2 + 3x + x^2 + 2x + 1 - 3x^4 - x^6\right)dx$$

$$= \int_{-1}^{1}\left(1 + 5x + 4x^2 - 3x^4 - x^6\right)dx = x + \frac{5}{2}x^2 + \frac{4}{3}x^3 - \frac{3}{5}x^5 - \frac{x^7}{7}\Big]_{-1}^{1}$$

$$= \left(1 + \frac{5}{2}1^2 + \frac{4}{3}(1^3) - \frac{3}{5}1^5 - \frac{1^7}{7}\right) - \left(-1 + \frac{5}{2}(-1)^2 + \frac{4}{3}(-1)^3 - \frac{3}{5}(-1)^5 - \frac{(-1)^7}{7}\right)$$

$$= 2 + 2\frac{4}{3} - 2\frac{3}{5} - 2\frac{1}{7} = 2\left(1 + \frac{4}{3} - \frac{3}{5} - \frac{1}{7}\right) = 2\,\frac{105 + 4.35 - 3.21 - 15}{105}$$

$$\boxed{I = \frac{334}{105}}$$

تذكير

(1) $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(2) $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0$$

$R = \{(x,y) : -1 \le x \le 1 ,\ x^3 \le y \le x+1\}$

$y = x^3$

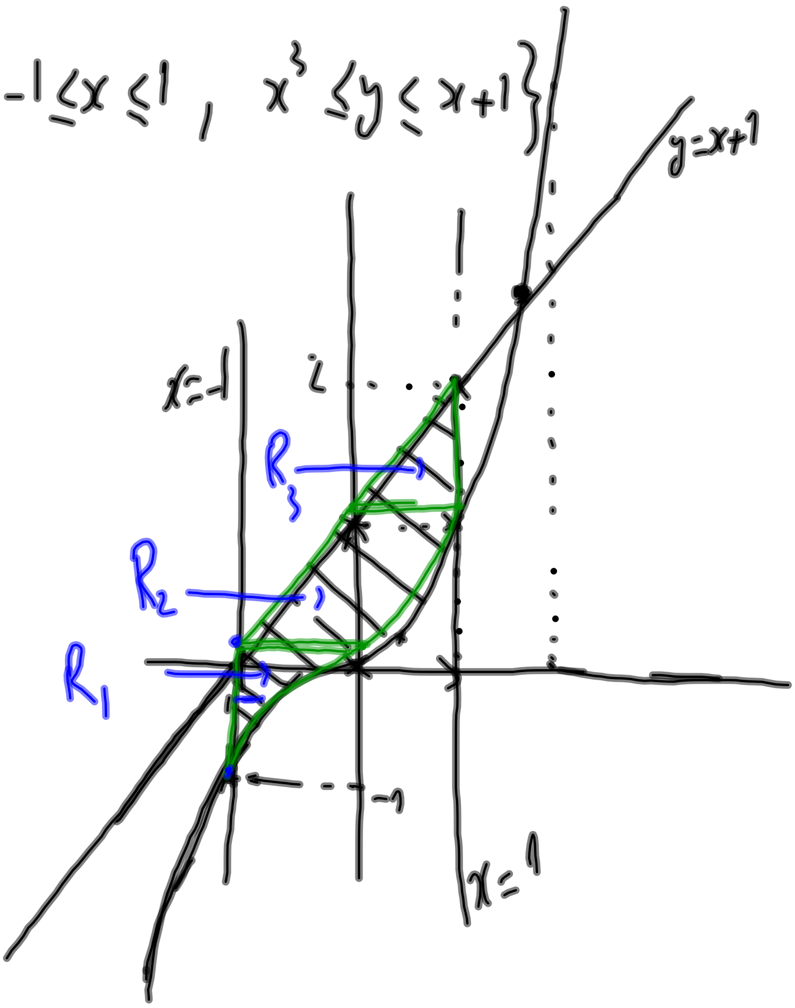
x	0	$\frac{1}{2}$	1	2
y	0	$\frac{1}{8}$	1	8

$y = x+1$

x	0	1
y	1	2

$x^3 = x+1$

$\Leftrightarrow x^3 - x = 1$



$R_y = R_1 \cup R_2 \cup R_3$

$= \{(x,y) : -1 \le y \le 0 ,\ -1 \le x \le \sqrt[3]{y} = y^{\frac{1}{3}}\} \cup \{(x,y) : 0 \le y \le 1 ,\ y-1 \le x \le y^{\frac{1}{3}}\}$

$\cup \{(x,y) : 1 \le y \le 2 ,\ y-1 \le x \le 1\}$

مثال 8 ص 167 : R المنطقة المحدودة بالمنحنيات

$y=\sqrt{x}$, $y=0$, $y=\sqrt{3x-18}$. اكتب $\iint_R f(x,y)\,dA$ كتكامل متعاقب

الحل :

$\sqrt{x}=\sqrt{3x-18}$

$\Rightarrow x=3x-18$

$\Rightarrow 2x=18$

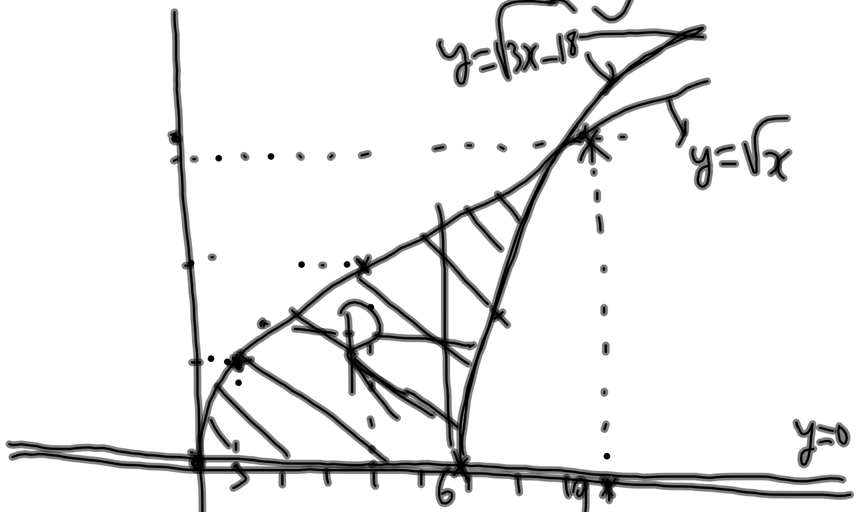
$\Rightarrow \boxed{x=9}$

$y=\sqrt{x}$

x	0	1	4	9
y	0	1	2	3

$y=\sqrt{3x-18}$

x	6	7	9
y	0	$\sqrt{3}$	3



$R_y=\left\{(x,y): 0\le y\le 3 ,\ y^2\le x\le \frac{y^2}{3}+6\right\}$

$R_x=R_1\cup R_2=\left\{(x,y): 0\le x\le 6 ,\ 0\le y\le \sqrt{x}\right\}$
$\cup\left\{(x,y): 6\le x\le 9 ,\ \sqrt{3x-18}\le y\le \sqrt{x}\right\}$

$\circ\ y=\sqrt{x}$
$\Rightarrow y^2=x$

$\circ\ y=\sqrt{3x-18}$
$\Rightarrow y^2=3x-18$
$\Rightarrow \frac{y^2+18}{3}=x$

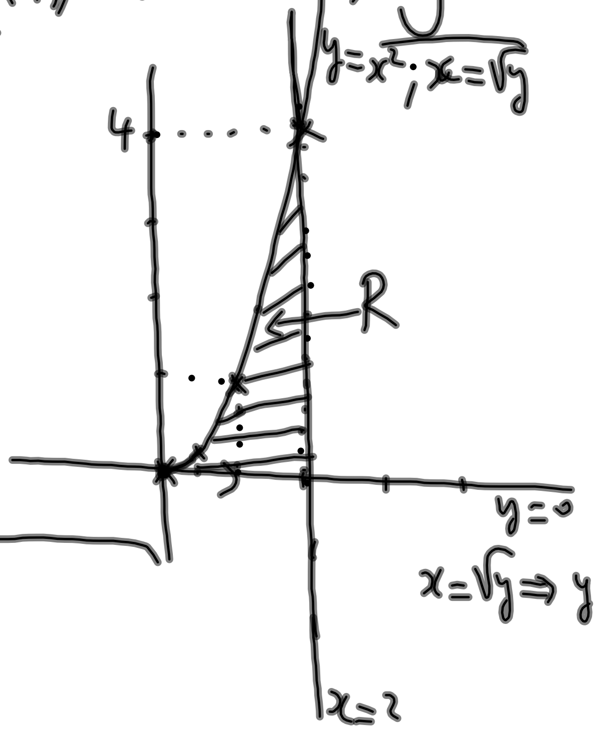
$$I=\iint_R f(x,y)\,dA=\int_0^6\int_0^{\sqrt{x}} f(x,y)\,dy\,dx+\int_6^9\int_{\sqrt{3x-18}}^{\sqrt{x}} f(x,y)\,dy\,dx \left(=\iint_{R_x} f(x,y)\,dA\right).$$

$$=\int_0^3\int_{y^2}^{\frac{y^2}{3}+6} f(x,y)\,dx\,dy$$

مثال 9 ص 169 : احسب التكامل : $I=\int_0^4\int_{\sqrt{y}}^2 y\cos x^5\,dx\,dy$

الحل : $R_y = R=\{(x,y): 0\le y\le 4, \sqrt{y}\le x\le 2\}$

$y=x^2 ; x=\sqrt{y}$



$x=\sqrt{y} \Rightarrow y=x^2$

بـ $R=R_x=\{(x,y): 0\le x\le 2, 0\le y\le x^2\}$

$$I=\int_0^2\left[\int_0^{x^2} y\cos x^5\,dy\right]dx$$

$$I=\int_0^2 \frac{y^2}{2}\cos x^5\Big]_0^{x^2}dx=\int_0^2\frac{(x^2)^2}{2}\cos x^5\,dx$$

$$=\int_0^2\frac{1}{2}\boxed{x^4\cos x^5}\,dx=\frac{1}{10}\sin x^5\Big]_0^2$$

$$I=\frac{\sin 2^5}{10}=\frac{\sin 32}{10}$$

تذكير

$(\sin x^5)'=5x^4\cos x^5$

$\frac{1}{5}(\sin x^5)'=x^4\cos x^5$

مثال ص 170: أحسب التكامل: $I=\int_0^8\int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx$

الحل:

$$R=R_x=\{(x,y): 0\le x\le 8,\ \sqrt[3]{x}\le y\le 2\}$$

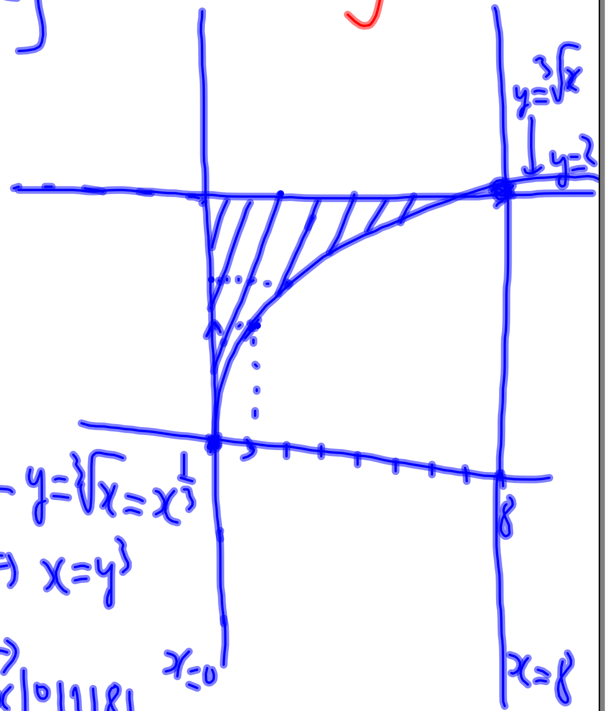
$$R_y=\{(x,y): 0\le y\le 2,\ 0\le x\le y^3\}$$

$$I=\iint_{R_y}\frac{1}{y^4+1}\,dA=\int_0^2\left[\int_0^{y^3}\frac{1}{y^4+1}\,dx\right]dy$$

$$=\int_0^2\frac{1}{y^4+1}\,x\Big|_0^{y^3}dy=\frac{1}{4}\int_0^2\frac{4y^3}{y^4+1}\,dy$$

$$=\frac{1}{4}\ln|y^4+1|\Big|_0^2=\frac{1}{4}\left(\ln(2^4+1)-\ln(0^4+1)\right)$$

$$\boxed{I=\frac{\ln(17)}{4}}$$



$y=\sqrt[3]{x}=x^{\frac{1}{3}}$

$\Leftrightarrow x=y^3$

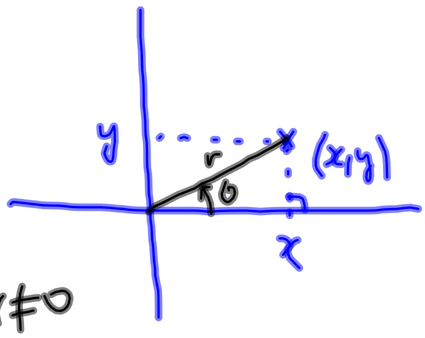
x	0	1	8
y	0	1	2

$8^{\frac{1}{3}}=(2^3)^{\frac{1}{3}}=2$

التكامل الثنائي باستعمال الاحداثيات القطبية

الاحداثيات القطبية

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} ; \quad r \geq 0,\ 0 \leq \theta \leq 2\pi$$



$$x^2+y^2=r^2$$

$$\tan\theta = \frac{y}{x},\ x \neq 0$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ x \neq 0 \end{cases}$$

$$R = \{(r,\theta) : r_1 \leq r \leq r_2,\ \theta_1 \leq \theta \leq \theta_2\}$$; مبرهنة (1)

$$\iint_R f(x,y)\,dA = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\,\boxed{r}\,d\theta\,dr$$

~~R = {(r,θ) : r₁ ≤ r ≤ r₂ ; θ₁ ≤ θ ≤~~

(٢)

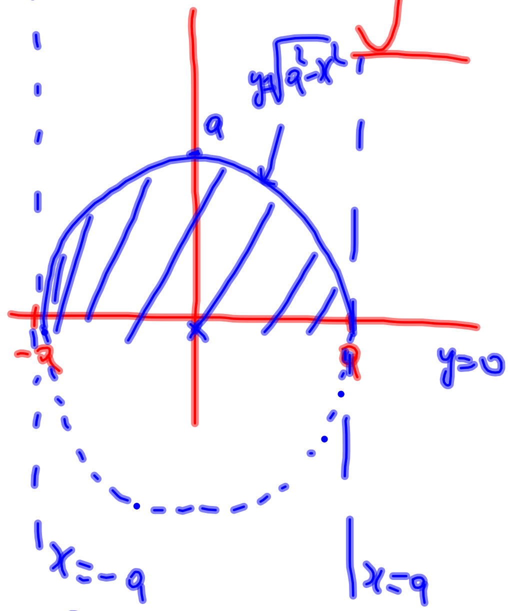
مثال 3 ص 186 :

احسب التكامل : $I=\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(x^2+y^2)^{\frac{3}{2}}\,dy\,dx$

الحل :

$$R_x=\left\{(x,y): -a\le x\le a,\ 0\le y\le \sqrt{a^2-x^2}\right\}$$

$y=$



$$y=\sqrt{a^2-x^2}\Longrightarrow \begin{cases} y^2+x^2=a^2 \\ y\geq 0 \end{cases}$$

$$R=\left\{(r,\theta): 0\le r\le a,\ 0\le\theta\le\pi\right\}$$

$$I=\int_0^a\left[\int_0^\pi (r^2)^{\frac{3}{2}}\, r\,d\theta\,dr\right.$$

$$=\int_0^a\int_0^\pi r^4\,d\theta\,dr=\int_0^a r^4\,\theta\Big]_0^\pi dr$$

$$=\int_0^a \pi r^4\,dr=\pi\frac{r^5}{5}\Big]_0^a$$

$$\boxed{I=\frac{\pi a^5}{5}}$$

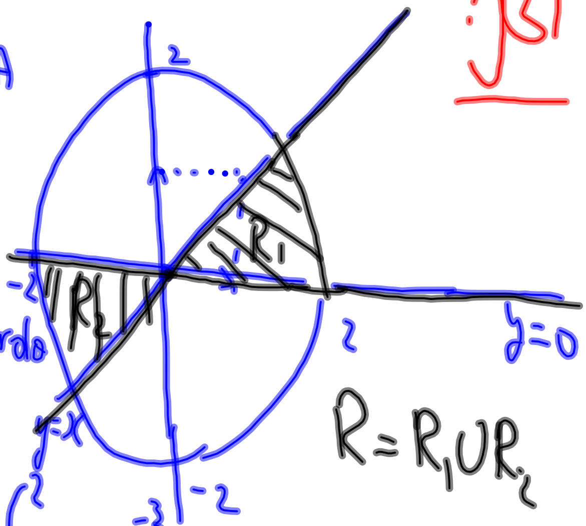
مثال: احسب التكامل

$$I=\iint_R \frac{1}{(1+x^2+y^2)^{3/2}}\,dA$$

حيث R المنطقة المحدودة بالمنحنيات التالية:

$x^2+y^2=4$; $y=x$; $y=0$

الحل:



$R=R_1\cup R_2$

$$I=\iint_{R_1}\frac{1}{(1+x^2+y^2)^{3/2}}\,dA+\iint_{R_2}\frac{1}{(1+x^2+y^2)^{3/2}}\,dA$$

$$=\int_0^{\pi/4}\int_0^2\frac{1}{(1+r^2)^{3/2}}\,r\,dr\,d\theta+\int_\pi^{5\pi/4}\int_0^2\frac{1}{(1+r^2)^{3/2}}\,r\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\left[\int_0^2 2r(1+r^2)^{-\frac{3}{2}}dr\right]d\theta+\frac{1}{2}\int_\pi^{5\pi/4}\left[\int_0^2 2r(1+r^2)^{-\frac{3}{2}}dr\right]d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{(1+r^2)^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg|_0^2+\frac{1}{2}\int_\pi^{5\pi/4}\frac{(1+r^2)^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg|_0^2 d\theta$$

$$=\int_0^{\pi/4}\left(1-\frac{1}{\sqrt5}\right)d\theta+\int_\pi^{5\pi/4}\left(1-\frac{1}{\sqrt5}\right)d\theta$$

$$=\frac{\sqrt5-1}{\sqrt5}\left(\theta\Big|_0^{\pi/4}+\theta\Big|_\pi^{5\pi/4}\right)=\frac{\sqrt5-1}{\sqrt5}\left(2\frac{\pi}{4}\right)$$

$$\boxed{I=\frac{\sqrt5-1}{\sqrt5}\,\frac{\pi}{2}=\frac{\sqrt5-1}{2\sqrt5}\pi}$$

أبريل 14-01:05 م

مثال نحول التكامل إلى تكامل بالإحداثيات القطبية ثم احسبه

$$I = \iint_R (x^2+y^2)^{\frac{3}{2}}\, dA$$

حيث R المنطقة المحدودة بالمنحنيات $y=0$, $y=\sqrt{1-x^2}$, $x=-1$, $x=1$

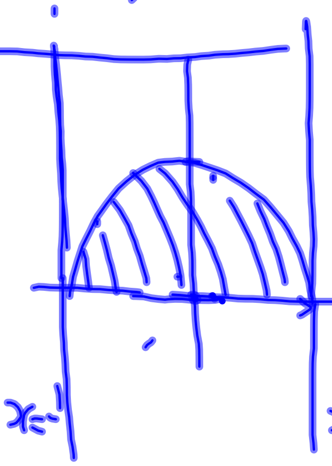
الحل:

$y=\sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2$

$\Leftarrow x^2+y^2=1$

$y=0$, $y \geq 0$



$R=\{(r,\theta) : 0 \leq r \leq 1 ,\ 0 \leq \theta \leq \pi\}$

$$I = \int_0^{\pi}\int_0^{1} (r^2)^{\frac{3}{2}}\, \underbrace{r\, dr\, d\theta}_{dA}$$

$$= \int_0^{\pi}\left[\int_0^{1} r^4\, dr\right] d\theta = \int_0^{\pi} \frac{r^5}{5}\Big|_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{5}\, d\theta = \frac{1}{5}\theta\Big|_0^{\pi} = \frac{\pi}{5}$$

$$\boxed{I = \frac{\pi}{5}}$$

مثال: استعمل الإحداثيات القطبية واحسب التكامل:

$$I = \iint_R \frac{1}{x^2+y^2+2}\, dA$$

حيث

$$R = \{(x,y) :\ x \geq 0 ,\ x^2+y^2 \leq 4\}$$

الحل:

$$\boxed{\ln a - \ln b = \ln \frac{a}{b}}$$

$$R = \{(r,\theta) : 0 \leq r \leq 2 ,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

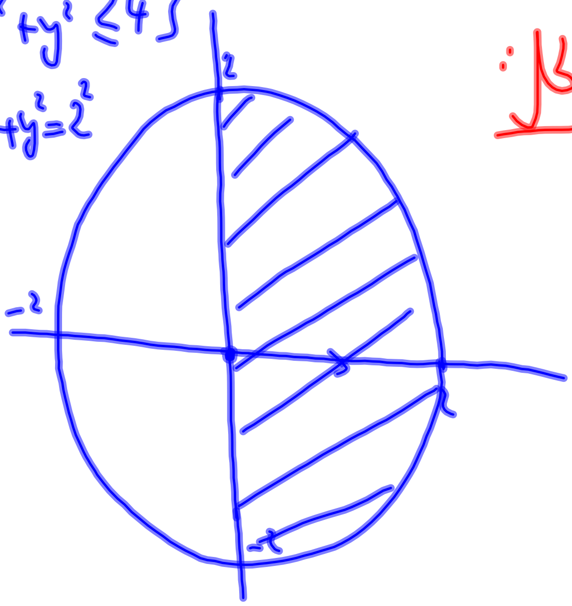
$$I = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \frac{1}{r^2+2}\, 2r\, dr\, d\theta$$

$$I = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln(r^2+2)\Big|_0^2 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\ln 6 - \ln 2)\, d\theta$$

$$I = \ln\left(\frac{6}{2}\right)\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \ln 3\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$\boxed{I = \pi \ln 3}$$

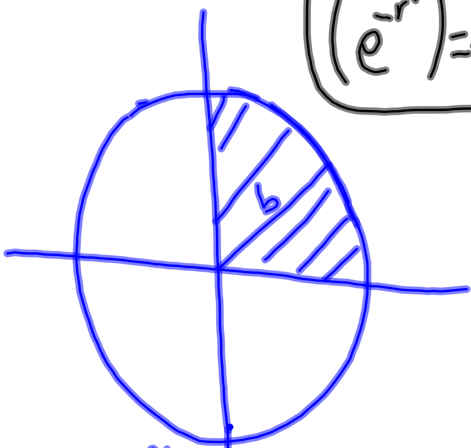
$x^2+y^2=2^2$



مثال ٢ ص 188 : احسب التكامل المعتل : $I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dA$

الحل : **ملحوظة :** $(e^{-r^2})' = -2re^{-r^2}$

$$I = \lim_{b \to +\infty} \iint_{R_b} e^{-(x^2+y^2)}\, dA ; \quad R_b :$$ ربع الموجب للقرص مركزه $(0,0)$ ونصف قطره b .



$$I_b = \iint_{R_b} e^{-(x^2+y^2)}\, dA = \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}\right) \int_0^b e^{-r^2}(-2r)\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\, e^{-r^2}\Big|_0^b\, d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} (e^{-b^2} - 1)\, d\theta$$

$$I_b = \frac{1}{2}(1 - e^{-b^2})\, \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}(1 - e^{-b^2})\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{I_b = \frac{\pi(1 - e^{-b^2})}{4}} \Longrightarrow I = \lim_{b \to \infty} I_b = \lim_{b \to \infty} \frac{\pi(1 - e^{-b^2})}{4} = \boxed{\frac{\pi}{4} = I}$$

$$J = \int_0^{\infty} e^{-x^2}\, dx = ?$$

ملحوظة :

$$J^2 = \left(\int_0^{\infty} e^{-x^2}\, dx\right)^2 = \int_0^{\infty} e^{-x^2}\, dx \cdot \int_0^{\infty} e^{-y^2}\, dy = \int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\, dx\, dy = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$J^2 = I \Longrightarrow J = \sqrt{I} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

$$\boxed{\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$$

د بالتالي