THE $\mathscr{L}_p$ SPACES*

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ABSTRACT

The $\mathscr{L}_p$ spaces which were introduced by A. Pełczyński and the first named author are studied. It is proved, e.g., that (i) X is an $\mathscr{L}_p$ space if and only if X^* is an $\mathscr{L}_q$ space ($p^{-1} + q^{-1} = 1$). (ii) A complemented subspace of an $\mathscr{L}_p$ space is either an $\mathscr{L}_p$ or an $\mathscr{L}_2$ space. (iii) The $\mathscr{L}_p$ spaces have sufficiently many Boolean algebras of projections. These results are applied to show that X is an $\mathscr{L}_\infty$ (resp. $\mathscr{L}_1$) space if and only if X admits extensions (resp. liftings) of compact operators having X as a domain or range space. We also prove a theorem on the "local reflexivity" of an arbitrary Banach space.

Section 1. Introduction. The purpose of this paper is to investigate the properties of the $\mathscr{L}_p$ spaces — i.e. those Banach spaces whose finite-dimensional subspaces are close to the finite-dimensional subspaces of $L_p(\mu)$ spaces. The $\mathscr{L}_p$ spaces seem to form the suitable framework in which the study of the isomorphic properties of the classical Banach spaces can be carried out. In the present paper we solve some questions concerning those spaces which were left open in earlier treatments, mainly [13, chaps. II, III] and [16, sec. 7]. Though many natural questions concerning these spaces remain open (see Section 5 below) the results presented here together with those of [13] and [16] give a rather clear picture on the basic properties of $\mathscr{L}_p$ spaces and their relation to $L_p(\mu)$ spaces. It turns out that these spaces behave in a nicer way than could be suspected a-priori and we feel that the presently known results already fully justify the introduction and detailed study of $\mathscr{L}_p$ spaces.

Before we can be more specific on the results proved here as well as the earlier results, we have to introduce some definitions and notations. First we recall some standard notations (cf. [3] and [4]). By $L_p(\mu) = L_p(\Omega, \mathscr{B}, \mu)$, $1 \leqq p \leqq \infty$, we denote the Banach space of equivalence classes of measurable functions on $(\Omega, \mathscr{B}, \mu)$

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whose p'th power is integrable (resp. are essentially bounded if $p = \infty$). If $(\Omega, \mathscr{B}, \mu)$ is the usual Lebesque measure space on $[0,1]$ we denote $L_p(\mu)$ by L_p. If $(\Gamma, \mathscr{B}, \mu)$ is the discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$ we denote $L_p(\mu)$ by $l_p(\Gamma)$. If $\Gamma = \{1, 2, \cdots, n\}$, $n < \infty$ we also denote $l_p(\Gamma)$ by l_p^n, while l_p will denote $l_p(\Gamma)$ with $\Gamma = \{\text{positive integers}\}$. The subspace of $l_\infty(\Gamma)$ of those functions which vanish at ∞ is denoted by $c_0(\Gamma)$ (resp. c_0 if $\Gamma = \{\text{positive integers}\}$). For a compact Hausdorff space K we denote by $C(K)$ the Banach space of continuous functions on K with the supremum norm.

By operator (resp. projection) we mean a *bounded linear* operator (resp. proection). Two Banach spaces X and Y are called isomorphic (denoted by $X \approx Y$) if there is an invertible operator from X onto Y. The distance coefficient $d(X, Y)$ of two isomorphic Banach spaces is defined by $\inf(\| T \| \, \| T^{-1} \|)$ where the inf is taken over all invertible operators T from X onto Y. The Banach spaces X and Y are called isometric if there is an operator T from X onto Y with $\| T \| = \| T^{-1} \| = 1$. A closed linear subspace Y of a Banach space X is said to be a complemented subspace if there is a projection from X onto Y, or what is the same, if there exists a closed linear subspace Z of X such that $X = Y \oplus Z$. A Banach space is said to be injective if it is complemented in any Banach space containing it. If we consider a Banach space X as a subspace of X^{**} we assume (unless stated otherwise) that X is embedded in X^{**} in the canonical manner.

We come now to the definition of the basic notion in this paper, i.e. of an $\mathscr{L}_p$ space (cf. [16]).

Definition. *Let $1 \leqq p \leqq \infty$ and $1 \leqq \lambda < \infty$. A Banach space X is said to be an $\mathscr{L}_{p,\lambda}$ space if for every finite-dimensional subspace B of X there is a finite-dimensional subspace C of X such that $C \supset B$ and $d(C, l_p^n) \leqq \lambda$ where $n = \dim C$. A Banach space is said to be an $\mathscr{L}_p$ space, $1 \leqq p \leqq \infty$, if it is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda < \infty$.*

It is easily seen and well-known, that the $\mathscr{L}_p$ spaces generalize the $L_p(\mu)$ and $C(K)$ spaces. Indeed, let X be an $L_p(\mu)$ space for some $1 \leqq p < \infty$ (resp. a $C(K)$ space), B a finite-dimensional subspace of X, and $\varepsilon > 0$ be given. Then there exists a projection from X, of norm $\leqq 1 + \varepsilon$, onto a subspace C of X with $C \supset B$ and $d(C, l_p^n) \leqq 1 + \varepsilon$ (resp. $d(C, l_\infty^n) \leqq 1 + \varepsilon$), where $n = \dim C$. This may be seen by using the span of the characteristic functions of finitely many disjoint measurable sets of finite measure (resp. by using partitions of unity) together with the argument of lemma 3.1 of [13].

The main results concerning the structure of $\mathscr{L}_p$ spaces which were proved in [16, Sec. 7] are summarized in

THEOREM I. (i) *Every $\mathscr{L}_p$ space, $1 \leqq p \leqq \infty$, is isomorphic to a subspace of an $L_p(\mu)$ for some measure μ.*

(ii) *An $\mathscr{L}_p$ space X, $1 \leqq p \leqq \infty$, is isomorphic to a complemented subspace of an $L_p(\mu)$ space if and only if X is complemented in X^{**} (this always holds if $1 < p < \infty$ since then X is reflexive by* (i)).

(iii) *If X is an $\mathscr{L}_1$ space then X^* is injective.*

(iv) *Every infinite-dimensional $\mathscr{L}_p$ space, $1 \leqq p < \infty$, has a complemented subspace isomorphic to l_p.*

We will be concerned here with the isomorphic theory of Banach spaces. The isometric analogues of most of the results we prove below are already known. Though we shall not need these isometric results we shall state them now since it is of interest to compare the isometric situation with the isomorphic one. For the proof of Theorem II see [16, sec. 7] and its references (cf. also [28].)

THEOREM II. (a) *A Banach space is isometric to an $L_p(\mu)$ space for some measure μ $(1 \leqq p < \infty)$ if and only if it is an $\mathscr{L}_{p,1+\varepsilon}$ space for every $\varepsilon > 0$.*

(b) *The dual X^* of a Banach space X is isometric to an $L_p(\mu)$ space for some measure μ $(1 \leqq p \leqq \infty)$ if and only if X is an $\mathscr{L}_{q,1+\varepsilon}$ space for every $\varepsilon > 0$ where $q^{-1} + p^{-1} = 1$ $(q = 1$,* resp. ∞ if $p = \infty$ resp. 1).

(c) *Let $Y \supset X$ be such that Y is an $\mathscr{L}_{p,1+\varepsilon}$ space for every $\varepsilon > 0$ $(1 \leqq p \leqq \infty)$ and there is a projection of norm 1 from Y onto X. Then X is an $\mathscr{L}_{p,1+\varepsilon}$ space for every $\varepsilon > 0$.*

(d) *A separable infinite-dimensional space X is an $\mathscr{L}_{p,1+\varepsilon}$ space for every $\varepsilon > 0$ $(1 \leqq p \leqq \infty)$ if and only if X can be represented as $\overline{\bigcup_{n=1}^{\infty} B_n}$ where $B_1 \subset B_2 \subset \cdots$ and B_n is isometric to l_p^n, $n = 1,2,\cdots$.*

(e) *For $1 \leqq p < \infty$, $p \neq 2$ there are no infinite-dimensional $\mathscr{L}_{p,1}$ spaces. The spaces which are $\mathscr{L}_{2,1+\varepsilon}$ spaces for every $\varepsilon > 0$ are also $\mathscr{L}_{2,1}$ spaces. A space which is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$ is an $\mathscr{L}_{\infty,1}$ space if and only if the unit cell of every finite-dimensional subspace of it is a polytope (c_0 is such a space).*

The main results concerning the structure of $\mathscr{L}_p$ spaces we prove here are summarized in

THEOREM III. (a) *A Banach space X is an $\mathscr{L}_p$ space $(1 \leqq p \leqq \infty)$ if and only if X^* is an $\mathscr{L}_q$ space $(p^{-1} + q^{-1} = 1)$.*

(b) *A complemented subspace X of an $\mathscr{L}_p$ space ($1 \leqq p \leqq \infty$) which is not isomorphic to a Hilbert space is an $\mathscr{L}_p$ space. (If $p = 1$ or ∞ X cannot be isomorphic to an infinite-dimensional Hilbert space.)*

(c) *Let X be an $\mathscr{L}_p$ space, $1 \leqq p \leqq \infty$. Then there is a constant ρ such that for every finite-dimensional subspace B of X there is a finite-dimensional subspace C of X such that $C \supset B$, $d(C, l_p^n) \leqq \rho$ ($n = \dim C$), and such that there is a projection of norm $\leqq \rho$ from X onto C.*

The proof of Theorem III is quite indirect. We do not see a simple and direct way to prove e.g. that if X is an $\mathscr{L}_p$ space then X^* is an $\mathscr{L}_q$ space. The difficulty lies in the fact that from the definition of an $\mathscr{L}_p$ space we have some information on the finite-dimensional quotient spaces of X^* but no obvious information on the finite-dimensional subspaces of X^*. Thus, we can view Theorem III(a) as a statement of the fact that if we define a class of spaces analogous to $\mathscr{L}_p$ spaces by considering finite-dimensional quotient spaces instead of subspaces we end up with exactly the same spaces. Similarly part (c) of Theorem III shows that the definition of $\mathscr{L}_p$ spaces is equivalent to the formally stronger statement in which the spaces C are not only required to be "close" to l_p^k spaces but also to admit a projection from X with a uniformly bounded norm. (This is an obvious fact only for $p = \infty$ and $p = 2$.) That these *a-priori* different definitions actually define the same class of spaces is the main reason for the fact that the theory of $\mathscr{L}_p$ spaces turns out to be so satisfactory.

Sections 2 and 3 are devoted to the proof of Theorem III and some of its corollaries. In Section 2 the reflexive case (i.e. $1 < p < \infty$) is proved while the cases $p = 1$ and ∞ are treated in Section 3. The additional arguments needed in these cases are caused by the difficulty of deducing properties of a Banach space X from its dual X^* in the non-reflexive situation (in general the isomorphic type of X is not uniquely determined by the isomorphic type of X^*). Our main tool for handling the cases $p = 1$ and ∞ is a theorem on general Banach spaces (Theorem 3.1). This theorem which we call the "local reflexivity principle" shows that in a strong sense the finite-dimensional subspaces of a general Banach space X are close to the finite-dimensional subspaces of X^{**}. The proof of this local reflexivity principle is based on a separation theorem of Klee [10].

Section 4 contains applications of the results of Sections 2 and 3. By completing the reasoning in [13, chaps. II, III] it is shown that the $\mathscr{L}_\infty$ (resp. $\mathscr{L}_1$) spaces are exactly those Banach spaces X which admit extensions (resp. liftings) of compact operators having X as a domain or range space.

Another application we give in Section 4 is the observation that a weakened version of the definition of an $\mathscr{L}_p$ space gives for $1 < p < \infty$ a joint characterization of $\mathscr{L}_p$ and $\mathscr{L}_2$ spaces and for $p = 1$ and ∞ characterizations of $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces respectively. We conclude Section 4 by a characterization of all $\mathscr{L}_p$ spaces and a joint charactefrization of $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_\infty$ spaces in terms of Boolean algebras of projections. These characterizations are restatements of results of [19] which were made possible by Theorem III(c).

Section 5 is devoted to some remarks and many open problems. The main part of the section is concerned with the question of functional representation and isomorphic classification of $\mathscr{L}_p$ spaces.

Section 2. The case $1 < p < \infty$. This section is devoted to the proof of Theorem III for $1 < p < \infty$. We begin with

THEOREM 2.1. *A complemented subspace of an* $L_p(\mu)$ *space,* $1 < p < \infty$, *which is not isomorphic to a Hilbert space, is an* $\mathscr{L}_p$ *space.*

Proof. Let X be a complemented subspace (which is not isomorphic to a Hilbert space) of the space $Y = L_p(\mu)$, $1 < p < \infty$. We assume, as we clearly may, that $\dim X = \infty$ and that $p \neq 2$ (recall that by Theorem I (a) the $\mathscr{L}_2$ spaces are exactly those spaces which are isomorphic to Hilbert spaces). The space X has a complemented subspace U isomorphic to l_p. Indeed, by [12, lemma 3] X has a separable subspace X_0 which is not isomorphic to a Hilbert space and by [8, cor. 3, p. 168] X_0 has a subspace U which is isomorphic to l_p and complemented in Y. We denote by Q a projection from Y onto X.

Let B be a finite-dimensional subspace of X. Since Y is an $L_p(\mu)$ space there is a finite-dimensional subspace W of Y and a projection P of Y onto W such that $W \supset B$, $d(W, l_p^n) \leqq 2\,(n = \dim W)$, and $\| P \| \leqq 2$. Since QW is a finite-dimensional subspace of X and U is isomorphic to l_p it follows easily that there is a constant K (depending only on $d(U, l_p)$ but not on W and for that matter not on B) and a subspace C of U such that $d(C, l_p^n) \leqq K$ and $\max(\| x \|, \| y \|) \leqq K \| x + y \|$ for every $x \in QW$ and $y \in C$. Let $\tau \colon W \to C$ be an isomorphism such that $\| \tau \| \leqq 1$ and $\| \tau^{-1} \| \leqq 2K$. Let $T \colon W \to X$ be the operator defined by $T = Q_{|W} + \tau(I - PQ_{|W})$. The restriction of T to B is the identity since on B both P and Q are the identity maps. We have that $\| T \| \leqq 1 + 3 \| Q \|$ and for every $x \in W$

$$\begin{aligned}\|Tx\| &\geqq K^{-1}\max(\|Qx\|, \|\tau(I-PQ)x\|)\\ &\geqq K^{-1}\max(\|Qx\|, (\|x-PQx\|)\|\tau^{-1}\|^{-1})\\ &\geqq K^{-1}\max(\|Qx\|, (\|x\|-2\|Qx\|)\|\tau^{-1}\|^{-1})\\ &\geqq K^{-1}(\|\tau^{-1}\|+2)^{-1}\|x\| \geqq \|x\|/(K^2+2K).\end{aligned}$$

It follows that $d(W, TW) \leqq (1+3\|Q\|)(K^2+2K)$, and thus $d(TW, l_p^n) \leqq 2(1+3\|Q\|)(2K+K^2)$. Since $B \subset TW$ this concludes the proof.

COROLLARY 1. *A Banach space is isomorphic to a complemented subspace of an $L_p(\mu)$ space for some measure μ, $1<p<\infty$, if and only if it is either an $\mathscr{L}_p$ or an $\mathscr{L}_2$ space.*

Proof. This follows from Theorem I, Theorem 2.1 and the well-known fact that for every Hilbert space H and every p, $1<p<\infty$ there is a measure μ and a complemented subspace of $L_p(\mu)$ which is isomorphic to H.

COROLLARY 2. *Let $1<p<\infty$. Then X is an $\mathscr{L}_p$ space if and only if X^* is an $\mathscr{L}_q$ space $(p^{-1}+q^{-1}=1)$.*

Proof. This follows immediately from Corollary 1 and the fact that $L_p^*(\mu) = L_q(\mu)$ for every measure μ.

We have thus proved parts (a) and (b) of Theorem III for $1<p<\infty$ and we turn now to the proof of part (c) of this theorem.

THEOREM 2.2. *Let X be an $\mathscr{L}_p$ space, $1<p<\infty$. Then there is a constant ρ such that for every finite-dimensional subspace B of X there is a finite-dimensional subspace $C \supset B$ of X and a projection π from X onto C such that $d(C, l_p^n) \leqq \rho$ ($n = \dim C$) and $\|\pi\| \leqq \rho$.*

Proof. By Theorem I(b) we may assume without loss of generality that X is a complemented subspace of an $L_p(\mu)$ space Y. We assume also that $\dim X = \infty$ (otherwise there is nothing to prove).

We proceed now as in the proof of Theorem 2.1 (we shall also follow the notations in that proof) and show that the subspace TW constructed there can be chosen so that there is a projection from X onto TW with a norm which is bounded by some constant depending only on X (but not on W and thus not on B). Constants depending only on X will be denoted below by $K_{i, i=1,2,\dots}$.

The subspace C of U (appearing in the proof of Theorem 2.1) is chosen now a little bit more carefully. Let R_0 be a projection from X onto U. (For future reference we note that only here we use the fact that U is complemented on X.)

Since U is isomorphic to l_p and $\dim R_0QW < \infty$ there is a projection R_1 of norm $\leqq K_1$ from U onto a subspace C of U such that $d(C, l_p^n) \leqq K_2$ and $R_1R_0QW = \{0\}$.* Having chosen C as above we define T as in the proof of Theorem 2.1 by $T = Q_{|W} + \tau(I - PQ_{|W})$. We define next an operator $S: X \to W$ such that $\| S \| \leqq K_3$ and $STw = w$ for every $w \in W$. This will conclude the proof for the case $1 < p < \infty$ since TS will then be a projection from X onto TW with $\| TS \| \leqq K_4$.

We claim that

$$S = P(I - R_1R_0) + \tau^{-1}R_1R_0$$

has the desired properties. Since R_1R_0 is a projection from X onto C, S is a well defined operator from X into W and clearly there exists a K_3 with $\| S \| \leqq K_3$. Let $w \in W$. Since $R_1R_0QW = \{0\}$ we get that

$$\tau^{-1}R_1R_0Tw = 0 + \tau^{-1} \cdot \tau \cdot (I - PQ)w = w - PQw.$$

Also,

$$P(I - R_1R_0)Tw = P(Tw - R_1R_0Tw) = PQw.$$

By adding these two equations we get that $STw = w$ as desired.

REMARK. The fact that in an $\mathscr{L}_p$ space X every finite-dimensional subspace is contained in another subspace which is not far from l_p^n for a suitable n and on which there is also a "good" projection, is far from obvious and is actually quite surprising. Indeed, if the $\mathscr{L}_p$ space X is given as $\overline{\bigcup_\alpha B_\alpha}$ with B_α a net of finite-dimensional subspaces of X, directed by inclusion, such that $d(B_\alpha, l_p^{n_\alpha}) \leqq K$ ($\dim B_\alpha = n_\alpha$), it is not true in general that there are "good" projections from X onto those B_α. For example it follows from the results of [25], [23], and [21] and a compactness argument (cf. also [17, Section 3]) that for $1 < p \leqq 4/3$ it is possible to write l_p as $\overline{\bigcup_{j=1}^\infty B_j}$ with $B_1 \subset B_2 \subset B_3 \cdots$ and $d(B_j, l_p^{n_j}) \leqq K$ for some K so that if

$$\lambda_j = \inf\{\| P \|;\ P \text{ a projection from } l_p \text{ onto } B_j\}$$

then $\lim_{j \to \infty} \lambda_j = \infty$.

* To see this, let A be a finite-dimensional subspace of l_p. Choose a projection P from l_p onto a subspace E of l_p with $E \supset A$, $d(E, l_p^n) \leqq 2$ ($n = \dim E$), and $\| P \| \leqq 2$. Then $(I - P)(l_p)$ is an infinite-dimensional subspace of l_p. Hence by a result of Pelczynski [21], given any positive integer m, there is a projection Q from l_p onto a subspace C of l_p with $C \subset (I - P)(l_p)$ such that $\| Q \| \leqq 2$ and $d(C, l_p^m) \leqq 2$. Then putting $R = Q(I - P)$; R is a projection from l_p onto C, with $\| R \| \leqq 6$. This argument holds also for $p = 1$, and with suitable notation changes for c_0 in place of l_p.

Section 3. The nonreflexive case. Our basic tool for extending the results of Section 2 to $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces is the following theorem which shows that all Banach speces are "locally reflexive".

THEOREM 3.1. *Let X be a Banach space (regarded as a subspace of X^{**}), let U be a finite-dimensional subspace of X^{**}, and let $\varepsilon > 0$. Then there exists a one-to-one operator $T: U \to X$ with $T(x) = x$ for all $x \in U \cap X$ and $\|T\| \|T^{-1}\| \leqq 1 + \varepsilon$.*

Proof. We first need the following preliminary observation: Let Y be a Banach space, K an open convex subset of Y, B a finite-dimensional Banach space, and $S: Y \to B$ a surjective operator. Let $\boldsymbol{K}$ denote the norm-interior of the weak* closure of K in Y^{**}. Then

$$S^{**}(\boldsymbol{K}) = S(K) \tag{1}$$

To see this, observe that since B is reflexive and S^{**} is weak* continuous, $\overline{S(K)} \supset S^{**}(\boldsymbol{K})$. Since S is surjective S is an open map, and thus $S(K)$ and $S^{**}(\boldsymbol{K})$ are convex open subsets of B, with $S(K) \subset S^{**}(\boldsymbol{K})$, which implies (1).

Now choose $\delta > 0$ (with $\delta < 1$) such that

$$\frac{1+\delta}{(1-\delta)(1-2\delta)-\delta(1+\delta)} < 1 + \varepsilon .$$

By the compactness of the unit ball of U, we may choose a finite number $y_1, \cdots, y_m$ of points of U satisfying $\|y_i\| = 1$ for all i, so that if $y \in U$ with $\|y\| = 1$, then for some i, $\|y - y_i\| < \delta$. We now claim that to complete the proof, it suffices to construct an operator $T: U \to X$ with $T_{|U \cap X} =$ identity, and satisfying

$$1 - 2\delta < \|Ty_i\| < 1 + \delta \quad \text{for all } i,\ 1 \leqq i \leqq m. \tag{2}$$

Indeed, suppose T satisfies (2), and let $y \in U$ with $\|y\| = 1$. Then by the Hahn-Banach Theorem (geometric version), there exist complex numbers $\alpha_1, \cdots, \alpha_m$ with $\sum_{i=1}^m |\alpha_i| \leqq \frac{1}{1-\delta}$ and $y = \sum_{i=1}^m \alpha_i y_i$. (Note that if $f \in U^*$, then

$$\|f\| \leqq (1-\delta) \sup_{1 \leqq i \leqq m} |f(y_i)|.)$$

Thus $\|T\| \leqq (1+\delta)/(1-\delta)$. Choosing y_i such that $\|y - y_i\| < \delta$, we have that

$$\begin{aligned} \|Ty\| &\geqq \|Ty_i\| - \|T(y - y_i)\| \\ &\geqq 1 - 2\delta - \frac{1+\delta}{1-\delta}\delta . \end{aligned}$$

Thus

$$\| T^{-1} \| \leqq \frac{1-\delta}{(1-\delta)(1-2\delta)-\delta(1+\delta)}$$

so $\| T \| \ \| T^{-1} \| < 1+\varepsilon$ by the definition of δ.

Now let $k = \dim U/U \cap X$, and choose independent vectors $u_1, \dots, u_k$ in U such that U equals the linear span of $U \cap X$ and $u_1, \dots, u_k$. Then for each i, we may choose scalars a_{ij} and a vector $b_i \in U \cap X$ such that

$$y_i = \sum_{j=1}^{k} a_{ij} u_j + b_i. \tag{3}$$

We may also choose $f_i \in X^*$ such that $|y_i(f_i)| \geqq 1-\delta$ with $\| f_i \| = 1$. Finally, let K_i and C_i be the subsets of the k-fold direct sum of X with itself defined by

$$K_i = \{(x_1, \cdots, x_k) \colon \ \Big\| \sum_{j=1}^{k} a_{ij} x_j + b_i \Big\| < 1+\delta\}$$

and

$$C_i = \left\{(x_1, \cdots, x_k) \colon \ \left| f_i \left\{ \sum_{j=1}^{k} a_{ij} x_j \right\} - \left\{ \sum_{j=1}^{k} a_{ij} u_j \right\} (f_i) \right| < \delta \right\}.$$

We shall now prove that $\bigcap_{i=1}^{m} (K_i \cap C_i)$ is non-empty. Once this is accomplished, we simply choose $(x_1, \cdots, x_k)$ belonging to all the K_i's and C_i's, and define $T \colon U \to X$ by $T(b + \sum_{i=1}^{k} \lambda_i u_i) = b + \sum_{i=1}^{k} \lambda_i x_i$ for all $b \in U \cap X$ and scalars $\lambda_1, \cdots, \lambda_k$. Then $T_{|U \cap X}$ is the identity and T satisfies (2). (We have, fixing i, that $|f_i(Ty_i) - y_i(f_i)| < \delta$ by (3), whence $\| T(y_i) \| > 1-2\delta$).

Now suppose that $\bigcap_{i=1}^{m} (K_i \cap C_i)$ is empty. Then since K_i and C_i are open convex subsets of X^k (the k-fold direct sum of X with itself) for all i, we have by a theorem of Klee [10] that there exists a finite-dimensional space B with $\dim B \leqq 2m-1$ and a surjective linear map $S \colon X^k \to B$ with $\bigcap_{i=1}^{m} (S(K_i) \cap S(C_i)) = \emptyset$. Now for each i, let K_i^{**} and C_i^{**} be the subsets of $(X^{**})^k$, the k-fold direct sum of X^{**} with itself, defined by

$$K_i^{**} = \{(x_1^{**}, \cdots, x_k^{**}) \colon \ \Big\| \sum_{j=1}^{k} a_{ij} x_j^{**} + b_i \Big\| < 1+\delta\}$$

and

$$C_i^{**} = \{(x_1^{**}, \cdots, x_k^{**}) \colon \ \Big| \sum_{j=1}^{k} a_{ij} x_j^{**}(f_i) - \sum_{j=1}^{k} a_{ij} u_j(f_i) \Big| < \delta\}.$$

Then evidently $(u_1, \cdots, u_k) \in K_i^{**} \cap C_i^{**}$ for all i. But fixing i, K_i^{**} (resp. C_i^{**}) is contained in the weak* closure of K_i (resp. C_i) in $(X^{**})^k$ (regarded as the dual of $(X^k)^*$). Indeed, this is immediate in the case of C_i^{**}; now fix $(x_1^{**}, \cdots, x_k^{**}) \in K_i^{**}$, and assume that for a certain j, $a_{ij} \neq 0$. (If $a_{ij} = 0$ all j, then this assertion is trivial). We may then choose nets (all indexed by the same directed set) $\{u_\alpha\}$ and $\{x_\alpha^r\}$ in X for all $1 \leqq r \leqq k$, $k \neq j$, such that $\| u_\alpha \| < 1 + \delta$ for all α with $u_\alpha \to \sum_{j=1}^k a_{ij} x_j^{**} + b_i$ weak* and $x_\alpha^r \to x_r^{**}$ weak* for all r, $1 \leqq r \leqq k$, $r \neq j$. Now for each α, define x_α^j by

$$x_\alpha^j = \frac{1}{a_{ij}}\Big(u_\alpha - \sum_{r \neq j} a_{ir} x_\alpha^r + b_i\Big).$$

Thus $(x_\alpha^1, \cdots, x_\alpha^k) \to (x_1^{**}, \cdots, x_k^{**})$ weak* and $(x_\alpha^1, \cdots, x_\alpha^r) \in K_i$ for all α.

Thus, in the notation we used at the beginning of this proof, $\boldsymbol{K}_i \supset K_i^{**}$ and $\boldsymbol{C}_i \supset C_i^{**}$, whence by (1), $S(\boldsymbol{K}_i) = S^{**}(K_i^{**})$ and $S(\boldsymbol{C}_i) = S^{**}(C_i^{**})$ for all i. But then since $(u_1, \cdots, u_k) \in K_i^{**} \cap C_i^{**}$ for all i, $S^{**}(u_1, \cdots, u_k) \in \bigcap_{i=1}^m (S(\boldsymbol{K}_i) \cap S(\boldsymbol{C}_i))$ contradicting the assumption that $\bigcap_{i=1}^m (S(\boldsymbol{K}_i) \cap S(\boldsymbol{C}_i)) = \emptyset$. Q.E.D.

REMARK. Theorem 3.1 has as a consequence the following

PROPOSITION. *Let $C \supset B$ be finite-dimensional Banach spaces. Let T be an operator from B into the infinite-dimensional Banach space X and let $\lambda \geqq \| T \|$. Then statements* (1) *and* (2) *are equivalent:*

(1) *For every $\varepsilon > 0$, there exists an extension $\tilde{T}$ of T from C into X with $\| \tilde{T} \| \leqq \lambda + \varepsilon$.*

(2) *For every finite-dimensional space W and every operator $S: X \to W$ there is an operator $\tilde{T}_s: C \to W$ which extends ST and satisfies $\| \tilde{T}_s \| \leqq \lambda \| S \|$.*

Proof. (2) and a compactness argument imply that there is an operator $\tilde{T}_1: C \to X^{**}$ with $\| \tilde{T}_1 \| \leqq \lambda$ and $\tilde{T}_1 \mid B = T$. Then (2) $\Rightarrow$ (1) follows immediately upon applying 3.1; (1) $\Rightarrow$ (2) is a simple compactness argument. In the special case $\dim C/B = 1$, this result is contained in [13, p. 60].

We return now to the the study of the $\mathscr{L}_p$ spaces. The following is an immediate consequence of Theorem 3.1:

COROLLARY. *Let $p = 1$ or ∞, let $\lambda \geqq 1$, and let X be a Banach space such that X^{**} is an $\mathscr{L}_{p,\lambda}$ space. Then X is an $\mathscr{L}_{p,\lambda+\varepsilon}$ space for all $\varepsilon > 0$.*

This result together with known results and an inspection of the proofs of Theorems 2.1 and 2.2, may be used to easily complete the proof of Theorem III. We first observe that the proofs of 2.1 and 2.2 yield

THEOREM 3.2. *Every complemented subspace of a $C(K)$ space is an $\mathscr{L}_\infty$ space. Every complemented subspace of an $L_1(\mu)$ space is an $\mathscr{L}_1$ space, which moreover satisfies the conclusion of* III(c).

Proof. Let X be an infinite-dimensional Banach space. If X is a complemented subspace of a $C(K)$ space, then X has a subspace U isomorphic to c_0 (cf. [21]). The proof that X is an $\mathscr{L}_\infty$ space now follows as in the proof of Theorem 2.1 (as we noted in Theorem 2.2, the proof of 2.1 does not require that U be complemented in X). Similarly, if X is a complemented subspace of an $L_1(\mu)$ space, then X is non-reflexive, and so X contains a complemented subspace isomorphic to l_1 (cf. [1], [8]). The proof that X is an $\mathscr{L}_1$ space satisfying the conclusion of III(c) now proceeds exactly as in the proofs of 2.1 and 2.2. Q.E.D.

It follows from Theorem I(ii) that if X is an $\mathscr{L}_\infty$ space, then X^{**} is injective. We thus obtain immediately, using the above corollary and Theorem 3.2, the

COROLLARY. *Every injective space is an $\mathscr{L}_\infty$ space. A Banach space X is an $\mathscr{L}_\infty$ space if and only if X^{**} is injective.*

COMPLETION OF THE PROOF OF THEOREM III. We consider first the $p = \infty$ case. Suppose that X is an $\mathscr{L}_\infty$ space. Then X^{**} is injective, and hence a complemented subspace of some $C(K)$ space. Then X^{***} is isomorphic to a complemented subspace of the dual of a $C(K)$ space, i.e. of some $L_1(\mu)$ space; and hence X^* being complemented in X^{***}, is also isomorphic to a complemented subspace of some $L_1(\mu)$ space. Hence X^* is an $\mathscr{L}_1$ space by Theorem 3.2. If we assume either that X^* is an $\mathscr{L}_1$ space or that X is a complemented subspace of an $\mathscr{L}_\infty$ space, then it follows from Theorem I that X^{**} is injective, and hence X is an $\mathscr{L}_\infty$ space by the above corollary. This completes the proof of III for the case $p = \infty$ (III(c) is trivial). Again by Theorem I, if X is an $\mathscr{L}_1$ space, then X^* is injective, hence X^* is an $\mathscr{L}_\infty$ space by the above corollary. If we assume either that X^* is an $\mathscr{L}_\infty$ space or that X is a complemented subspace of an $\mathscr{L}_1$ space, then it follows from what we have proved that X^{**} is an $\mathscr{L}_1$ space, and consequently X is an $\mathscr{L}_1$ space by the corollary to Theorem 3.1; the only thing remaining to be proved is

THEOREM 3.3. *Let X be an $\mathscr{L}_1$ space. Then X satisfies the conclusion of Theorem III(c).*

Proof. We assume that X is of infinite-dimension. Now by Theorem I, X^* is injective, whence X^{**} is isomorphic to a complemented subspace of an $L_1(\mu)$ space. Hence X is isomorphic to a non-reflexive subspace of an $L_1(\mu)$ space, and thus (cf. [1], [8]) there is a projection R_0 from X onto a subspace U of X isomorphic to l_1. Moreover, by Theorem 3.2, X^{**} is an $\mathscr{L}_1$ space satisfying the conclusion of III(c). Thus there is a constant $\rho > 0$ such that if B is a finite-dimensional subspace of X^{**}, then there exists a projection P_1 from X^{**} onto a subspace W with $W \supset B$, $d(W, l_1^n) \leqq \rho$ (where $n = \dim W$), and $\| P_1 \| \leqq \rho$.

Now let B be a finite-dimensional subspace of X. Then $B \subset X^{**}$, so we may choose W and P_1 as above with $n = \dim W$; then we define P by $P = P_{1|X}$. Finally by Theorem 3.1, there is a map $Q: W \to X$ with $\| Q \| \leqq 2$ and $Q_{|W \cap X}$ the identity on $W \cap X \supset B$. The remainder of the proof now proceeds exactly as in the proof of 2.2. Q.E.D.

Section 4. Other characterizations of the $\mathscr{L}_p$ spaces. The first result in this section is a characterization of $\mathscr{L}_\infty$ spaces by extension properties for compact operators.

THEOREM 4.1. *The following five assertions concerning a Banach space X are equivalent:*

1. *X is an $\mathscr{L}_\infty$ space.*
2. *For all Banach spaces $Z \supset Y$ every compact operator $T: Y \to X$ has an extension to a compact operator $\tilde{T}: Z \to X$.*
3. *Same as 2 but without the requirement that $\tilde{T}$ be compact.*
4. *For all Banach spaces Y and Z with $Z \supset X$ every compact operator $T: X \to Y$ has an extension to a compact operator $\tilde{T}: Z \to Y$.*
5. *The same as 4 but without the requirement that $\tilde{T}$ be compact.*

Proof. The proof of Theorem 4.1 is actually contained already in [13] if we take into account the results concerning $\mathscr{L}_\infty$ spaces which were proved on Section 3, in particular, the fact that X is an $\mathscr{L}_\infty$ space if and only if X^{**} is injective. We make here only some comments concerning the proof.

It is obvious that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). In [13, theorems 2.1 and 3.3] it is proved that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4). Hence it remains to prove only that (3) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). The first step in such a proof is to observe that if (3) (resp. (5)) hold there is a $\lambda < \infty$ such that for every Y, Z and compact T there is a suitable $\tilde{T}$ with $\| \tilde{T} \| \leqq \lambda \| T \|$ (cf. the proof of [13, theorem 2.2]). The fact that (3) $\Rightarrow$ (1) is a consequence of (4) $\Rightarrow$ (1) in [13, theorem 2.1], and that X^{**} injective $\Rightarrow$ X is an $\mathscr{L}_\infty$ space. The proof that (5) $\Rightarrow$ (1) is also contained in [13] but since this fact is explicitly stated there only under the hypothesis that X has the approximation property we give here an outline of the argument.

We assume now that X satisfies (5) and let $\lambda < \infty$ be such that $\tilde{T}$ can always be chosen so that $\| \tilde{T} \| \leqq \lambda \| T \|$. Let Z be any $C(K)$ space containing X. Since for every finite-dimensional space B every operator from B into X^* is the adjoint of an operator from X into B^* it follows from our assumption that for every finite-dimensional subspace B of X^* there is an operator $T_B: B \to Z^*$ such that $\| T_B \| \leqq \lambda$ and $\phi T_B(b) = b$ for every $b \in B$, where $\phi: Z^* \to X^*$ is the natural restriction map (i.e. the adjoint of the embedding of X into Z). We consider T_B as a (non-linear) map from X^* into Z^* by putting $T_B x^* = 0$ if $x^* \in X^* \sim B$. We consider the collection $\{T_B\}$ as a net by ordering the finite-dimensional subspaces of X^* by inclusion. Since the unit ball of Z^* is w^* compact it follows by Tychonoff's theorem that there is a subnet $\{T_{B'}\}$ of $\{T_B\}$ such that the limit of $T_{B'}x^*$ exists in the w^* topology for every $x^* \in X^*$. Put $\tilde{T}x^* = w^* \lim T_{B'}x^*$. It is easy to verify that $\tilde{T}: X^* \to Z^*$ is a bounded linear operator such that $\phi\tilde{T}$ is the identity of X^*. Hence $\tilde{T}$ is an isomorphism and $\tilde{T}X^*$ is a complemented subspace of Z^*. Since Z^* is an $L_1(\mu)$ space we get from the results of Section 3 that X^* is an $\mathscr{L}_1$ space and thus X is an $\mathscr{L}_\infty$ space. This concludes the proof.

REMARKS.

(1) It also follows from our results that a Banach space X is an $\mathscr{L}_\infty$ space if and only if there is a constant $\lambda > 0$ such that for all finite-dimensional Banach spaces $Z \supset Y$, every operator $T: Y \to X$ has an extension $\tilde{T}: Z \to X$ with $\| \tilde{T} \| \leqq \lambda \| T \|$. (The assumption that X has this property implies that X^{**} is injective.) Also the proof of (1) $\Rightarrow$ (4) shows that X is an $\mathscr{L}_\infty$ space if and only if there is a constant $\lambda > 0$ such that for all Banach spaces Y and Z with $Z \supset X$ and Y finite-dimensional, every operator $T: X \to Y$ has an extension $\tilde{T}: Z \to Y$ with $\| \tilde{T} \| \leqq \lambda \| T \|$. A similar remark applies to Theorem 4.2 below.

(2) The question of which spaces X have property (3) or (5) for every T (i.e. also non-compact bounded operators) is clearly (cf. [3, p. 94]) equivalent to the question of characterizing the injective spaces. Our results together with those of [13] imply that a Banach space X is an injective space if and only if X is an $\mathscr{L}_\infty$ space and X is isomorphic to a complemented subspace of some conjugate space. (An example is given in [24] of an injective space which is not isomorphic to a conjugate space.) Although much is known concerning the injective spaces (cf. [24]) the question of their characterization is still far from having a satisfactory solution (see the next Remark).

(3) A Banach space X is called a $\mathscr{P}_\lambda$ space if for every Banach space Y with $X \subset Y$ there is a projection from Y onto X or norm $\leqq \lambda$. It is easily seen that an injective Banach space is a $\mathscr{P}_\lambda$ space for some $\lambda \geqq 1$. Using a result of James,

Theorem 3.1 and more care in the proof of Theorem 3.2 it follows that if X is an infinite-dimensional Banach space with X^{**} a $\mathscr{P}_\lambda$ space then X is an $\mathscr{L}_{\infty,10\lambda}$ space. (Indeed James proved in [7] that if B is a Banach space isomorphic to c_0 and if $\varepsilon > 0$ is given then there exists $B_1 \subset B$ with $d(B_1\ c_0) < 1 + \varepsilon$. Using this, it follows by our proof of 2.1 that if X is a subspace of a $C(K)$ space such that there is a projection of norm $\leqq \lambda$ from $C(K)$ onto X, then X is an $\mathscr{L}_{\infty,9\lambda+\varepsilon}$ space for all $\varepsilon > 0$.) We do not know, however, if there exists a function f from the positive real numbers into themselves, such that if X is a finite-dimensional $\mathscr{P}_\lambda$ space, then X is an $\mathscr{L}_{\infty,f(\lambda)}$ space. The best presently-known information on this question seems to be corollary 7, p. 298 of [16], where a function f is given depending not only on λ but also on the "coordinate asymmetry" of X.

(4) In [13], a Banach space was defined to be an $\mathscr{N}_\lambda$ space if there exists a set $\{B_\tau\}$ of finite-dimensional subspaces of X directed by inclusion such that $X = \overline{\bigcup_\tau B_\tau}$, and every B_τ is a $\mathscr{P}_\lambda$ space. Evidently every $\mathscr{L}_{\infty,\lambda}$ space is an $\mathscr{N}_\lambda$ space. It is shown in [13] that if X is an $\mathscr{N}_\lambda$ space, then X^{**} is a $\mathscr{P}_\lambda$ space; we thus obtain that if X is an infimite-dimensional $\mathscr{N}_\lambda$ space, then X is an $\mathscr{L}_{\infty,10\lambda}$ space. (Thus a space is an $\mathscr{N}$ space in the terminology of [13] if and only if it is an $\mathscr{L}_\infty$ space.)

(5) The isometric analogues of Theorem 4.1, i.e. the case where the extension $\tilde{T}$ is required to satisfy $\|\tilde{T}\| = \|T\|$ or $\|\tilde{T}\| \leqq \|T\| + \varepsilon$ for every given $\varepsilon > 0$, are discussed in detail in [13], chaps. V and VI).

(6) Besides extension properties for compact operators, there are also other extension properties with characterize $\mathscr{L}_\infty$ spaces, for example, the possibility of extending weakly-compact operators defined on X (cf. [13, theorem 2.2]). A similar remark applies to Theorem 4.2 below.

The $\mathscr{L}_1$ spaces are characterized, naturally, by properties which are the duals of Properties (2)–(5) of Theorem 4.1. We recall first the definition of the main notion involved in these dual properties.

Let X, Y and Z be Banach spaces and let $\phi: Z \to Y$ be a surjective operator. An operator $T: X \to Y$ is said to admit a lifting to Z (with respect to ϕ) if there is an operator $\tilde{T}: X \to Z$ such that $\phi\tilde{T} = T$. Such a $\tilde{T}$ is called a lifting of T to Z.

THEOREM 4.2. *Let X be a Banach space. Then the following five statements are equivalent:*

1. *X is an $\mathscr{L}_1$ space.*

2. *For all Banach spaces Z and Y and any surjective operator $\phi: Z \to Y$, every compact operator $T: X \to Y$ has a compact lifting $\tilde{T}$ to Z.*

3. *Same as 2, but without the requirement that the lifting $\tilde{T}$ be compact.*

4. *For all Banach spaces Z and Y and any surjective operator $\phi: Z \to X$, every compact operator $T: Y \to X$ has a compact lifting $\tilde{T}$ to Z.*

5. *Same as 4, but without the requirement that the lifting $\tilde{T}$ be compact.*

Proof. Clearly (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). Again it is easy to prove that if (3) (resp. (5)) hold for a space X then there is a λ such that for every quotient map ϕ a suitable $\tilde{T}$ in (3) (resp. (5)) can be chosen so that $\| \tilde{T} \| \leqq \lambda \| T \|$. In view of this remark the proof of (3) $\Rightarrow$ (1) of Theorem 4.1 (i.e. of [13, theorem 2.1]) shows that if X satisfies (3) of the present theorem X^* is an $\mathscr{L}_\infty$ space and hence by Theorem III X is an $\mathscr{L}_1$ space. An argument very similar to the proof of (5) $\Rightarrow$ (1) of Theorem 4.1 proves also that (5) $\Rightarrow$ (1) in the present theorem. We shall indicate now the proofs of (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4).

Proof of (1) $\Rightarrow$ (2). Assume that X is an $\mathscr{L}_{1,\lambda}$ space and thus $X = \bigcup_\alpha B_\alpha$ where the $\{B_\alpha\}$ form a net of finite-dimensional subspaces of X directed by inclusion such that $d(B_\alpha, l_1^{n\alpha}) \leqq \lambda$ for all α. Each B_α thus contains vectors $\{e_{i,\alpha}\}_{i=1}^{n_\alpha}$ such that $\| e_{i,\alpha} \| = 1$ and $b \in B_\alpha$, $\| b \| \leqq 1 \Rightarrow b = \sum_i \gamma_i e_{i,\alpha}$ with $\sum | \gamma_i | \leqq \lambda$. Let $\phi: Z \to Y$ be a surjective homomorphism and let $T: X \to Y$ be a compact operator of norm 1. We denote the unit ball of the Banach space X by S_X (and similarly the unit balls of other Banach spaces). Since $\overline{TS_X}$ is a compact subset of Y it follows from the open mapping theorem that there is a compact symmetric subset U of Z such that $\phi U \supset \overline{TS_X}$. Since $e_{i,\alpha} \in S_X$ for every i and α there is a $u_{i,\alpha} \in U$ such that $\phi u_{i,\alpha} = Te_{i,\alpha}$. For each α define now $T_\alpha: B_\alpha \to Z$ by $T_\alpha(\sum_i \gamma_i e_{i,\alpha}) = \sum_i \gamma_i u_{i,\alpha}$. Clearly $\phi T_\alpha = T_{|B\alpha}$ and $T_\alpha S_{B\alpha} \subset \lambda U$. By a compactness argument like the one in the proof of (5) $\Rightarrow$ (1) in Theorem 4.1 it follows that there is a $\tilde{T}: X \to Z$ such that $\phi \tilde{T} = T$ and $\tilde{T} S_X \subset \lambda U$. This proves that (2) holds.

Proof of (1) $\Rightarrow$ (4). Assume that X is an $\mathscr{L}_{1,\lambda}$ space. It follows from Theorem III(c) that every compact operator into X is the limit in the norm topology of operators with a finite-dimensional range. Hence (4) will be proved once we show that for every $\phi: Z \to X$ with ϕ surjective there is a λ such that for every Y and every $T: Y \to X$ with $\dim TY < \infty$ there is a compact lifting $\tilde{T}$ of T into Z with $\| \tilde{T} \| \leqq \lambda \| T \|$.

Assume now that $\dim TY < \infty$. Let the subspace B of X be such that $TY \subset B$ and $d(B, l_1^n) \leqq \lambda$. By a simple and well known property of l_1^n spaces it follows that there is a $\tilde{T}: Y \to \phi^{-1}(B) \subset Z$ such that $\dim \tilde{T}Y < \infty$, $\phi\tilde{T} = T$ and $\| \tilde{T} \| \leqq K \| T \|$ where K depends only on λ and ϕ but not on T. This concludes the proof.

REMARKS. 1. If in statements 2 or 4 the requirements that T be compact (and of course also that $\tilde{T}$ be compact) are dropped one gets properties which characterize $l_1(\Gamma)$ spaces. This result is due to Pełczyński [21] and Köthe [11].

2. The isometric analogues of Theorem 4.2 (i.e. the discussion of the lifting properties when ϕ is a quotient map and $\tilde{T}$ is required to be so that $\| \tilde{T} \| = \| T \|$ or $\| \tilde{T} \| \leqq \| T \| + \varepsilon$) are treated in [6] and [15].

We pass now to characterizations of the $\mathscr{L}_p$ spaces for all p. The results of Sections 2 and 3 together with a compactness argument, show that a considerably weaker form of the requirements of the Definition in Sec. 1 yields a characterization of the $\mathscr{L}_p$ spaces for $p = 1$ or ∞, and a joint characterization of the $\mathscr{L}_p$ and $\mathscr{L}_2$ spaces for $1 < p < \infty$.

THEOREM 4.3. *Let $1 \leqq p \leqq \infty$, and let X be a Banach space satisfying the following property: There is a $\lambda > 0$, so that for every finite-dimensional subspace E of X, there exists an n, and operators $T_E: E \to l_p^n$ and $S_E: l_p^n \to X$ satisfying $\| T_E \| \leqq 1$, $\| S_E \| \leqq \lambda$, with $S_E T_E$ equal to the identity operator on E. Then X is an $\mathscr{L}_p$ space or an $\mathscr{L}_2$ space if $1 < p < \infty$, and X is an $\mathscr{L}_p$ space if $p = 1$ or ∞.*

(If $p = 1$ or $p = \infty$, and X is of infinite-dimension X cannot be an $\mathscr{L}_2$ space. However for all p with $1 < p < \infty$, l_2 is isomorphic to a complemented subspace of L_p; hence every $\mathscr{L}_2$ space X satisfies these hypotheses.)

Proof. The proof is very similar to the proof of theorem 7.1 of [16], for the case of $1 \leqq p < \infty$. (One has to correct a slight error in this proof: the functionals $T_E^*(\xi_i)$ must be replaced by suitable Hahn-Banach extensions y_i^E, to all of X.) Following the argument given there and using the weak* topology in X^{**}, one obtains an $L_p(\mu)$ space $\tilde{Z}$, a subspace $\tilde{X}$ of $\tilde{Z}$ isomorphic to X, and a bounded linear operator $\tilde{P}: \tilde{Z} \to \tilde{X}^{**}$ with $\tilde{P}(x) = x$ for all $x \in \tilde{X}$. If $1 < p < \infty$, $\tilde{Z}$ and hence X is reflexive, thus $\tilde{p}$ is a projection from $\tilde{Z}$ onto $\tilde{X}$. Hence X is isomorphic to a complemented subspace of an $L_p(\mu)$ space, so X is an $\mathscr{L}_p$ space or an $\mathscr{L}_2$ space by Theorem 2.1. If $p = 1$, it follows that $\tilde{X}^{**}$ is complemented in $\tilde{Z}^{**}$ (cf. [13]) and hence X is an $\mathscr{L}_1$ space by Theorem 3.3.

Finally, suppose $p = \infty$. It then follows that if Y and Z are finite-dimensional Banach spaces with $Z \supset Y$, then every operator $T: Z \to X$ has an extension $\tilde{T}: Z \to X$ with $\|\tilde{T}\| \leqq \lambda \|T\|$. Thus by Theorem 4.1 (cf. the first remark following its proof) X is an $\mathscr{L}_\infty$ space. Q.E.D

We now apply Theorem III to questions treated in [19]. We recall first the definition of the basic notion introduced in [19]. *A Banach space X is said to have sufficiently many Boolean algebras of projections* if there is a number λ such that for every finite-dimensional subspace B of X there is a Boolean algebra $\mathscr{B}$ of projections on X satisfying

$$(*) \qquad \|\mathscr{B}\| = \sup\{\|P\|,\ P \in \mathscr{B}\} \leqq \lambda \quad \text{and}$$

$$B \subset \overline{\operatorname{span}}\{Py; P \in \mathscr{B}\} \text{ for some } y \in X.$$

Let us say that *all Boolean algebras of projections on a Banach space X are of the same type.* if there exists a monotone positive real-valued function $g(t)$ defined for $t \geqq 1$ and a positive real-valued function $f(u_1, u_2, \cdots)$ defined for all sequences $\{u_i\}_{i=1}^{\infty}$ of non-negative numbers with only finitely many $u_i \neq 0$ such that for every Boolean algebra $\mathscr{B}$ of projections on X, every $x \in X$ and every disjoint $\{P_i\}_{i=1}^{n}$ in X

$$(**) \qquad g(\|\mathscr{B}\|)^{-1} f(\|P_1x\|, \|P_2x\|, \cdots, \|P_n(x)\|, 0, 0, 0, \cdots)$$

$$\leqq \Big\|\sum_{i=1}^{n} P_i x\Big\| \leqq g(\|\mathscr{B}\|) f(\|P_1x\|, \|P_2x\|, \cdots, \|P_nx\|, 0, 0, \cdots).$$

Finally let us say that *a Banach space X has sufficiently many Boolean algebras of projections of the same type* if there is a number $\lambda < \infty$ and functions $g(t), f(u_1, u_2, \cdots)$ as above so that for every finite-dimensional subspace B of X there is a Boolean algebra of projections $\mathscr{B}$ on X so that (*) and (**) hold (of course here g may be chosen to be constant).

The proof of the main result of [19] together with Theorem IIIc show that

THEOREM 4.4. *A Banach space X is an $\mathscr{L}_p$ space for some p, $1 \leqq p \leqq \infty$, if and only if X has sufficiently many Boolean algebras of the same type.*

In view of Theorem IIIc (for $p = 1$) we may now restate the main result of [19] as a joint characterization of $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_\infty$ spaces.

THEOREM 4.5. *A Banach space X is an $\mathscr{L}_p$ space for $p = 1$, 2 or ∞ if and only if*

(i) *X has sufficiently many Boolean algebras of projections.*

(ii) *all Boolean algebras of projections on X are of the same type.*

REMARK. L. Tzafriri has obtained [28] a characterization of spaces isomorphic to $L_p(\mu)$ spaces $1 \leqq p < \infty$ or $c_0(\Gamma)$. This characterization is in terms of Boolean algebras of projections and is closely related to Theorem 4.4.

Section 5. Remarks and open problems. The main open question concerning $\mathscr{L}_p$ spaces is certainly the problem of finding a functional representation of these spaces. We shall sum up here the known results in this direction. The only case where the functional representation presents no problem is the case $p = 2$. By Theorem I a Banach space is an $\mathscr{L}_2$ space if and only if it is isomorphic to a Hilbert space.

For $1 < p < \infty$, $p \neq 2$, the problem of functional representation of the $\mathscr{L}_p$ spaces is by Corollary 1 to Theorem 2.1, equivalent to the problem of isomorphic classification of complemented subspaces of $L_p(\mu)$ spaces. By duality it is clearly enough to consider the range $2 < p < \infty$. This fact may be helpful since for $p > 2$ some information on the structure of an arbitrary subspace of $L_p(\mu)$ is known (cf. [8]). In the separable (infinite-dimensional) case four different isomorphics type of $\mathscr{L}_p$ spaces ($1 < p < \infty$, $p \neq 2$) are known [16]: l_p, L_p, $l_p \oplus l_2$ and $(l_2 \oplus l_2 \oplus \cdots)_p$. It is remarked in [16] that other "natural candidates" are either not $\mathscr{L}_p$ spaces or are isomorphic to one of those four spaces. Thus, e.g. $(l_p \oplus l_p \oplus \cdots)_2$ is not an $\mathscr{L}_p$ space ($p \neq 2$) and $L_p(l_2)$ (= the space of measurable $f: [0,1] \to l_2$ whose pth power is Bochner integrable) is isomorphic to L_p. Let us repeat now two questions posed in [16].

PROBLEM 1a. Are there only four different isomorphism types of separable infinite-dimensional $\mathscr{L}_p$ spaces ($1 < p < \infty$, $p \neq 2$)?

PROBLEM 1b. Let X be an infinite-dimensional $\mathscr{L}_p$ space which is isomorphic to a subspace of l_p. Is $X \approx l_p$ ($1 < p < \infty$, $p \neq 2$)?

Clearly Problem 1b is a special case of Problem 1a. The next Problem, on the other hand, generalizes Problem 1a.

PROBLEM 1c. Is every $\mathscr{L}_p$ space X isomorphic to a direct sum of an $L_p(\mu)$ space and a sum (with an arbitrary number, finite or infinite, of summands) of Hilbert spaces with the l_p norm, i.e. $X \approx L_p(\mu) \oplus (\sum_\gamma \oplus L_2(\mu_\gamma))_p (1 < p < \infty, p \neq 2)$? (Every space of the above form is isomorphic to a complemented subspace of some

$L_p(v)$ space, and hence is an $\mathscr{L}_p$ space by Theorem III; for a direct proof, see [16].)

We pass now to the case $p = 1$. We know of five different isomorphism types of separable infinite-dimensional $\mathscr{L}_1$ spaces. These are l_1, L_1 D, $L_1 \oplus D$ and $L_1 \otimes D$, where D is the subspace of l_1 spanned by $\{u_k\}_{k=1}^{\infty}$ where $u_k = e_k - (e_{2k} + e_{2k+1})/2$, $k = 1,2,\cdots$, and $\{e_k\}_{k=1}^{\infty}$ denotes the usual basis vectors of l_1. (In the discussion below of $\mathscr{L}_1$ spaces, D will always denote this particular Banach space). By $A \otimes B$ we denote here the completion of the algebraic tensor product of A and B normed by the greatest cross norm. It is known (cf. [5]) that $L_1 \otimes X$ can be identified with $L_1(X)$, the space of X-valued Bochner integrable functions on $[0,1]$. The space D was introduced in [14]; it is the kernel of a certain quotient map $T: l_1 \to L_1$. By Theorem 2 of [17] an infinite-dimensional subspace U of l_1 is determined up to isomorphism by l_1/U. Hence the fact that D is the kernel of an operator from l_1 onto L_1 already determines the isomorphism type of D. It is proved in [14] that D is not isomorphic to a complemented subspace of an $L_1(\mu)$ space. That D is an $\mathscr{L}_1$ space (actually an $\mathscr{L}_{1,\lambda}$ space for every $\lambda > 2$) is easily checked directly and follows also from Proposition 5.2 below. It is easy to verify that if A and B are $\mathscr{L}_1$ spaces then $A \oplus B$ and $A \otimes B$ are also $\mathscr{L}_1$ spaces. In view of these facts the only part of the statement made above concerning l_1, L_1, D $L_1 \oplus D$ and $L_1 \otimes D$ which still requires a proof is that $L_1 \oplus D \not\approx L_1 \otimes D$. This fact is an immediate consequence of the following proposition.

PROPOSITION 5.1. *Let P be a projection of $L_1 \otimes D$ onto a subspace isomorphic to D. Then the kernel of P is isomorphic to $L_1 \otimes D$.*

Proof. Let $Y = L_1 \otimes D$ and let P be a projection of Y onto a subspace isomorphic to D. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements in L_1 such that $\|u_n\| = 1$ all n and $u_n \to 0$ weakly. Let $Z_n = u_n \otimes D$, $n = 1,2,\cdots$. Every Z_n is a subspace of Y isometric to D and there is a projection Q_n from Y onto Z_n with $\|Q_n\| = 1$. We claim that for large enough n, $\|P(u_n \otimes x)\| \leqq \frac{1}{2}\|u_n \otimes x\|$ for every $x \in D$. Indeed, for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in D we have by the choice of u_n that $u_n \otimes x_n \to 0$ weakly in Y. Hence $P(u_n \otimes x_n) \to 0$ weakly and thus, since PY is isomorphic to the subspace D of l_1, $\|P(u_n \otimes x_n)\| \to 0$ and this proves our assertion. Let I denote the identity operator of Y and I_n the identity operator of Z_n, $n = 1,2,3,\cdots$. Since for large n $\|P_{|Z_n}\| \leqq \frac{1}{2}$, $I_n - Q_nP_{|Z_n}$ is an invertible operator in Z_n. It follows easily that $(I-P)(I_n - Q_nP_{|Z_n})^{-1}Q_n$ is a projection from Y onto its subspace $(I-P)Z_n$ which is isomorphic to Z_n and thus to D. We have thus shown that $(I-P)Y \approx D \oplus U$ for a suitable U. Since $D \oplus D \approx D$ (see below) we get that

$$Y \approx (I-P)Y \oplus PY \approx (I-P)Y \oplus D \approx D \oplus D \oplus U \approx D \oplus U \approx (I-P)Y$$

as desired.

It follows from theorem 2 of [17] cited above that $D \approx D \oplus D \approx D \oplus l_1$. Indeed, let $T: l_1 \to L_1$ be the quotient map whose kernel is D. Then $T \oplus T: l_1 \oplus l_1 \to L_1 \oplus L_1$ and $T \otimes I: l_1 \otimes l_1 \to L_1 \otimes l_1$ are quotient maps whose kernels are $D \oplus D$ and $D \otimes l_1$ respectively. Since $l_1 \approx l_1 \oplus l_1 \approx l_1 \otimes l_1$ and $L_1 \oplus L_1 \approx L_1 \otimes l_1 \approx L_1$ we can apply Theorem 2 of [17] to get the desired result. We do not know, however, whether $D \otimes D \approx D$. Another candidate for a new isomorphism type of a separable $\mathscr{L}_1$ space is obtained in the following way. Let $T_1: l_1 \to D$ be a quotient map and let D_1 be the kernel of T_1. By Theorem 2 of [17] the isomorphism type of D_1 does not depend on the particular choice of T_1. By Proposition 5.2 below D_1 is an $\mathscr{L}_1$ space and by the reasoning of [14] D_1 is not isomorphic to a complemented subspace of an $L_1(\mu)$ space.

PROBLEM 2a. Are any two of the spaces D, $D \otimes D$ and D_1 isomorphic?

Let us remark that it is not hard to give concrete representations of $D \otimes D$ and D_1. Let $\{e_{n,k}\}_{n,k=1}^{\infty}$ be an enumeration of the unit vectors of l_1 as a double sequence. Put

$$u_{n,k} = e_{n+1,k} - e_{n,k} - 2^{-n} \sum_{i=0}^{2^n-1} e_{n,2^nk+i}$$

and

$$\begin{aligned} v_{n,k} = \; & e_{n,k} - \tfrac{1}{2}(e_{2n,k} + e_{2n+1,k} + e_{n,2k} + e_{n,2k+1}) \\ & + \tfrac{1}{4}(e_{2n,2k} + e_{2n+1,2k} + e_{2n,2k+1} + e_{2n+1,2k+1}) \end{aligned}$$

for $n,k = 1,2,\cdots$. Then D_1 is isomorphic to the closed linear span of the $\{u_{n,k}\}$ while $D \otimes D$ is isomorphic to the closed linear span of the $\{v_{n,k}\}$. If it turns out that either $D \otimes D$ or D_1 give a new isomorphism type of $\mathscr{L}_1$ spaces it would seem very likely that there are infinitely many isomorphism types of separable infinite-dimensional $\mathscr{L}_1$ spaces. Let us mention in this connection that by Theorems I and III and the results of [21] and [24] a Banach space Z is a separable infinite-dimensional $\mathscr{L}_1$ space if and only if Z^* is isomorphic to l_∞.

Another problem concerning $\mathscr{L}_1$ spaces is

PROBLEM 2b. Is every complemented subspace of an $L_1(\mu)$ space isomorphic to $L_1(\nu)$ for some ν? In particular is every infinite-dimensional complemented subspace of L_1 isomorphic to either l_1 or L_1? (It follows from Theorem III that a space X is isomorphic to a complemented subspace of an $L_1(\mu)$ space if and only if X is complemented in X^{**} and X is an $\mathscr{L}_1$ space.)

We turn to the case $p = \infty$. In this case even the problem of functional representation and isomorphic classification of the separable spaces which are $\mathscr{L}_{\infty,1,+\varepsilon}$ for every $\varepsilon > 0$ is not completely solved. For information concerning the functional representation of spaces which are $\mathscr{L}_{\infty,1+\varepsilon}$ for every $\varepsilon > 0$ we refer to [18]. The $C(K)$ spaces are the most important class of spaces which are $\mathscr{L}_{\infty,1+\varepsilon}$ spaces for every $\varepsilon > 0$ and therefore it is natural to ask

PROBLEM 3a. Is every Banach space Y whose dual is isometric to an $L_1(\mu)$ space (i.e. Y is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$) isomorphic to a $C(K)$ space? In particular what is the case if Y is separable?

The question of isomorphic classification of separable $C(K)$ spaces has been completely solved by Bessaga and Pełczyński [2] and Milutin [20] (cf. also [22]). There are uncountably many different isomorphic types of such spaces and thus in particular of separable infinite-dimensional $\mathscr{L}_\infty$ spaces. The present knowledge concerning the isomorphic classification of non-separable $C(K)$ spaces, though quite large, is still very fragmentary. The main sources for information concerning this question are [22] and [24].

It follows from theorems 2.3 and 5.1 of [24] and the continuum hypothesis that if X is a separable $\mathscr{L}_\infty$ space, then X^{**} is isomorphic either to l_∞ or to $l_\infty(\Gamma)$ where Γ is a set of cardinality the continuum. Let us restate a question raised in [24]:

PROBLEM 3b. Let X be an $\mathscr{L}_\infty$ space. Is X^{**} isomorphic to $l_\infty(\Gamma)$ for some set Γ?

It is conjectured in [24] that 3b has an affirmative answer.

A third natural question concerning $\mathscr{L}_\infty$ spaces is

PROBLEM 3c. Is every $\mathscr{L}_\infty$ space isomorphic to a space which is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$?

The case $p < \infty$ seems to suggest that the answer to Problem 3c might be negative.) However we do not know at present even of a possible candidate for a counterexample. The case $p < \infty$ also suggests (cf. Theorem I) that the following problem should have a positive answer.

PROBLEM 3d. Does every infinite-dimensional $\mathscr{L}_\infty$ space have a subspace isomorphic to c_0?

Zippin [29] has shown that a space which is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$ contains a subspace isometric to c_0. Thus a positive answer to Problem 3c would imply that also Problem 3d has a positive answer.

PROBLEM 3e. Is every $\mathscr{L}_\infty$ space isomorphic to a complemented subspace of a $C(K)$ space?

It is even unknown whether every space which is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$ is isomorphic to a complemented subspace of a $C(K)$ space. The spaces Y which are isometric to a subspace of a $C(K)$ space on which there is a projection of norm 1 were characterized in [18]. By [21] a positive solution to Problem 3e would imply a positive solution to Problem 3d. We state finally the analogue of Problem 1b.

PROBLEM 3f. Is every infinite-dimensional $\mathscr{L}_\infty$ subspace of c_0 isomorphic to c_0?

Another class of problems concerning $\mathscr{L}_p$ spaces stems from considering the following question. Let $X \supset Y$ be Banach spaces. Assume that two out of the three spaces $X, Y, X/Y$ are $\mathscr{L}_p$ spaces for a given $1 \leqq p \leqq \infty$. Is the third space also an $\mathscr{L}_p$ space? As far as positive results concerning this question are concerned we have

PROPOSITION 5.2. (a) *If X/Y and either one of Y or X are an $\mathscr{L}_1$ space then also the third space is an $\mathscr{L}_1$ space.*

(b) *If X and either Y or X/Y are $\mathscr{L}_2$ spaces then also the third space is an $\mathscr{L}_2$ space.*

(c) *If Y and either X or X/Y are $\mathscr{L}_\infty$ spaces then also the third space is an $\mathscr{L}_\infty$ space.*

Proof. (a) Since X/Y is an $\mathscr{L}_1$ space it follows by Theorem I that $(X/Y)^*$ is injective. Hence $X^* \approx (X/Y)^* \oplus Y^*$. Consequently X^* is injective if and only if Y^* is injective and by Theorem III this concludes the proof.

Part (b) of the Proposition is of course trivial and well known. It is clearly enough to assume only that X is an $\mathscr{L}_2$ space. We chose to write (b) in the present form just for reasons of symmetry. Part (c) follows from Part (a) by passing to the dual.

PROBLEM 4a. Is it true that in every case which is not covered by Proposition 4.2 the assumption that two of the spaces X, Y, X/Y are an $\mathscr{L}_p$ space for some $1 \leqq p \leqq \infty$ does not imply in general that the third space is also an $\mathscr{L}_p$ space?

We cannot settle this problem in any of the cases not covered by Proposition 5.2. The most interesting special cases of Problem 4a are for $p = 1$, 2 and ∞. For $p = 2$ this problem has been around for some time (it is attributed to R. Palais). Let us state it explicitly.

PROBLEM 4b. Let X be a Banach space and let Y be a subspace of X such that Y and X/Y are both isomorphic to Hilbert space. Is X isomorphic to a Hilbert space?

For $p = 1$ Problem 4a is equivalent to the same problem for $p = \infty$ (by duality). It can be shown that Problem 4 for $p = 1$ is also equivalent to the more concrete

PROBLEM 4c. Let Y be a subspace of l_1 such that $Y \approx l_1$. Is Y complemented in l_1?

We conclude the paper with the following well-known open problem.

PROBLEM 5. Let X be a Banach space such that every closed subspace of X is complemented. Is X an $\mathscr{L}_2$ space?

We mention this problem here since it is, in a sense, the "missing link" in the circle of questions treated in Section 4. It is well known that the isometric version of Problem 5 is true (Kakutani [9]).

It is also well known that the common Banach spaces which are not $\mathscr{L}_2$ spaces have an uncomplemented subspace. For general classes of spaces, however, little is known concerning Problem 5. Results of general type exist under some symmetry conditions on X. For example, by using the results of [26] the first named author proved (unpublished) that every Banach space with a symmetric basis in which every closed subspace is complemented is isomorphic to Hilbert space.

Added in proof. (1) Since the completion of the present paper the second named author found two new isomorphic types of separable infinite-dimensional $\mathscr{L}_p$ spaces, $1 < p < \infty$, $p \neq 2$. Thus, for every such value of p there are now known six different isomorphic types of separable infinite-dimensional $\mathscr{L}_p$ spaces. The answer to problems 1a and 1c above is therefore negative. It is very likely that there are more isomorphic types (perhaps even infinitely many) of such spaces. For $p > 2$ the new types of $\mathscr{L}_p$ spaces are obtained as follows. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $w_n \to 0$ as $n \to \infty$ and $\sum_n w_n^{2p/(p-2)} = \infty$. Let $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ denote the unit vector bases of l_2 and l_p respectively. Let X_p be the closed linear span of $\{w_n \cdot e_n + f_n\}_{n=1}^{\infty}$ in $l_2 \oplus l_p$. Then X_p and $Y_p = (X_p \oplus X_p \oplus \ldots)_p$ are new isomorphic types of $\mathscr{L}_p$ spaces. The isomorphic type of X_p (and thus also of Y_p) does not depend on the particular choice of the sequence $\{w_n\}_{n=1}^{\infty}$. For $1 < p < 2$ the new isomorphic types are obtained by duality. An examination of the new examples gives also a partial solution to problem 4a. Details will appear elsewhere.

(2) The separation theorem of Klee can be deduced from the usual separation theorem. Let $\{C_i\}_{i=0}^{n}$ be convex open sets in a locally convex space X such that $\cap_{i=0}^{n} C_i = \varnothing$. Consider the set

$$C = \{(x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_n);\ x_i \in C_i,\ 0 \le i \le n\}$$

in $X \oplus X \oplus \ldots \oplus X$ (n times). The set C is convex, open, and does not contain the origin. Hence by the usual separation theorem there are $\{f_i\}_{i=1}^{n} \in X^*$ such that $\sum_{i=1}^{n} f_i(x_0 - x_i) > 0$ whenever $x_i \in C_i$, $0 \le i \le n$. Let $T: X \to R^n$ be defined by $Tx = (f_1(x), \ldots, f_n(x))$. Then $\cap_{i=0}^{n} TC_i = \varnothing$. Q.E.D.

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