Chapter 1: Systems of linear equations & matrices

linear equation:

An equation is called linear if the powers of all variables in the equation are 1, and it does not involve any product of variables

Examples:

$\sqrt{2}x+3y-2z=3$ → linear

$3x+\sqrt{y}=5$ → not linear

$x-yz+2y=1$ → not linear

$x_1-2x_2+3x_3=7$ → linear

$\boxed{\text{General form for linear equation}}$

$a_1x_1+a_2x_2+\dots+a_nx_n=b$

→ Where $x_1, x_2, \dots, x_n$ called variables (unkowns)

$a_1, a_2, \dots, a_n$ and b called coefficents (constans)

Special parts:

$2x+3y=0$ is called homogeneous linear equation

وذلك لأن قيمة المعادلة تساوي 0

linear Systems:

A finite set of linear equations

$$2x_1+3x_2-x_3=1$$
$$x_1-2x_2+3x_3=2$$
$$x_1+2x_3=5$$

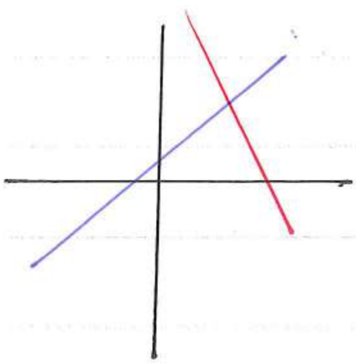
→ system of linear equations or linear system

System of linear equation

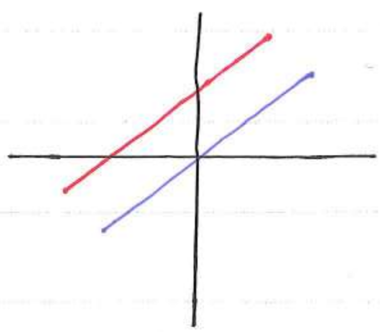
- one solution
- no solution
- infinitely many solutions

* if there is at least one solution of the system it is called consistent
* if the system has no solution it's called inconsistent

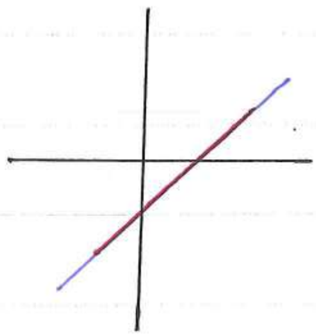
$$\left.\begin{aligned} x-2y&=0\\ 2x+y&=0\\ x+y&=0 \end{aligned}\right\}$$ homogeneous linear system



One solution



no solution



infinitely many solutions

Exercise 8

1. find the solution for the system

$$x+2y=4$$
$$2x+4y=9$$

يمكن حل هذا النوع من الاسئلة بالحذف او التعويض

$\rightarrow -2*\text{equation } 1 \rightarrow$ $-2x-4y=-8$

$$\underline{2x+4y=9}$$

$0 \neq 1$ بما أن الناتج عبارة رياضية خاطئة

$\rightarrow$ The system has no solution

2. find the solution for the system

$$3x-y=3$$
$$x+y=5$$

$$3x-\cancel{y}=3$$
$$\underline{x+\cancel{y}=5}$$
$$4x=8 \rightarrow \boxed{x=2} \; \boxed{y=3}$$

$\rightarrow$ $(2,3)$ is a solution, the system has one solution

3. find the solution for the system

$$3x-2y=9$$
$$6x-4y=18$$

$\rightarrow -2\,eq1 \rightarrow$ $-6x+4y=-18$

$$\underline{6x-4y=18}$$

$0=0$ $\rightarrow$ The system has infinitely many solutions

let $\boxed{y=t} \rightarrow \boxed{x=\frac{2}{3}t+3} \rightarrow \left(\frac{2}{3}t+3,\ t\right)$

at $t=0 \rightarrow (3,0)$ is solution

how we use the matrices to solve linear system ?

Elementary Row Operations (e.r.o)

1. multiply a row by non zero constant
2. Interchange two rows
3. Add a multiple of one equation to another

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
\qquad
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m
\end{bmatrix}
$$

Augmented matrix

← يمكننا حل linear system باستخدام الـ matrix اذا تمكنا من تحويل Augmented matrix للشكل التالي عن طريق elementary row operations

$$
\begin{bmatrix}
1 & a_{12} & a_{13} & b_1 \\
0 & 1 & a_{23} & b_2 \\
0 & 0 & 1 & b_3
\end{bmatrix}
$$

Triangular form

diagonal

حتى لو لم تساوي القيم 1 لا توجد مشكلة ولكن اذا كانت 1 يكون اسرع للحل

Q1 Solve the following system of linear equation::

$$2x - y + 3z = 13$$
$$x + 2y = -3$$
$$3x + y - z = -2$$

step 1 $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 1 & 2 & 0 & -3 \\ 3 & 1 & -1 & -2 \end{array}\right]$ → Augmented Matrix

Step 2 0 نهدف الى جعل هذه القيمة

$* R_2 \to -2R_2 + R_1$

$* R_3 \to -3R_2 + R_3$

$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 0 & -5 & 3 & 19 \\ 0 & -5 & -1 & 7 \end{array}\right)$

Note!!

يمكن عمل خطوتين في خطوة واحدة

ويمكن تقسيم العمليتين لخطوتين

step 3 نهدف لجعل هذه القيمة 0 ايضاً

$R_3 \to R_2 - R_3$

$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 0 & -5 & 3 & 19 \\ 0 & 0 & 4 & 12 \end{array}\right)$

step 4 Back substitution

$$2x - y + 3z = 13$$
$$-5y + 3z = 19$$
$$4z = 12 \rightarrow \boxed{z = 3}$$
$$-5y + 9 = 19 \rightarrow \boxed{y = -2}$$
$$2x + 2 + 9 = 13 \rightarrow \boxed{x = 1}$$

(3) Solve the following system of linear equation

$$X_1 + 4X_2 + X_3 = 5$$

$$X_1 + X_2 + 5X_3 = 12$$

$$3X_1 + X_2 + X_3 = 8$$

Step 1 Augmented Matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 5 \\ 1 & 1 & 5 & 12 \\ 3 & 1 & 1 & 8 \end{array}\right)$$

step 2 $R_2 \to -R_1 + R_2$

$R_3 \to -3R_1 + R_3$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 5 \\ 0 & -3 & +4 & +7 \\ 0 & -11 & -2 & -7 \end{array}\right)$$

step 3 $R_3 \to -\frac{11}{3} R_2 + R_3$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 5 \\ 0 & -3 & 4 & 7 \\ 0 & 0 & -\frac{50}{3} & -\frac{98}{3} \end{array}\right)$$

Step 4 $X_1 + 4X_2 + X_3 = 5$

$-3X_2 + 4X_3 = 7$

$-\frac{50}{3} X_3 = \frac{-98}{3} \to X_3 = 1.96$

$\to X_2 = -2.33$

$\to X_1 = 5$

Remarks:

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

في حال واجهنا هذا الشكل اثناء محاولة الوصل
لـ triangular form فهذا الشكل يعني أن
$\Rightarrow$ (linear system has infinitely many solutions)

← التفسير الرياضي لهذا الامر هو ان عدد المعادلات اصبح اقل من عدد المجاهيل

$x+3y+2z=4$

$y-3z=2$

let $z=t \rightarrow y=2+3t \rightarrow x=-2-11t$

$(-2-11t, 2+3t, t)$ ← ويكتب الحل بهذا الشكل

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

اما هذا الشكل فهو يعني أن
$\Rightarrow$ linear system has no solution

← السبب هو وجود عبارة رياضية خاطئة $1 \neq 0$

Q_3 for which values of "a" will the following system has: no solution? exactly one solution? infinitely many sol.?

$x + 2y - 3z = 4$

$3x - y + 5z = 2$

$4x + y + (a^2 - 14)z = a + 2$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right)$$

→ step 1 Augmented Matrix

→ step 2 $R_2 \to -3R_1 + R_2$

$R_3 \to -4R_1 + R_3$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

→ step 3 $R_3 \to -R_2 + R_3$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

→ if $a = 4$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

→ infinitely many solutions

→ if $a = -4$

$$\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

→ no solution

→ $a \neq \pm 4$ exactly one solution (unique)

Gaussian Elimination

→ Reduced Row Echelon form (r.r.e.f)

we say the matrix is in reduced row echelon form (r.r.e.f) if the following conditions holds:

1. The first nonzero number in the row is a 1, and we call it **leader 1**

leader1 $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

leading 1

(تحقق الشرط)

$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ → not leader 1

leading 1

(لا تحقق الشرط)

* تقرأ الاعداد من اليسار الى اليمين

2. Any Rows with All zeros are placed at the bottom of the matrix

$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(تحقق الشرط)

$\begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 & 1 \end{pmatrix}$ → يجب ان يكون آخر سطر

(لا تحقق الشرط)

* لا مانع من وجود سطرين اصفار ولكن المهم ان يكونا بالاسفل

3. Every leading 1 is to the right of the one above it

$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

(تحقق الشرط)

$\begin{pmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ →

* المهم ان يكون على يمينه ولا يشترط ان يكون في العمود الذي يليه مباشرة

(تحقق الشرط)

4. Each column that contains a leader 1, has zeros everywhere else.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

(تحقق الشرط) (تحقق الشرط) (لا تحقق الشرط)

العمود الثاني لا يحتوي على leader1 لذلك لا يهم باقي القيم

→ A matrix that has the first three properties is called row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

↓ r.r.e.f ↓ r.e.f ↓ r.r.e.f

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

↓ r.e.f ↓ r.r.e.f

Gaussian Elimination

Used to convert Matrix to row echelon form

Gaussian Jordan Elimination

Used to convert Matrix to reduced row echelon form

وجود المصفوفة لنظام خطي ما في وضعية r.r.e.f سياعدكم حل النظام الخطي بسهولة بالغة

Q_1 Suppose that the augmented matrix for a linear system in the unknowns x, y, z has been reduced by elementary row operations to:

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{pmatrix}$$

what is the solution of the system.

← نلاحظ أن المصفوفة في وضعية r.r.e.f

back substitution: $x=5,\ y=3,\ z=7 \rightarrow (5,3,7)$ unique solution

Q_2 if the augmented Matrix for a linear system with unknowns x_1, x_2, x_3, x_4 is:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

what is the solution of the system

$0 \neq 1$ so the system has no solution

free variables and leading variables:

→ **leading variables:** The variables whose columns in the r.r.e.f contain leading 1 → اي المتغيرات التي تحتوي عمودها على leading 1

→ **free variables:** The variables whose columns in the r.r.e.f does not contain a leading 1

$$\begin{array}{ccc} x & y & b \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

x → leading variable

y → free variable

Q3 what is the solution of the system

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

بما أن عدد المتغيرات أكبر من عدد المعادلات

the system has infinitely many solutions

→ back sub $x+3z=4$

$y+2z=5$

since Matrix in r.r.e.f

x, y → leading variables

z → free variabe

let $z=t$, $x=4-3t$, $y=5-2t$ ← اسهل فرض دائماً هو free variable

$(4-3t, 5-2t, t)$

← يمكن فرض اي متغير t ولكن الاسهل هو free variable

Q_4 solve the linear system by Gauss Jordan elimination

$$3x-y+z+7w=13$$
$$-2x+y-z-3w=-9$$
$$-2x+y-7w=-8$$

$$\rightarrow \left(\begin{array}{cccc|c} 3 & -1 & 1 & 7 & 13 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{array}\right)$$

$$\tfrac{1}{3}R_1 \rightarrow \begin{pmatrix} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{pmatrix}$$

$$\begin{array}{l} \\ R_2 \rightarrow 2R_1+R_2 \\ R_3 \rightarrow 2R_1+R_3 \end{array} \begin{pmatrix} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ 0 & 1/3 & -1/3 & 5/3 & -1/3 \\ 0 & 1/3 & 2/3 & -7/3 & 2/3 \end{pmatrix}$$

$$3R_2 \begin{pmatrix} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 1/3 & 2/3 & -7/3 & 2/3 \end{pmatrix}$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{3}R_2+R_1 \\ R_3 \rightarrow \frac{-1}{3}R_2+R_3 \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & 4 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 1 & -4 & 1 \end{pmatrix}$$

$$R_2 \rightarrow R_3+R_2 \begin{pmatrix} 1 & 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \end{pmatrix}$$

$$x+4w=4$$
$$y+w=0$$
$$z-4w=1$$

the sys has inf. many solutions

w is free variable

let $w=t$, $x=4-4t$, $y=-t$, $z=1-4t$

Homogeneous Linear Systems

A system of linear equations is said to be homogeneous if the constant terms are all zero

Examples: $5x+2y=0$

$$2x-y=0$$

→ Every homogeneous system of linear equation is consistent

اي ان اي نظام خطي "homogeneous" يوجد له اما حل واحد او عدد لانهائي من الحلول

هذ يعتبر الحل الاول دائماً ويسمى → $x_1=0,\ x_2=0, \dots, x_n=0$

"trivial solution"

→ if there are another solutions, they are called nontrivial solutions

← لا يوجد الا احتمالين لحل homo linear sys وهو اما حل واحد فقط صفري "trivial"

او عدد لانهائي من الحلول بالاضافة لحل الصفري

← اذا كانت عدد المتغيرات اكبر من عدد المعادلات ← infinitely many solutions

Q1 Solve the System

$$x+2y-z=0$$
$$2x-y+3z=0$$
$$3x+y+2z=0$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{pmatrix}$$

$$\begin{array}{c} \\ \\ -R_2+R_3 \end{array} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

← يمكن التوقف وعمل Back sub من
عند هذه الخطوة او الوصول الى r.r.e.f

$$\begin{array}{c} \\ -\frac{1}{5}R_2 \\ \\ \end{array} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

← في كلتا الحالتين الاجابة inf many solutions

لان عدد المعادلات اقل من عدد المجاهيل
ولكن يجب كتابة الجواب النهائي على شكل
(, ,) بدلالة t

$$\begin{array}{c} -2R_2+R_1 \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z free variabe

let $z=t$

$$x+z=0$$
$$y-z=0$$

$$x=-t,\ y=t \rightarrow (-t,t,t)$$

Q_2 Solve the system

$$x - 2y + z = 0$$
$$x + y + 3z = 0$$
$$2x + y - z = 0$$

Q_3 solve the system

$$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = 1$$

$$\frac{2}{x} + \frac{3}{y} + \frac{8}{z} = 0$$

$$-\frac{1}{x} + \frac{9}{y} + \frac{10}{z} = 5$$

Matrices and Matrix operations

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

m: number of rows

n: number of columns

$A = [a_{ij}]_{m \times n}$

Ex: if A is 2×3 matrix such that $a_{ij} = 2i - j$, find A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$a_{11} = 2 - 1 = 1$, $a_{13} = -1$, $a_{22} = 2$

$a_{12} = 2 - 2 = 0$, $a_{21} = 3$, $a_{23} = 1$

so $A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

→ Matrix product

$$\underset{m \times r}{A} \qquad \underset{r \times n}{B} \quad \Rightarrow \quad \underset{m \times n}{AB}$$

حتى نتمكن من إيجاد قيمة AB يجب ان تكون عدد الاعمدة في المصفوفة الاولى يساوي عدد الصفوف في المصفوفة الثانية

المصفوفة الناتجة من عملية الضرب تكون مكونة (حجمها) من عدد صفوف الاولى وعدد اعمدة الثانية

Q_1 Let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix}$ find AB, BA

$AB \to \underbrace{A_{2\times3} \ B_{2\times2}}_{\text{غيرمتساويات}} \to$ not possible

$BA \to \underbrace{B_{2\times2} \ A_{2\times3}}_{\text{متساويات}} \to$ Possible $= BA_{2\times3} = C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix}$

$\to C_{11} = 10 - 1 = 9$ Dot product بين الصف الاول لـB والعمود الاول لـ A

$\to C_{12} = 5 + 4 = 9$ Dot produkt بين الصف الاول لـB والعمود الثاني لـ A

$\to C_{13} = 15 + 2 = 17$ Dot product بين الصف الاول لـB والعمود الثالث لـ A

$\to C_{21} = 6 - 2 = 4$ Dot product بين الصف الثاني لـB والعمود الاول لـ A

$\to C_{22} = 3 + 8 = 11$ Dot product بين الصف الثاني لـB والعمود الثاني لـ A

$\to C_{23} = 9 + 4 = 13$ Dot product بين الصف الثاني لـB والعمود الثالث لـ A

Q_2 let $A = \begin{pmatrix} 1 & 4 & 3 \\ -2 & 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 6 & -2 \end{pmatrix}$ find AB and BA if possible

$A_{2\times3} \ B_{3\times2} \to$ Possibe $(AB)_{2\times2} = \begin{pmatrix} 32 & 18 \\ 23 & 13 \end{pmatrix}$

$B_{3\times2} \ A_{2\times3} \to$ Possible $(BA)_{3\times3} = \begin{pmatrix} -6 & 28 & 14 \\ -7 & 37 & 19 \\ 10 & 14 & 14 \end{pmatrix}$

Note that $AB \neq BA$

Q_3 $A = \begin{pmatrix} 2 & 4 & 1 \\ 5 & 2 & -1 \\ 4 & 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 2 & -2 \\ 7 & 5 \end{pmatrix}$ if $C = AB$

find C_{11}, C_{32}, C_{21}

$C = \begin{pmatrix} 19 & 3 \\ 7 & 6 \\ 26 & 18 \end{pmatrix}$ $C_{11} = 19$, $C_{32} = 18$, $C_{21} = 7$

→ Trace of Matix

If A is a square matrix, then trace of A $tr(A)$ is the sum of the entries on the main diagonal of A

Ex: if $A = \begin{pmatrix} 6 & 2 & 1 \\ 8 & -3 & 6 \\ 4 & 9 & 5 \end{pmatrix}$ find $tr(A)$

$tr(A) = 6 - 3 + 5 = 8$

→ Transpose of a matrix A^t

if $A = \begin{pmatrix} 3 & -2 \\ 5 & 4 \\ 6 & 2 \end{pmatrix}$ then $A^t = \begin{pmatrix} 3 & 5 & 6 \\ -2 & 4 & 2 \end{pmatrix}$

Ex: if $A = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 9 \\ 2 & -6 \end{pmatrix}$ find:

1. $(A+B)^t$

$$A+B = \begin{pmatrix} 11 & 14 \\ 3 & -2 \end{pmatrix} \rightarrow (A+B)^t = \begin{pmatrix} 11 & 3 \\ 14 & -2 \end{pmatrix}$$

2. $A^t + B^t$

$$A^t = \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix} , \quad B^t = \begin{pmatrix} 8 & 2 \\ 9 & -6 \end{pmatrix}$$

$$A^t + B^t = \begin{pmatrix} 11 & 3 \\ 14 & -2 \end{pmatrix}$$

$\boxed{\text{Note } (A+B)^t = A^t + B^t}$

3. $tr(AB)$

$$AB = \begin{pmatrix} 34 & -3 \\ 16 & -15 \end{pmatrix} \rightarrow tr(AB) = 34 - 15 = 19$$

4. $tr(BA)$

$$BA = \begin{pmatrix} 33 & 76 \\ 0 & -14 \end{pmatrix} \rightarrow tr(BA) = 33 - 14 = 19$$

$\boxed{\text{Note } tr(AB) = tr(BA)}$

5. $tr(A+B) = 9$

6. $tr(A) + tr(B)$

$7 + 2 = 9$

$\boxed{\text{Note } tr(A) + tr(B) = tr(A+B)}$

Properties of transpose

1. $(A^t)^t = A$
2. $(A+B)^t = A^t + B^t$
3. $(KA)^t = K A^t$, K is constant
4. $(AB)^t = B^t A^t$ → نعكس ترتيب المصفوفات

General Properties of Matrix

1. $A+B = B+A$
2. $A+(B+C) = (A+B)+C$
3. $A(BC) = (AB)C$
4. $A(B \pm C) = AB \pm AC$
5. $(B \pm C)A = BA \pm CA$ ← الترتيب مهم
6. if $KA = 0$ then $K=0$ or $A=0$

عادة ما تكون الاسئلة
على هذه القواعد على شكل
which of the following statement
is always correct

True or False

1. if A and B are two Matrices such that $AB=0$, then $A=0$ or $B=0$

$\boxed{\text{False}}$ → for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A \neq 0$, $B \neq 0$ but $AB = 0$

2. if $A \neq 0$, B and C are Matrices such that $AB = AC$, then $B = C$

$\boxed{\text{False}}$ → for example $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$

$AB = AC$ but $B \neq C$

$= \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$

Diagonal Matrix

let A be a square Matrix, we say A is diagonal Matrix if all the entries not on the main diagonal are zeros

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \to \text{Diagonal Matrix}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{pmatrix} \to \text{Not Diagonal}$$

Identity Matrix I

All entries on the main diagonal are 1, and all other entries are zeros

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Very important Note !!

عند ضرب أي مصفوفة بـ I لا تتغير قيمتها

Ex: $A = \begin{pmatrix} 7 & 8 \\ -2 & 5 \end{pmatrix}$

$$AI = \begin{pmatrix} 7 & 8 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ -2 & 5 \end{pmatrix} = A$$

Note that $IA = AI = A$

Inverse of a matrix

Definition: if A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be invertible (non singular) and b called invere of A

A is said to be singular if it does't have inverse

In another form $AA^{-1} = A^{-1}A = I$ where I is Identity Matrix

Rules for 2x2 Matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

تبديل اشارات   تبديل اماكن

→ if $\det(A) \neq 0$ then A is invertible (non singular) → اي له انفرس

→ if $\det(A) = 0$ then A is singular → اي ليس له انفرس

Q1 let $A = \begin{pmatrix} 6 & -5 \\ 1 & 2 \end{pmatrix}$ find A^{-1} if exist

$\det(A) = 6 \times 2 - (-5)(1) = 17 \neq 0$ (invertible)

$$A^{-1} = \frac{1}{17} \begin{pmatrix} 2 & 5 \\ -1 & 6 \end{pmatrix}$$

Linear system in Matrices form

Example: write the following system in Matrices form

$2x - y + 3z = 5$

$8x + 2y - z = 1$

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 8 & 2 & -1 \end{pmatrix} \rightarrow \text{coefficient Matrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

→ we can write the system as $AX = b$

$$\begin{pmatrix} 2 & -1 & 3 \\ 8 & 2 & -1 \end{pmatrix}_{2\times3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{3\times1} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}_{2\times1}$$

Example: if $A^{-1} = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

find: the solution for the system $AX = b$

Step 1 $\quad A^{-1}AX = A^{-1}b \rightarrow$ since $A^{-1}A = I$

نضرب الطرفين بـ A^{-1}

$IX = A^{-1}b \rightarrow IX = X$ عند ضرب I بأي مصفوفة لا يتغير ناتجها

$X = A^{-1}b$

$$A^{-1}b = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}_{2\times2} \begin{pmatrix} 7 \\ -3 \end{pmatrix}_{2\times1} = \begin{pmatrix} -1 \\ -19 \end{pmatrix} \rightarrow X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -19 \end{pmatrix}$$

$x = -1$, $y = -19$

Properties of Inverse of the Matrix

1. If B and C are both inverses of A, then $B = C$
2. If A and B are invertible matrices $\rightarrow (AB)^{-1} = B^{-1}A^{-1}$

→ Example: if $A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & -1 & 5 \\ 3 & 4 & 2 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 4 & 1 & 4 \\ 2 & 5 & 7 \\ -1 & 2 & 6 \end{pmatrix}$

and $C = (AB)^{-1}$, find C_{23}

$C = (AB)^{-1} = B^{-1}A^{-1}$

C_{23} = حاصل dot product بين الصف الثاني لـ B والعمود الثالث لـ A

$= (2)(1) + (5)(5) + (7)(2) = 41$

3. $(A^T)^{-1} = (A^{-1})^T$

→ Example: if $A^{-1} = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$, find $(A^T)^{-1}$

$(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix}$

4. $(A^{-1})^{-1} = A$

5. $A^n = \underbrace{AA\ldots A}_{n \text{ factor}} \quad (n>0)$

$\boxed{\text{Note } A^0 = I}$

6. $A^{-n} = (A^{-1})^n = (A^n)^{-1} \quad (n>0)$

7. $(KA)^{-1} = \frac{1}{K} A^{-1}$, where K non zero constant

Q if $(A^T + 2I)^{-1} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$, find A, a_{21}

Step 1 : نضع انفرس للطرفين للتخلص من الانفرس عاليسار

$$((A^T + 2I)^{-1})^{-1} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}^{-1} \rightarrow \det = 12 - 10 = 2$$

$$A^T + 2I = \frac{1}{2}\begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$$

Step 2
جد A^T

$$A^T = \begin{pmatrix} 2 & -5/2 \\ -1 & 3/2 \end{pmatrix} - 2I$$

$$A^T = \begin{pmatrix} 2 & -5/2 \\ -1 & 3/2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -5/2 \\ -1 & -1/2 \end{pmatrix}$$

Step 3 $A = (A^T)^T = \begin{pmatrix} 0 & -1 \\ -5/2 & -1/2 \end{pmatrix}$

$a_{21} = -5/2$

Q if $A, B, (A+B)$ are invertible Matrices, show that

$$A(A^{-1}+B^{-1})B(A+B)^{-1} = I$$

في حال كان الامتحان multiple choice
يكون نفس السؤال find value of

$$A(A^{-1}+B^{-1})B(A+B)^{-1} =$$

$$= (AA^{-1}+AB^{-1})B(A+B)^{-1}$$

$$= (I+AB^{-1})B(A+B)^{-1}$$

$$= (IB+AB^{-1}B)(A+B)^{-1} = (B+AI)(A+B)^{-1}$$

$$= (\overset{c}{A+B})(\overset{c}{A+B})^{-1}$$

$$CC^{-1} = I$$

True or false :

A, B are two Matrices of same size, $(A+B)^2 = A^2+2AB+B^2$

False $(A+B)^2 = (A+B)(A+B)$

$$= A^2+\underbrace{AB+BA}_{\neq 2AB}+B^2$$

ليس بالضرورة ان تكون $AB=BA$

Elementary Matrices

→ Matrix A,B are said to be row equivalent if one can be obtained from other by e.r.o

← اي اذا تمكنا من الحصول على المصفوفة B عن طريق اجراء e.r.o على المصفوفة A

فانه يمكن تسمية المصفوفتان A,B row eq

Example: $A = \begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, Are A,B row eq.?

for Matrix A

$\frac{1}{3}R_1 \rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

$\begin{matrix} \\ -2R_1+R_2 \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = B$ so A,B are row equivalent matrices

← المصفوفة الناتجة عن اجراء e.r.o واحدة فقط على Identity matrix

تسمى Elementary Matrix

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow 2r_1$ ✓

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{matrix} \rightarrow 3r_1 \\ \rightarrow 3r_2 \end{matrix}$ not elementary

$\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \rightarrow -5r_1+r_2$ ✓

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $r_1 \rightleftarrows r_3$ ✓

← حاصل ضرب EA (elementary matrix في مصفوفة مصفوفة) هو نفس ناتج اجراء عملية e.r.o (التي ادت لتشكيل E) على المصفوفة A

Example: $E\begin{pmatrix}5 & 0\\ 0 & 1\end{pmatrix}$, $A\begin{pmatrix}3 & 4\\ 2 & -1\end{pmatrix}$

حصلنا عليها عن طريق $5r_1$ من I

$$EA=\begin{pmatrix}15 & 20\\ 2 & -1\end{pmatrix}$$

← لو قمنا بعمل $5r_1$ على المصفوفة A سنحصل على نفس الناتج

how to find inverse of the Matrix using e.r.o

Step 1 $(A \mid I_n)$

← اي نضع بجانب المصفوفة I_n التي تقابلها

Step 2 ← نحول الطرف اليسار الى I عن طريق e.r.o

Step 3 $(I_n \mid A^{-1})$

← اي اذا اصبح الطرف اليسار I_n يصبح الطرف اليمين هو A^{-1}

Q. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

Step 1 $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$

Step 2 $\begin{array}{r} \\ -2r_1 + r_2 \\ -r_1 + r_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$ دائمًا في كل خطوة ننجز عمود

$\begin{array}{r} -2r_2 + r_1 \\ \\ 2r_2 + r_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$

$\begin{array}{r} \\ \\ -r_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right)$

$\begin{array}{r} -9r_3 + r_1 \\ 3r_3 + r_2 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$

Step 3 $A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$

Q_2 find inverse of $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

Note :

1. Every Elementary Matrix is invertible(اي لها انفرس)
2. The Inverse of E is E

Theorm : Equivalent statements

If A is an $n \times n$ matrix, then the following statements are equivalent, all True or All false

1. A is invertible
2. $A\bar{X}=0$ has only the trivial solution
3. the r.r.e.f of A is I_n
4. The Matrix A can be written as product of elemetary matrices

To Solve linear System Using Inverse

Step 1. find A^{-1}

step 2 from $A\bar{X}=b$

$$A^{-1}A\bar{X}=A^{-1}b$$

$$\bar{X}=A^{-1}b$$

Theorm : if $A_{n \times n}$ is invertible Matrix, then for every $n \times 1$ matrix b, the system $A\bar{X}=b$ has one solution which is $\bar{X}=A^{-1}b$

Q_3 $A = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$ find

1. A^{-1}

step1 $\left(\begin{array}{ccc|ccc} 1 & 4 & 4 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$

step2 by e.r.o $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{array}\right)$

step3 $A^{-1} = \begin{pmatrix} -2 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$

2. if $AB = \begin{pmatrix} 4 & 3 & 4 & 12 \\ 1 & 2 & 6 & 34 \\ 2 & 5 & 1 & -21 \end{pmatrix} \to C$ find b_{23}

$AB = C$

$A^{-1}AB = A^{-1}C$

$IB = A^{-1}C \to B = A^{-1}C$

b_{23} = dot Product بين الصف الثاني لـ A^{-1} والعمود الثالث لـ C

$= (4)(\frac{1}{2}) - (\frac{1}{2})(6) + 0 = \boxed{-1}$

3. if $\bar{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$, find the solution of $A\bar{X} = b$

$A^{-1}A\bar{X} = A^{-1}b$

$\bar{X} = A^{-1}b = \begin{pmatrix} -2 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \\ -1 \end{pmatrix}$ $x = 12, y = -1, z = -1$

Q_4 True or False

1. let A and B be two matrices, if A is invertible and $AB=0$ then $B=0$

$\therefore$ True $\quad AB=0$

$$A^{-1}AB = A^{-1}0$$

$$IB=0 \rightarrow B=0$$

2. let A, B and C be matrices, if A is Invertible and $AB=AC$ then $B=C$

True $\quad AB = AC$

$$A^{-}AB = A^{-}AC$$

$$IB = IC \rightarrow B=C$$

Q_5 consider the system

$$-x_1+4x_2+x_3=b_1$$

$$x_1+9x_2-2x_3=b_2$$

$$6x_1+4x_2-8x_3=b_3$$

Solve the system if $b_1=0, b_2=1, b_3=0$

$b_1=-3, b_2=4, b_3=-5$

Diagonal, Triangular and symmetric matrices

Diagonal Matrices:

The square matrix A is Diagonal if all entries not on the main diagonal are zeroes

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \rightarrow \text{Diagonal Matrix}$$

→ A Diagonal Matrix is invertibe if all of it's diagonal entries are nonzero

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \dots & 0 \\ 0 & 1/d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1/d_n \end{pmatrix}$$

where D is Diagonal Matrix

In General $D^K = \begin{pmatrix} d_1^K & 0 & \dots & 0 \\ 0 & d_2^K & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & d_n^K \end{pmatrix}$

Example $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$ $A^{-} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1/9 \end{pmatrix}$

Triangular Matrices

→ Upper triangular

if all entries below the main Diagonal are zeros

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \rightarrow \text{Uppertriangular}$$

→ lower triangular

if all entries above the main Diagonal are zeros

Properties :

1. If A is upper $\rightarrow$ A^T is lower
2. If A is lower $\rightarrow$ A^T is upper
3. The product of lower Matrices is lower and the product of upper matrices is upper
4. Triangular Matrix is invertible iff it's diagonal entries are nonzeros
   - → if A is upper invertible then A^{-} is upper
   - → if A is lower invertible then A^{-} is lower

Symmetric Matrices

A square Matrix A is symmetric if $A = A^T$

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 8 & -4 \\ 3 & -4 & 9 \end{pmatrix} \rightarrow$$ نلاحظ ان العناصر حول القطر متماثلة

$$A^T = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 8 & -4 \\ 3 & -4 & 9 \end{pmatrix}$$

Skew Symmetric

A square Matrix A is skew symmetric if $A^T = -A$

$$B = \begin{pmatrix} 0 & -5 & 4 \\ 5 & 0 & 3 \\ -4 & -3 & 0 \end{pmatrix} \rightarrow B^T = \begin{pmatrix} 0 & 5 & -4 \\ -5 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix} = -B$$

True or false

If A is skew symmetric then $tr(A)=0$

True والسبب انه في الـ skew sym عناصر

القطر دائماً تساوي 0 وعليه فان $tr(A) = 0$

Theom: if A and are symmetric Matrices with the same size then:

1. A^T is symmetric
2. $A+B$ and $A-B$ are symmetirc
3. KA is symmetric

True or False

1. If A is a square Matrix, then $A+A^T$ is symmetric

اذا كانت transpose لا تغير المصفوفة symmetric

True $(A+A^T)^T = A^T + (A^T)^T = A + A^T$

2. $A - A^T$ is symmetric

False $(A-A^T)^T = A^T - A = -(A-A^T)$

skew symmetric not symmetric

3. AA^T is symmetric

True $(AA^T)^T = (A^T)^T A^T = AA^T$

تذكر ان T يعكس عند الضرب

4. if A is invertible symmetric matrix, then A^{-1} is symmetric

True $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ $\qquad A = A^T$

so $A^{-1} = (A^{-1})^T$

Transpose لا يغير القيمة

Find a Diagonal Matrix A that satisfies

$$A^{-2} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow A^2 = \begin{pmatrix} 1/9 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow A = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

فقط غيرنا الصيغة ولم نغير القيمة

Q) Find all values of a and b which A and B are both not invertible

$$A = \begin{pmatrix} a+b-1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 0 \\ 0 & 2a-3b-7 \end{pmatrix}$$

not invertible means det = 0

$\det(A) = 3a+3b-3 = 0 \rightarrow a+b=1 \rightarrow a = 1-b$

$\det(B) = 10a - 15b - 35 = 0 \rightarrow 2a-3b=7$

$$2(1-b) - 3b = 7$$

$$2 - 2b - 3b = 7$$

$$-5b = 5 \rightarrow \boxed{b=-1}, \boxed{a=2}$$

True or False

1. $[1]$ is an Identity matrix

True

2. The system $x+2y=3$

$-x+ay=b$ has no solution if $a=-2$

False, if $b=-3$ then system has infinitely many solutions

Chapter 2 Determinants

2.1 Determinants by Cofactor Expansion

→ Minor of entry a_{ij} (M_{ij}):

Determinants of the submatrix that remains after deleting the ith row and jth column

→ Cofactor of entry a_{ij} (C_{ij})

$C_{ij} = (-1)^{i+j} M_{ij} = \pm M_{ij}$

Note!!
M_{ij}, C_{ij} only for square Matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det(A) = ad - bc$

Q1 let $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 5 & 2 \\ -4 & 6 & 1 \end{pmatrix}$, find M_{21}, M_{22}, M_{32} and C_{21}, C_{22}, C_{32}

$M_{21} = \begin{vmatrix} 3 & -1 \\ 6 & 1 \end{vmatrix} = 9 \rightarrow C_{21} = (-1)^3 9 = -9$

$M_{22} = \begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} = -2 \rightarrow C_{22} = (-1)^4 -2 = -2$

$M_{32} = \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 8 \rightarrow C_{32} = -8$

→ If A is $n\times n$ Matrix, then :

Rule 1 : $\det(A) = a_{ij}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$ ايجاد المحددة عن طريق الصفوف

Rule 2 : $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$ ايجاد المحددة عن طريق الاعمدة

Q2 let $A = \begin{pmatrix} 2 & 4 & 3 \\ -5 & 6 & 1 \\ 3 & 4 & 2 \end{pmatrix}$, find $\det(A)$

1. $i=1$

$$\det A = \overset{a_{11}}{2}\ \overset{C_{11}}{\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix}} - \overset{a_{12}}{4}\ \overset{C_{12}}{\begin{vmatrix} -5 & 1 \\ 3 & 2 \end{vmatrix}} + \overset{a_{13}}{3}\ \overset{C_{13}}{\begin{vmatrix} -5 & 6 \\ 3 & 4 \end{vmatrix}}$$

$$= 2(8) - 4(-13) + 3(-38) = -46$$

2. $i=2$

$$\det(A) = +5\begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix}$$

$$= 5(-4) + 6(-5) + 4 = -46$$

3. $j=1$

$$\det(A) = 2\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} + 5\begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 3 \\ 6 & 1 \end{vmatrix}$$

$$= 2(8) + 5(-4) + 3(-14) = -46$$

← نختار عادة الصف او العمود الذي يعطي الحل الاسرع

Q_3 let $A = \begin{pmatrix} 3 & 0 & 7 \\ 5 & 8 & 6 \\ -4 & 0 & 4 \end{pmatrix}$, find $\det(A)$

Use $j=2$ $\quad \det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$$= 0 + 8(-1)^4 \begin{vmatrix} 3 & 7 \\ -4 & 4 \end{vmatrix} + 0$$

$$= 8 * 40 = 320$$

Q_4 let $B = \begin{pmatrix} 1 & 2 & 6 & 0 \\ 3 & 2 & 5 & 4 \\ 0 & 6 & 0 & 4 \\ 3 & 0 & 7 & 1 \end{pmatrix}$ find $\det(B)$

use $i=3$ $\quad \det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$

$$= 0 + 6(-1)^5 \begin{vmatrix} 1 & 6 & 0 \\ 3 & 5 & 4 \\ 3 & 7 & 1 \end{vmatrix} + 0 + 4(-1)^7 \begin{vmatrix} 1 & 2 & 6 \\ 3 & 2 & 5 \\ 3 & 0 & 7 \end{vmatrix}$$

$$= -50$$

Theorm: If A is $n\times n$ diagonal or upper triangular or lower triangular Matrix then:

$$\det(A) = a_{11}a_{22}a_{33}\dots a_{nn}$$

اي تكون المحددة حاصل ضرب العناصر الموجودة عم القطر الرئيسي

Q_{15} If $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ find $|A|$

A is upper triangular $\rightarrow |A| = (1)(2)(3)(4) = 24$

Q_{16} If $\begin{vmatrix} 2x-4 & 0 & 0 \\ 0 & x+3 & 0 \\ 0 & 0 & x \end{vmatrix} = 0$ find x

$\det = (2x-4)(x+3)(x) = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$x=2 \quad x=3 \quad x=0$

True or False:

1. for any square Matrix A and scalar K, $\det(KA) = K\det(A)$

False for Example $A = \begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}$, $3A = \begin{pmatrix} 6 & 12 \\ 3 & 12 \end{pmatrix} \rightarrow \det(A) = 4$, $\det(3A) = 36$

2. for any square Matrices A, B, $\det(A+B) = \det(A) + \det(B)$

False for Example $A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $A+B = \begin{pmatrix} 3 & 13 \\ 6 & 10 \end{pmatrix}$

$\det(A) = -3$, $\det(B) = 5$, $\det(A+B) = 12$

2.2 Evaluating determinants by row reduction

→ Properties of determinants :

1. Let A a square Matrix, if A has a row of zeros or column of zeros, then $\det(A) = 0$
2. let A a square Matrix, $\det(A) = \det(A^T)$
3. If Matrix B results from A by multiply one row or one column of A by scalar K, then $\det(B) = K\det(A)$

Example if $A = \begin{pmatrix} 2 & 5 \\ 4 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 12 & -9 \end{pmatrix}$

$\det(A) = -26$, $\det(B) = -78 = 3\det(A)$

4. $\det(KA) = K^n \det(A)$, n = number of rows

Example if $A = \begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix} \rightarrow 3A = \begin{pmatrix} 6 & 12 \\ 3 & 12 \end{pmatrix}$

$\det(A) = 4$, $\det(3A) = 36 \rightarrow \det(3A) = 3^2 \det(A)$ عدد الصفوف

5. If Matrix B results from A by Interchange two rows or two colulmns, then $\det(B) = -\det(A)$

Example $A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$

$\det A = 2$, $\det B = -2$

6. If Matrix B results from A by adding a multiple of one row to another row or adding a multiple of one column to another then $\det(A) = \det(B)$

Example $A = \begin{pmatrix} 6 & 3 \\ -4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 3 \\ 14 & 14 \end{pmatrix} \to 3r_1 + 3r_2$ from A

$\det(A) = 30 + 12 = 42$, $\det B = 84 - 42 = 42$

7. If A is a square Matrix with two proportional rows or two proportiona columns , then $\det(A) = 0$

اي اذا كانت مصفوفة تحتوي على صف من مضاعفات صف اخر او عمود من مضاعفات عمود آخر

Example $A = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$

$\det A = 12 - 12 = 0 \to (r_2 = \frac{2}{3} r_1)$

8. The Matrix A is invertible iff $\det A \neq 0$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

9. If A,B are square Matrices of same size $\to \det(AB) = \det(A)\det(B)$

Q_1 if A and B are 3x3 matrices, $\det(A)=2$, $\det(B)=-4$ find

1. $\det(AB^{-})$

$$\det(AB^{-}) = \det(A)\det(B^{-})$$
$$= \det(A)\cdot\frac{1}{\det(B)} = 2\cdot\frac{-1}{4} = \frac{-1}{2}$$

2. $\det(5A)$

$$\det(5A) = 5^3\det(A) = (125)(2) = 250$$

3. $\det(2A^2B^{-3})$

$$\det(2A^2B^{-3}) = \det(2A)\det(A)\det(B^{-1})^3$$
$$= 2^3\det(A)\det(A)\left(\frac{1}{\det(B)}\right)^3$$
$$= 8\times 2\times 2\times\frac{1}{-64} = \frac{-1}{2}$$

Q_2 If $A = \begin{pmatrix} 3 & 1 & 5 \\ 6 & 4 & -8 \\ 3 & 2 & -4 \end{pmatrix}$, find $\det(A)$

$$\det(A) = 0 \quad (r_2 = 2r_3)$$

Q_3 $A = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 3 \\ 2 & 1 & 6 & 1 \\ 2 & 4 & 8 & 11 \end{pmatrix}$, find det(A)

$$B = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 3 \\ 2 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad -2r_1 + r_4 \to$$

using $i=4$

$$\det(B) = (1)(-1)^8 \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 2 & 1 & 6 \end{vmatrix}$$

$\det(B) = \det(A)$

حسب خاصية 6

Q_4 let $A = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, Show that $\det(A) = -a_{13}a_{22}a_{31}$

$$B = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{13} \end{pmatrix}$$

$\det(B) = a_{31}a_{22}a_{13}$ (upper triag)

← حصلنا B عن طريق تبديل صفين من A

→ so $\det(A) = -\det(B) = -a_{13}a_{22}a_{31}$

Q_{15} if $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

1. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 6$ (تبديل سطرين)

2. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -6$ (تبديل سطرين مرتين
كل مرة ضرب بـ (-1))

3. $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$ $\rightarrow r_1 + r_2$ (det لا تؤثر على)
$\rightarrow -r_2$ (نضرب بـ(-1))
det = 6

4- $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{vmatrix} = (-10)(-6) = 60$

2.3 Cramer's Rule and finding inverse

$C_{ij} = (-1)^{i+j} M_{ij}$ $\quad$ C_{ij} : cofactor of entry a_{ij}

→ C : Matrix of cofactor

→ C^T = adjoint of A adj(A)

Theorm : if A is invertible matrix then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Q1 $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 5 & 4 \\ 2 & -4 & 1 \end{pmatrix}$, use adjoint method to find A^{-1} if exist

→ step1 $\det(A) = 2(21) - (1)(-9) + 3(-6) = 33$

$\det(A) \neq 0$ → exist

→ step2 find C

$$C = \begin{pmatrix} 21 & 9 & -6 \\ -13 & -4 & 10 \\ -11 & -11 & 11 \end{pmatrix}$$

→ step3 find $C^T = \operatorname{adj}(A)$

$$C^T = \begin{pmatrix} 21 & -13 & -11 \\ 9 & -4 & -11 \\ -6 & 10 & 11 \end{pmatrix}$$

→ step4 $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$

* $C_{21} = (-1)^3 M_{21} = (-1)(13) = 13$

Q_2 find A^{-1} if exist for $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$

Note: $A \, adj(A) = det(A) \, I$

Q_3 $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$, if $B = A^{-1}$, find b_{32}

$$b_{32} = \frac{C_{23}}{det(A)} = \frac{(-2)(-1)^5}{det A}$$

Cramer's Rule

Remember $\Rightarrow$ if the Matrix A is invertible then the system $A\bar{X}=b$ has unique solution

$\rightarrow$ we can use Cramer's Rule to find Solution

$X = \dfrac{\det(A_x)}{\det(A)}$, A_x : المصفوفة الناتجة عن استبدال عمود معاملات x بـ b

$y = \dfrac{\det(A_y)}{\det(A)}$, A_y : المصفوفة الناتجة عن استبدال عمود معاملات y بـ b

$z = \dfrac{\det(A_z)}{\det(A)}$, A_z : المصفوفة الناتجة عن استبدال عمود معاملات z بـ b

Q. $A = \begin{pmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 6 \\ -1 \\ -20 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solve the system using Cramer's Rule

$\rightarrow$ Step 1 $\det A = -55$

$\rightarrow$ Step 2 $A_x \begin{pmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -2 & 2 & -3 \end{pmatrix} \rightarrow \det(A_x) = 144 \rightarrow x = -\dfrac{144}{55}$

$$A_y = \begin{pmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & 3 \end{pmatrix} \rightarrow \det(A_y) = 61 \rightarrow y = -\frac{61}{55}$$

$$A_z \begin{pmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{pmatrix} \rightarrow \det(A_z) = -230 \rightarrow z = \frac{\det A_z}{\det A} = \frac{230}{55}$$

Q2 if $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $\det A \neq 0$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{pmatrix}$

find solution for $A\bar{X} = b$

$$A_x = \begin{pmatrix} 3b_1 & b_1 & c_1 \\ 3b_2 & b_2 & c_2 \\ 3b_3 & b_3 & c_3 \end{pmatrix} \rightarrow \det(A_x) = 0$$

$$A_y = \begin{pmatrix} a_1 & 3b_1 & c_1 \\ a_2 & 3b_2 & c_2 \\ a_3 & 3b_3 & c_3 \end{pmatrix} \rightarrow \det(A_y) = 3\det(A)$$

$$A_z = \begin{pmatrix} a_1 & b_1 & 3b_1 \\ a_2 & b_2 & 3b_2 \\ a_3 & b_3 & 3b_3 \end{pmatrix} \rightarrow \det(A_z) = 0$$

$$x = 0, \quad z = 0, \quad y = \frac{\det(A_y)}{\det(A)} = \frac{3\det(A)}{\det(A)} = 3$$

Q_3 $A = \begin{pmatrix} 2 & 5 & a \\ 1 & 4 & b \\ 0 & 7 & c \end{pmatrix}$, $\det(A) = 8$, $\bar{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

If $x = S_1$, $y = S_2$, $z = S_3$ is solution of $A\bar{X} = b$, find S_3

$$Z = \frac{\det(A_z)}{\det(A)} = S_3$$

$$A_z = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 4 & 1 \\ 0 & 7 & 2 \end{pmatrix} \rightarrow \det(A_z) = 13$$

$$\rightarrow S_3 = \frac{13}{8}$$

Q_4 find values of K for which A is invertible

$$A = \begin{pmatrix} 2 & 1 & 0 \\ K & 2 & K \\ 2 & 4 & 2 \end{pmatrix}$$

$$\det(A) = 2(4-4K) - 1(2K-2K) + 0$$

$$= 8 - 8K = 8(1-K)$$

for invertible $8(1-K) \neq 0$

A is invertible if $K \neq 1$

A is not invertibe if $K = 1$

Chapter 4: General Vector space

4.1 Real Vector spaces

→ **Vector Space:** A non empty set of objects on which two operations are defined (addition, Scalar Multiplication).

→ **Object:** Anything that could be defined (numbers, functions, Matrices..)

If the following conditions are satisfies by all objects U, V, W in V and all scalars K and l, then we call V a **vector space** and we call the objects in V **vectors**.

1. $U+V$ in V

← اي عند جمع اي عنصرين داخل V يكون الناتج عنصر موجود في V ايضاً

2. $U+V = V+U$

3. $U+(V+W) = (U+V)+W$

يعني العنصر المحايد لـ V

4. There exists an element 0_V called a zero vector for V such that

   $U+0 = 0+U = U$ for all U in V

5. for all U in V, there is an object $(-U)$ in V called a negative of U such that $U+(-U) = (-U)+U = 0$

يعني النظير الجمعي لـ U

6. KU in V

7. $K(U+V) = KU + KV$

8. $(K_1+K_2)U = K_1U + K_2U$

9. $K_1(K_2U) = K_1K_2U$

10. $1U = U$

Remember

$\mathbb{R}$: the real numbers

$\mathbb{R}^2$: the vectors in xy-plane الازواج المرتبة

$\mathbb{R}^3$: the vectors in 3-space

Q_1 Let $V = \mathbb{R}^2$, show that $(\mathbb{R}^2, +, .)$ is a vector space

يعني انه عناصر V هي ازواج مرتبة   جمع ازواج مرتبة   ضرب زوج مرتب بثابت

1- let $U, V \in \mathbb{R}^2$, $U = (U_1, U_2)$, $V(V_1, V_2)$

$U+V = (U_1+V_1, U_2+V_2) \in V$

2- $U+V = V+U$

3- let $W = (W_1, W_2)$ $U+(V+W) = (U+V)+W$

4- $0_V = (0,0)$ $U+0 = 0+U = U$

5- $-U = (-U_1, -U_2) \rightarrow (-U)+U = U+(-U) = 0$

6- $KU = (KU_1, KU_2) \in V$

7- $K(U+V) = (KU_1+KV_1, KU_2+KV_2)$

$KU+KV = (KU_1, KU_2) + (KV_1, KV_2)$

8-

9-

10-

Q_2 Let $W = \{(x,1) : x \in \mathbb{R}\}$ IS W vector space or not

نلاحظ ان W عبارة عن ازواج مرتبة ولكن قيمة y دائما تساوي 1

Q_3 Let $V = \mathbb{R}^3$, IS $(\mathbb{R}^3, +, .)$ a vector space or not

$V = \{(x,y,z) : x,y,z \in \mathbb{R}\}$

Note!!

$(\mathbb{R}^n, +, .)$ is a vector space

Q_4 IS $W = \{(x,x) : x \in \mathbb{R}\}$ a vector space?

تعني زوج مرتب بحيث عنصري x,y دائماً متساويان

let $U = (u,u)$, $V = (v,v)$, $W = (w,w)$

1. $U+V = (u+v, u+v)$

2. $U+V = V+U$

3. $U+(V+W) = (U+V)+W$

4. $0_W = (0,0) \in W$

   $0_W + U = U + 0_W = U$

5. $-U = (-u,-u) \in W$

   $U+(-U) = (0,0) = (-U)+U$

6. $KU = (Ku, Ku)$

7. $K(U+V) = KU+KV$

8. $(K_1+K_2)U = ((K_1+K_2)u, (K_1+K_2)u) = (K_1u+K_2u, K_1u+K_2u) = K_1U+K_2U$

9. $K_1(K_2U) = K_1K_2U$

10. $1U = U$

∴ So W is vector space

Q_5 Is $W = \{(0,y) : y \in \mathbb{R}\}$ a vector space

Q_6 Let $V = \mathbb{R}^2$, if $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $ku = (ku_1, 0)$

Is V a vector space or not

$1u = (u_1, 0) \neq u$ so Axiom 10 fails to hold

$\rightarrow$ V is not vector space

Note!

Every plane pass through the origin is a vector space

Q_7 Let $V = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$ Is V vector space?

تكلي مجموعة تحتوي جميع المصفوفات 2x2

let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

1- $A+B = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix} \in V$

2- $A+B = B+A$ 3- $A+(B+C) = (A+B)+C$

4- $0_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow 0+A = A+0 = A$

5- $-A = \begin{pmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{pmatrix} \rightarrow A+(-A) = (-A)+A = 0$

6- $KA = \begin{pmatrix} Ka_1 & Kb_1 \\ Kc_1 & Kd_1 \end{pmatrix} \in V$

7-

8-

9-

10- $1A = A$ So V is a vector space

? What is the difference between M_n and $M_n(\mathbb{R})$

$M_n(\mathbb{R})$: the set of All square matrices with real entries

اي ان جميع عناصر المصفوفة تنتمي الى $\mathbb{R}$ ← (لا يوجد حد ثابت في العناصر)

$\rightarrow \begin{pmatrix} a & b \\ c & 3 \end{pmatrix}$ not $M_2(\mathbb{R})$

Note!!

The set V of all $\underbrace{m\times n \text{ matrices with real entries}}_{M_{mn}(\mathbb{R})}$, is a vector space

Q_8 Is $W = \left\{ \begin{pmatrix} a & b \\ 2 & c \end{pmatrix} : a,b,c \in R \right\}$ a vector space?

$0_W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin W$, not vector space

homework

Q_9 Is $W \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$ a vector space?

Q_{10} Is $M = \left\{ \begin{pmatrix} a & a & a \\ b & b & b \end{pmatrix}, a,b \in \mathbb{R} \right\}$ a vector space?

4.2 Subspaces

ال subspace عبارة عن vector space موجود داخل vector space آخر

Theorm: If W is a nonempty subset of the vector space V, Then W is a subspace of V iff the following conditions hold

1. if w_1, w_2 in W then w_1+w_2 in W ← الشرط الاول في ال vector space
2. if $kw_1 \in W$ ← الشرط السادس في ال vector space

← اذا تحقق الشرطين الاول والسادس تكون باقي الشروط متحققة دون اثبات

Q1 Let $W = \{(x,0) : x \in \mathbb{R}\}$. show that W is subspace of $\mathbb{R}^2$

Let $U, V \in W$, $U = (x_1, 0)$, $V = (x_2, 0)$

① $U+V = (x_1+x_2, 0) \in W$ → condition 1 holds

② $kU = (kx_1, 0) \in W$ → condition 2 holds

So W is subspace of $\mathbb{R}^2$

Q_2 Let $W = \{(x,x) : x \in \mathbb{R}\}$ Is W subspace of $\mathbb{R}^2$

Let $U, V \in W$, $U = (u,u)$, $V = (v,v)$

① $U+V = (u+v, u+v) \rightarrow$ condition 1 holds
$\downarrow$
$\in W$

② $kU = (ku, ku) \rightarrow$ condition 2 holds
$\downarrow$
$\in W$

so W is subspace of $\mathbb{R}^2$

Q_3 let $W = \{(x,0,z) : x,z \in \mathbb{R}\}$ is W subspace of $\mathbb{R}^3$

let $U, V \in W$, $U = (x_1, 0, z_1)$, $V = (x_2, 0, z_2)$

① $U+V = (x_1+x_2, 0, z_1+z_2) \in W$

② $kU = (kx_1, 0, kz_1) \in W$

so W is subspace of $\mathbb{R}^3$

Q_4 let $W = \{(a,b,c,1), a,b,c \in \mathbb{R}\}$ is W subspace of $\mathbb{R}^4$

$U = (x_1, x_2, x_3, 1)$

$3U = (3x_1, x_2, 3x_3, 3) \notin W$ so no subspace $\mathbb{R}^4$

Q_5 let $M = \{(a,b,c) : c = 2a - b,\ a,b \in \mathbb{R}\}$ Is M subspace of $\mathbb{R}^3$?

$U, V \in M \rightarrow U = (a_1, b_1, c_1) = (a_1, b_1, 2a_1 - b_1)$

$V = (a_2, b_2, c_2) = (a_2, b_2, 2a_2 - b_2)$

1. $U + V = (a_1 + a_2,\ b_1 + b_2,\ 2(a_1 + a_2) - (b_1 + b_2)) \in M$

2. $KU = (Ka_1,\ Kb_1,\ 2Ka_1 - Kb_1) \in M$

so M is subspace of $\mathbb{R}^3$

Q_6 Let $W = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, a,b,c \in \mathbb{R} \right\}$ is W subspace of $M_{22}(\mathbb{R})$?

let $A, B \in W$, $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$

1. $A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix} \in W$

2. $KA = \begin{pmatrix} Ka_1 & Kb_1 \\ Kc_1 & 0 \end{pmatrix} \in W$

so W is subspace of $M_{22}(\mathbb{R})$

Q_7 let $W = \{ A \in M_{nn}(\mathbb{R}) : A$ is upper triagular $\}$ Is W subspace of $M_{nn}(\mathbb{R})$

let $A, B \in W$, A upper, B upper

1. $A+B \in W$ (upper+upper = upper)
2. $KA \in W$ (K upper = upper)

so W is subspace

Q_8 let $W = \{ A \in M_{nn} : A$ is symmetric $\}$ Is W subspace of $M_{nn}(\mathbb{R})$

Q_9 One of the following is Not True

A) $W = \{(a,b,c) : a < b\}$ is subspace of $\mathbb{R}^3$

B) $W = \{(a,b,c) : b = 2c + a\}$ is subspace of $\mathbb{R}^3$

C) $W = \{A \in M_{mn} : tr(A) = 0\}$ is subspace of M_{nm}

D) $W = \{A \in M_{nn} : A \text{ is lower}\}$ is subspace of M_{nn}

Q_{10} let W be the set of all point (x,y) in $\mathbb{R}^2$ such that

$x \geq 0, y \geq 0$ is W subspace of $\mathbb{R}^2$

let $U, V \in W \rightarrow U = (1,1)$, $V = (2,2)$

1- $U + V = (3,3) \in W$

2- $kU \rightarrow -U = (-1,-1) \notin W \rightarrow$ not subspace

Note !!

Every nonzero vector space V has at least two subspaces:
V itself, and the set $\{0\}$ which called zero subspace

Subspaces of M_{nm}

* the set of $n \times n$ upper triangular matrices
* the set of $n \times n$ lower triangular matrices
* the set of $n \times n$ diagonal matrices
* the set of $n \times n$ symmetric matrices

Subspace of polynomials

$P_n(\mathbb{R})$ means the set of all polynomials of degree n or less

where $n \geq 0$

Example $P_2(\mathbb{R})$: The set of all poly of degree 2 or less

Note !!

$V = P_n$ is a vector space for all $n \geq 0$

Q_{11} let W be the set of all polynomials of the form $a_0+a_1x+a_2x^2+a_3x^3$

where $a_1 = a_2$, Is W a subspace of P_3

$$W = \{ a_0 + a_1x + a_2x^2 + a_3x^3 ; a_1 = a_2 \}$$

let $p, q \in W \rightarrow p = a_0 + a_1x + a_1x^2 + a_3x^3$

$q = b_0 + b_1x + b_1x^2 + b_3x^3$

1- $p+q = (a_0+b_0) + (a_1+b_1)x + (a_1+b_1)x^2 + (a_3+b_3)x^3$ ✓

2- $kp = ka_0 + ka_1x + ka_1x^2 + ka_3x^3$ ✓

so W is subspace

linear Combination

A vector w is linear combination of the vectors $v_1, v_2, \ldots$

if it can be expressed in the form $w = k_1 v_1 + k_2 v_2 + \ldots$

Example: $U = \langle 3,1 \rangle$, $V = \langle 2,4 \rangle$

$3U - 2V \rightarrow$ linear combination

Q_1 let $S = \{(1,0),(0,1)\}$, let $U = (7,5)$, is U l.C of S

اذا تمكنا من كتابة U باستخدام عناصر S فبإمكاننا تسمية U ← linear compination of S

$(7,5) = 7(1,0) + 5(0,1) \rightarrow$ so $U \Rightarrow$ linear Compination of S

Q_2 let $S = \{(1,2),(-1,1)\}$, $U = (3,5)$, is U l.C of S

$(3,5) \overset{?}{=} k_1(1,2) + k_2(-1,1)$

$3 = k_1 - k_2$

$5 = 2k_1 + k_2$

$8 = 3k_1 \rightarrow k_1 = \frac{8}{3} \rightarrow k_2 = \frac{-1}{3}$

$(3,5) = \frac{8}{3}(1,2) - \frac{1}{3}(-1,1)$

so U is l.C of S

Q_3 let $S=\{(1,1),(2,2)\}$, $U=(4,5)$, is U L.C of S?

$$(4,5) \stackrel{?}{=} K_1(1,1)+K_2(2,2)$$

$$4 = K_1+2K_2$$

$$5 = K_1+2K_2$$

$-1=0$ (no solution ↔ معلومة رياضية خاطئة)

so U is not L.C of S

Q_4 let $S=\{(1,1,0),(0,1,1),(1,0,1)\}$, $U=(2,5,3)$, Is U L.C of S?

$$(2,5,3) \stackrel{?}{=} K_1(1,1,0)+K_2(0,1,1)+K_3(1,0,1)$$

$$2 = K_1+K_3$$
$$5 = K_1+K_2$$
$$3 = K_2+K_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$K_3=0$, $K_2=3$, $K_1=2$

So U is L.C of S

Q_5 let $S=\{(1,2,-1),(2,3,1),(1,3,-4)\}$, $U=(5,4,7)$, Is U L.C of S?

$$(5,4,7) \stackrel{?}{=} K_1(1,2,-1)+K_2(2,3,1)+K_3(1,3,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 3 & 3 & 4 \\ -1 & 1 & -4 & 7 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & -16 \end{pmatrix}$$

$0 \neq -16$ (no solution)

U not L.C of S

Q6 let $S = \left\{ \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix} \right\}$, $U = \begin{pmatrix} 2 & 5 \\ -2 & 4 \end{pmatrix}$, Is U l.C of S

$$\begin{pmatrix} 2 & 5 \\ -2 & 4 \end{pmatrix} \stackrel{?}{=} k_1 \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$$

$2 = 3k_1 + k_3$

$5 = 2k_1 + 2k_2 + k_3$

$-2 = -2k_2 - 2k_3$

$4 = k_1 + 4k_2 + 5k_3$

Q_7 let $S = \{2+x+4x^2,\ 1-x+3x^2,\ 3+2x+5x^2\}$

let $q = 6+11x+6x^2$, Is q L.C of S

$$6+11x+6x^2 = K_1(2+x+4x^2) + K_2(1-x+3x^2) + K_3(3+2x+5x^2)$$

$6 = 2K_1 + K_2 + 3K_3$ الحد الثابت

$11 = K_1 - K_2 + 2K_3$ معامل x

$6 = 4K_1 + 3K_2 + 5K_3$ معامل x^2

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 11 & 1 & -1 & 2 \\ 6 & 4 & 3 & 5 \end{array}\right)$$

Span(S)

Rule: S spans V if every element in V can be written as L.C of S → we denote this by $V = span(S)$

Q1 let $S = \{(1,0),(0,1)\}$, Is S spans $\mathbb{R}^2$

معنى السؤال هو هل نستطيع كتابة جميع عناصر $\mathbb{R}^2$ باستخدام عناصر S

let $v \in \mathbb{R}^2$, $v = (a,b)$

$(a,b) \stackrel{?}{=} k_1(1,0) + k_2(0,1)$

$a = k_1 \quad , b = k_2$

$(a,b) = a(1,0) + b(0,1)$

نلاحظ انه مهما تغيرت قيمة A , B المعادلة تتحقق

so S spans $\mathbb{R}^2$

Q2 let $S = \{(1,1,1),(3,2,-1),(1,0,-3)\}$ is S spans $\mathbb{R}^3$

let $V \in \mathbb{R}^3$, $V = (a,b,c)$

$$(a,b,c) = k_1(1,1,1) + k_2(3,2,-1) + k_3(1,0,-3)$$

$$a = k_1 + 3k_2 + k_3$$
$$b = k_1 + 2k_2$$
$$c = k_1 - k_2 - 3k_3$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 1 & a \\ 1 & 2 & 0 & b \\ 1 & -1 & -3 & c \end{pmatrix}$$

$$\begin{array}{r} \\ R_1 - R_2 \rightarrow \\ R_1 - R_3 \rightarrow \end{array} \begin{pmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 1 & a-b \\ 0 & 4 & 4 & a-c \end{pmatrix} \rightarrow \begin{array}{r} \\ \\ -4R_2 + R_3 \end{array} \begin{pmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & a-c-4a+4b \end{pmatrix}$$

$$0 \neq -3a + 4b - c$$

The system has no solution

So S not spans $\mathbb{R}^3$ or $\mathbb{R}^3 \neq \text{span}(s)$

Note !!

if the system is consistant then $\mathbb{R}^3 = \text{span}(s)$

بدون اي شروط عل a,b,c,d

Q_3 let $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ Is S spans $M_2(\mathbb{R})$?

let $A \in M_2(\mathbb{R}) \rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = K_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + K_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + K_4 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Q_4 let $S = \{x^2+1, 2x+3, 2x^2+2x+5\}$, I S spans P_2

$p \in P_2 \rightarrow p = ax^2+bx+c$

$ax^2+bx+c = k_1(x^2+1) + k_2(2x+3) + k_3(2x^2+2x+5)$

$a = k_1 + 2k_3$

$b = 2k_2 + 2k_3$

$c = k_1 + 3k_2 + 5k_3$

4.3 linear Independence

Let $S = \{V_1, V_2, V_3, \dots\}$, if $K_1V_1 + K_2V_2 + \dots = 0$ has only trivial solution then $S \rightarrow$ Linearly Independents

if otherwise (infinity many solution) $S \rightarrow$ Linearly dependent

Q_1 let $S = \{(1,2), (-1,3)\}$ Is S L.D or L.I set in $\mathbb{R}^2$

$$K_1(1,2) + K_2(-1,3) = 0$$

$$K_1 - K_2 = 0$$

$$2K_1 + 3K_2 = 0 \quad \rightarrow K_1 = 0, K_2 = 0$$

So S = L.I

Q_2 let $S = \{(1,2,1), (-1,1,-3), (1,5,-1)$, L.D or L.I in $\mathbb{R}^3$?

$$K_1(1,2,1) + K_2(-1,1,-3) + K_3(1,5,-1) = (0,0,0)$$

$$\begin{aligned} K_1 - K_2 + K_3 &= 0 \\ 2K_1 + K_2 + 5K_3 &= 0 \\ K_1 - 3K_2 - K_3 &= 0 \end{aligned} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & -3 & -1 & 0 \end{pmatrix}$$

امامنا خيارين للحل: det(A) (الأسهل) / solution for system

$\det(A) = 0 \rightarrow$ Not invertible $\rightarrow$ inf many solution

Linearly dependent

Q_3 $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ Is S L.I set in M_2

$$K_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + K_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + K_3 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

$K_1 + K_3 = 0$

$K_1 + K_2 = 0$

$K_3 = 0$ $\rightarrow$ $K_2 = 0, K_1 = 0, K_3 = 0$ so unique solution (trivial) $\rightarrow$ L.I

$K_2 + K_3 = 0$

Q_4 $S = \{ x^2+3, 2x^2-x+1, x+5 \}$ Is L.I set in P_2

$K_1(x^2+3) + K_2(2x^2-x+1) + K_3(x+5) = 0$

$K_1 + 2K_2 = 0$

$-K_2 + K_3 = 0$

$3K_1 + K_2 + 5K_3 = 0$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 5 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the sys has infinitely many solution $\rightarrow$ S is L.d

Note: If we can write one element of S as a linear compination of the other elements then S is linearly dependent

In Q_4 Note that $(x+5) = 2(x^2+3) - (2x^2-x+1)$

so S L.d

Q_5 Let $S = \{(-3,0,4), (5,-1,2), (1,1,3)\}$ Is S L.I in $\mathbb{R}^3$

$k_1(-3,0,4) + k_2(5,-1,2) + k_3(1,1,3) = 0$

$$\begin{vmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{vmatrix} = 39 \neq 0$$

A is Invertible $\rightarrow$ Unique solution $\rightarrow$ L.I

VIQ Q_6 for which real values of λ do the following vectors form linearly dependent set in $\mathbb{R}^3$

$V_1 = (\lambda, \frac{-1}{2}, \frac{-1}{2})$, $V_2 = (\frac{-1}{2}, \lambda, \frac{-1}{2})$, $V_3 = (\frac{-1}{2}, \frac{-1}{2}, \lambda)$

$$A = \begin{vmatrix} \lambda & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \lambda \end{vmatrix}$$

$\det(A) = \lambda^3 - \frac{3}{4}\lambda - \frac{2}{8} = 0$

$8\lambda^3 - 6\lambda - 2 = 0$

$(\lambda - 1)(4\lambda + 2)(2\lambda + 1) = 0$

$\lambda = 1 \ , \frac{-1}{2}$

Q_7 let $S = \{(\lambda, 2, 5), (3, \lambda, 4), (-1, 3, 6)\}$ then one of the following is True

A) If $\lambda = \frac{1}{6}$, then s is linearly dependent

B) If $\lambda = 1$, then s is linearly Independent

c) If $\lambda = \frac{1}{6}$, then S is linearly Independent

D) If $\lambda = 2$, then S is linearly dependent

$$\begin{vmatrix} \lambda & 3 & -1 \\ 2 & \lambda & 3 \\ 5 & 4 & 6 \end{vmatrix} = \lambda(6\lambda - 12) - 3(-3) - 1(8 - 5\lambda) = 0$$

$$6\lambda^2 - 12\lambda + 9 - 8 + 5\lambda = 0$$

$$6\lambda^2 - 7\lambda + 1 = 0$$

$\lambda = 1$, $\lambda = \frac{1}{6}$ → linearly dependent

ans is A

Summary

let $S = \{V_1, V_2, V_3, \ldots\}$ in R^n

→ If $K_1V_1 + K_2V_2 + \ldots = 0$ has trivial solution → S linearly Independent

→ If $K_1V_1 + K_2V_2 + \ldots = (a, b, \ldots)$ has any solution → S spans R^n

4.4 Coordinates and Basis

We say S is a Basis for V iffs

1- S is linearly Independent

2- S spans V

Q_1 $S = \{(1,0),(0,1)\}$, Is S basis of $\mathbb{R}^2$

1- $k_1(1,0) + k_2(0,1) = 0$

$\det A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ $\det A \neq 0$ so S linearly Ind

2- $k_1(1,0) + k_2(0,1) = (a,b)$

$k_1 = a$, $k_2 = b$ so S spans V

$a(1,0) + b(0,1) = (a,b)$

S is basis for $\mathbb{R}^2$

Q_2 $S = \{(1,1),(3,3)\}$, Is S basis of $\mathbb{R}^2$

1- $k_1(1,1) + k_2(3,3) = 0$

$k_1 + 3k_2 = 0$

$k_1 + 3k_2 = 0$

$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \rightarrow \det(A) = 0$ so L.d

$\rightarrow$ not basis

Q3 $S = \{(1,2), (1,1)\}$, Is S basis for R^2

1. $K_1(1,2) + K_2(1,1) = 0$

$K_1 + K_2 = 0$
$2K_1 + K_2 = 0$

$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \rightarrow \det = -1 \neq 0$ so L.I

2. $K_1(1,2) + K_2(1,1) = (a,b)$

$K_1 + K_2 = a$ → بما انا يهمنا فقط وجود solution على الاقل →
$2K_1 + K_2 = b$ فيكفي التأكد عدم وجود عبارة رياضية خاطئة

مثل $0 = \text{number}$

$\rightarrow S$ spans V $\rightarrow S$ is basis for R^2

Q4 $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ Is S basis for R^3

1. $\det \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ so L.I

2. $K_1(1,0,0) + K_2(0,1,0) + K_3(0,0,1) = (a,b,c)$

$K_1 = a$, $K_2 = b$, $K_3 = c$ so S spans $V \rightarrow S$ is basis for R^3

Q5 $S = \{(1,2,1), (1,0,1), (0,1,1)\}$ Is S basis for R^3

Very Important Notes

1. $S = \{(1,0),(0,1)\} \rightarrow$ standard basis of $\mathbb{R}^2$
2. $S = \{(1,0,0),(0,1,0),(0,0,1)\} \rightarrow$ Standard basis of $\mathbb{R}^3$
3. $S = \{1, x, x^2\} \rightarrow$ standard basis for P_2
4. $S = \{1, x, x^2, x^3\} \rightarrow$ standard basis for P_3
5. $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \rightarrow$ Standard basis of M_{22}

Q6 $S = \{x+1, x^2+1, x^2+2x\}$, Is S basis for P_2

Q7 Is $S = \left\{ \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$ basis for M_{22}?

Coordinate Vector

If $S = \{V_1, V_2, V_3, \ldots\}$ is a basis of V

and $U = k_1V_1 + k_2V_2 + k_3V_3 + \ldots$ then

$U_S = (k_1, k_2, k_3, \ldots)$ is coordinate vector of U relative to S

Q_1 let $S = \{(1,0),(0,1)\}$ be a basis for R^2, find U_S where

$U = (7,3)$

$(7,3) = k_1(1,0) + k_2(0,1)$

$k_1 = 7, \ k_2 = 3 \rightarrow U_S = (7,3)$

Q_2 let $S = \{(1,2),(1,1)\}$ be a basis for R^2, find U_S where

$U = (2,5)$

$(2,5) = k_1(1,2) + k_2(1,1)$

Q_3 $S = \{(1,1,0), (2,1,1), (1,3,1)\}$ is basis for $\mathbb{R}^3$

find U_S when $U = (8,3,-2)$

Q_4 $S = \{3x+1, x^2+4, 8\}$ is basis for P_2, find U_S when

$U = 3x^2+5x-7$

Q_5 $S = \{(1,3,5), (1,4,8), (2,-1,5)\}$ is basis for $\mathbb{R}^3$ and $U_s = (2,-1,3)$, find U

$U = 2(1,3,5) - (1,4,8) + 3(2,-1,5)$

$= (7,-1,17)$

Q_6 $S = \{x^2+x-2,\ x^2+1,\ 3x-2\}$ is a basis for P_2 $V_s = (3,-2,4)$, find V

$V = 3(x^2+x-2) - 2(x^2+1) + 4(3x-2)$

$= x^2 + 15x - 16$

4.5 Dimension

Notes:

← نستطيع ايجاد اكثر من Basis لـ vector space معين، ولكن جميع هذه Basis يكون داخلها نفس عدد العناصر

Example $S = \{(1,0),(0,1)\}$ basis for R^2

$S = \{(1,2),(1,1)\}$ basis for R^2

: ملاحظة ان عدد العناصر متساوي لـ S_1, S_2

Dimension:

1. number of vectors in a basis of V

Or

2. number of free parameters in V

→ free parameters

if $v = (a,b,c) \rightarrow$ free parameters $= 3$

if $v = (a,b,a) \rightarrow$ free parameters $= 2$

if $v = (a,b,c),\ c = 2a \rightarrow$ free parameters $= 2$

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow$ free parameters $= 4$

if $A = \begin{pmatrix} a & b \\ 1 & a \end{pmatrix} \rightarrow$ free parameters $= 2$

→ $Dim(R^2) = 2$

→ $Dim(R^3) = 3$

→ $Dim(R^n) = n$

→ $Dim(P_n) = n+1$

→ $Dim(M_{nn}) = n^2$

→ $Dim(M_{mn}) = mn$

Q1 find $dim(P_2)$, $dim(P_3)$, $dim(M_{22})$

$dim(P_2) = 3$ since $S = \{1, x, x^2\}$ (3 elements)

$dim(P_3) = 4$

$dim(M_{22}) = 4$

Q2 let $W = \{(a,b,c,d) : a = 2b,\ c = 3d\}$ find dim(w) and basis of w

→ $dim(W) = 2$ (2 free parameters)

→ to find basis

$U \in W \rightarrow U = (2b, b, 3d, d)$

$(2b, b, 3d, d) = b(2,1,0,0) + d(0,0,3,1)$

$\{(2,1,0,0), (0,0,3,1)\}$ is basis for W

Q3 let $M = \{(a,b,c) : a+b+c = 0\}$ find dim(M) and basis of m

→ $dim(M) = 2$

→ $U = (a, b, -(a+b))$

$(a, b, -a-b) = a(1,0,-1) + b(0,1,-1)$

$\{(1,0,-1), (0,1,-1)\}$ is basis of M

Q_4 $W = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a,b \in R \right\}$ find $\dim(W)$

$\dim W = 2$

Q_5 find $\mathrm{Dim}(W)$

a- $W = \{ (a,b,c,d,e) : a+b+c+d+e = 0 \}$

b- $W = \{ A \in M_{33} :$ A is upper triangular $\}$

c- $W = \{ A \in M_{55} :$ A is lower triangular $\}$

d- $W = \{ A \in M_{nn} :$ A is diagonal $\}$

e- $W = \{ A \in M_{nn} : A = A^T \}$

Q6 let w be the set of all polynomials q in P_2 such that $q(1)=0$, find dim(w)

$q \in W \rightarrow q = a_0 + a_1 x + a_2 x^2$

$q(1) = a_0 + a_1 + a_2 = 0$

$a_2 = -a_0 - a_1$

so $\dim(W) = 2$

Theorms

1. If set S has more than n elelements of V, then S L.d
2. If set S has fewer than n elements, then S doesn't span V
3. If $\dim(V)=n$, elements of $S = n$ then S is a basis iff S is L.I or spans V

اي اذا كان dim V = n وكان عدد عناصر S يساوي n ايضاً فيكفي اثبات شرط واحد من شروط basis والشرط الاخر متحقق تلقائياً

Example: $S = \{(2,3),(1,5)\}$, Is S basis for $\mathbb{R}^2$

$\dim \mathbb{R}^2 = 2$, and S has 2 elements

$k_1(2,3) + k_2(1,5) = 0$

$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \rightarrow \det(A) = 7 \neq 0$ so L.I

so its a basis

4. let W subspace of V

$\dim(W) \leqslant \dim(V)$

$\rightarrow$ if $\dim(V) = \dim(W)$ then $W = V$

4.6 Change of Basis

Let V be a vector space, $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$ be two basis for V then we can write $w \in V$ as

$w = c_1 a_1 + c_2 a_2 + \dots$

and

$w = k_1 b_1 + k_2 b_2 + \dots$

نستطيع كتابة اي عنصر داخل V على

شكل L.C لعناصر Basis

Also $\rightarrow$

$w_A = (c_1, c_2, \dots, c_n)$ $\rightarrow$ coordinate vector of w relative to Basis A

$w_B = (k_1, k_2, \dots, k_n)$ $\rightarrow$ coordinate Vector of w relative to basis B

What Is the relation between W_A, W_B ?

$W_A = P_{B \to A} W_B$ $\quad$ $P_{B \to A}$: transition matrix from B to A

$W_B = P_{A \to B} W_A$ $\quad$ $P_{A \to B}$: transition matrix from A to B

how to find $P_{B \to A}$, $P_{A \to B}$?

Q_1 let $B = \{(1,0), (0,1)\}$ and $A = \{(1,1), (1,2)\}$ be two Basis for R^2

find $P_{A \to B}$, $P_{B \to A}$

$\boxed{P_{A \to B}}$

Step 1 $\quad [B \mid A] \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$

step 2 $\quad$ نحول B الى I $\quad [I \mid P_{A \to B}]$ $\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$

نلاحظ انها

جاهزة على شكل I

so $P_{A \to B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\boxed{P_{B\to A}}$ Step 1 $\left(A \,\vdots\, B\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$

Step 2 حول ال A ل I

$$\xrightarrow{-r_1+r_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{-r_2+r_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right) \quad P_{B\to A}$$

Note!! $(P_{B\to A})^{-1} = P_{A\to B}$

Q_2 let $B = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$, $A = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ be two Basis for $\mathbb{R}^2$

let $u = \begin{pmatrix}3\\5\end{pmatrix}$, find U_B, U_A and explain the relation between them

$\rightarrow \begin{pmatrix}3\\5\end{pmatrix} = k_1\begin{pmatrix}1\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\end{pmatrix}$

$k_1 = 3$, $k_2 = 5$ $\rightarrow U_B = (3,5)$ or $\begin{bmatrix}3\\5\end{bmatrix}$

$\rightarrow \begin{pmatrix}3\\5\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix}$

$3 = c_1 + c_2$ $\rightarrow c_1 = 1, c_2 = 2 \rightarrow U_A = (1,2)$ or $\begin{pmatrix}1\\2\end{pmatrix}$

$5 = c_1 + 2c_2$

from 1st example $P_{A\to B} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$

$U_B = P_{A\to B}\, U_A = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$

Note: If P is the trasition Matrix from A to B for finite dimensional Vector space V, then P^{-1} is the transition Matrix from B to A

Q3 $S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \right\}$, $S_2 = \{ V_1, V_2, V_3 \}$, S_1 and S_2 basis for R^3

$U = \begin{pmatrix} 12 \\ 1 \\ 19 \end{pmatrix}$, find U_{S_2} given that $P_{S_1 \to S_2} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$

$$U_{S_2} = \underset{\text{given}}{P_{S_1 \to S_2}} \; \underset{\text{يجب إيجادها}}{U_{S_1}}$$

$$\begin{pmatrix} 12 \\ 1 \\ 19 \end{pmatrix} = K_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + K_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + K_3 \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

$$K_1 = 2, \; K_2 = -1, \; K_3 = 3$$

$$\to U_{S_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\to U_{S_2} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$$

Q4 $B = \{ U_1, U_2 \}$, $\bar{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix} \right\}$, let $W \in R^2$, $W_B = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$P_{\bar{B} \to B} = \begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$, find $W_{\bar{B}}$, W

$$W_{\bar{B}} = P_{B \to \bar{B}} \, W_B$$

$$P_{B \to \bar{B}} = (P_{\bar{B} \to B})^{-1} \to \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 2 & 0 & -1 \end{array} \right)$$

$$r_2/2 \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 0 & -1/2 \end{array} \right) \xrightarrow{-5r_2 + r_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 5/2 \\ 0 & 1 & 0 & -1/2 \end{array} \right)$$

$$P_{B \to \bar{B}} = \begin{pmatrix} 1 & 5/2 \\ 0 & -1/2 \end{pmatrix} \to W_{\bar{B}} = \begin{pmatrix} 1 & 5/2 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$W = -3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

Q5 let $B = \{P_1, P_2\}$ and $A = \{q_1, q_2\}$ be bases for P_1 where

$P_1 = 1+2x$, $P_2 = 3-x$, $q_1 = 2-2x$, $q_2 = 4+3x$

a- find transition Matrix from B to A

b- find transition Matrix from A to B

4.7 Row space, column space and null space

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} \to r_1 \\ \to r_2 \\ \\ \to r_n \end{matrix}$

$\begin{matrix} \downarrow & \downarrow & & \downarrow \\ c_1 & c_2 & & c_n \end{matrix}$

Row space = span $\{r_1, r_2, \ldots, r_m\}$

Column space = span $\{c_1, c_2, \ldots c_n\}$

null space = solution of $AX = 0$

Q1 $A = \begin{pmatrix} 1 & 2 & 5 & 4 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 1 & 6 \end{pmatrix}$, find

1. row space — span $\{(1,2,5,4), (2,-1,1,3), (3,1,1,6)\}$
2. column space — span $\{(1,2,3), (2,-1,1), (5,1,1), (4,3,6)\}$
3. null space

$x_1 + 2x_2 + 5x_3 + 4x_4 = 0$

$2x_1 - x_2 + x_3 + 3x_4 = 0$

$3x_1 + x_2 - x_3 + 6x_4 = 0$

Note!! for this Q

→ row space is subspace of $\mathbb{R}^4$

→ Column space is subspace of $\mathbb{R}^3$

Bases for row space and column space

let $A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$, find basis for row space, column space and null space

$$\begin{array}{r} \\ -2r_1+r_2 \to \\ r_1+r_3 \to \end{array} \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{pmatrix} \longrightarrow \begin{array}{r} \\ r_2/{-7} \\ r_2+r_3 \end{array} \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

← الصفوف غير الصفرية بعد Gaussian Elemenation تعتبر Basis لـ row space

$\{(1,4,5,2),(0,1,1,4/7)\}$ basis for row space

← الأعمدة التي تحتوي leading يكون العمود الأصلي لها Basis لـ column space

$\{(1,2,-1),(4,1,3)\}$ basis for column space

from $AX=0$, $x_1 = -9s - \frac{30}{7}t$, $x_2 = -s - \frac{4}{7}t$, $x_3 = s$, $x_4 = t$

null space $= \{(-9s - \frac{30t}{7}, -s - \frac{4}{7}t, s, t)\}$

basis for null space $= \{(-9,-1,1,0), (\frac{-30}{7}, \frac{-4}{7}, 0, 1)\}$

4.8 Rank, Nullity

→ Dim (row space) = Dim (column space)

rank(A) = dim (row space)

nullity(A) = dim (null space)

→ $\text{rank}(A) + \text{nullity}(A) = n$ → number of columns of A

→ $\text{rank}(A) = \text{rank}(A^T)$

→ $\text{rank}(A^T) + \text{nullity}(A^T) = m$ → num of rows (A) or num of col (A^T)

Also: rank(A) = number of leading in the matrix result when we write A in r.e.f

nullity(A) = number of free parameters in $A\underline{X} = 0$

Q: let A be 5x3 matrix, if rank(A) = 2, find nullity(A), nullityA^T

$2 + \text{nullity}(A) = 3 \rightarrow \text{nullity}(A) = 1$

$2 + \text{nullity}(A^T) = 5 \rightarrow \text{nullity}(A^T) = 3$

Q_2 A is 3x6 Matrix, what is the largest value for rank(A)

3, A has at most 3 leading

Q_3 A is 6x3 Matrix, what is the largest value for rank(A)

3, $rank(A) = rank(A^T)$, A^T has at most 3 leading

Q_4 A is 3x5 Matrix, what is the largest value of nullity(A) and smallest value of nullity(A)

$rank(A) + nullity(A) = 5$

rank(A) → smallest value = 0
→ largest value = 3

nullity → largest = 5
→ smallest = 2

Q_5 find largest possible value of rank(A) and smallest Possible Value of nullity(A) for

A_{4x6}

A_{5x5}

A_{6x4}

Q16 Complete the table

	a	b	c	d	e	f	g
size of A	3x3	3x3	3x3	5x6	6x5	4x4	6x5
rank(A)	3	2	1	2	2	0	5
dim(row)							
dim(column)							
nullity(A)							
nullity(A^T)							

5.1 Eigen Values and Eigen Vectors

how to find Eigen Values ?

$\rightarrow \boxed{\det(\lambda I - A) = 0}$

λ : eigen value A: $n \times n$ matrix

I: Identity Matrix

Note!!

$\lambda I_{2\times2} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\lambda I_{3\times3} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

Q_1 Let $A = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$, find eigen values of A

$$\lambda I - A = \begin{pmatrix} \lambda-3 & -2 \\ 0 & \lambda-5 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda-3)(\lambda-5) - 0 = 0$$

$$\lambda = 3, 5$$

Q_2 Let $A = \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}$, find eigen values of A

$$\det(\lambda I - A) = 0$$

$$\rightarrow \begin{vmatrix} \lambda-1 & -5 \\ -2 & \lambda+2 \end{vmatrix} = 0$$

$$\rightarrow (\lambda-1)(\lambda+2) - 10 = 0$$

$$\lambda^2 + 2\lambda - \lambda - 2 - 10 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda+4)(\lambda-3) = 0$$

$$\lambda = -4, 3$$

Note!!

$\lambda^2 + \lambda - 12 = 0$ called Characarestic equation

how to find Eigen Vectors?

$\rightarrow \boxed{(\lambda I - A)X_\lambda = 0}$ $\quad X_\lambda$: Eigen Vector

Note!!

$(X_\lambda) \leftarrow$ هي الحل الغير صفري للنظام $(\lambda I - A)X_\lambda = 0$

$(\lambda) \leftarrow$ هي الثوابت التي تجعل للنظام $(\lambda I - A)X_\lambda = 0$ حل غير صفري

Q_3 find eigen values and eigen vectors for $A = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$

$$\rightarrow \lambda I - A = \begin{pmatrix} \lambda - 2 & -1 \\ 0 & \lambda - 4 \end{pmatrix}$$

$$\rightarrow \det(\lambda I - A) = (\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = 2, \lambda = 4$$

$\rightarrow$ for $\lambda = 2$, let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$(\lambda I - A)X = 0$$

$$\begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$0x_1 - x_2 = 0$ اي انها قد تكون اي قيمة

$x_2 = 0, \quad x_1 = t$

so $X_{\lambda_1} = \begin{pmatrix} t \\ 0 \end{pmatrix}$ let $t = 1 \rightarrow X_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\rightarrow$ for $\lambda = 4$ let $X_{\lambda_2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$2x_1 - x_2 = 0$

$x_2 = 2x_1$

$\rightarrow X_{\lambda_2} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix}$ let $x_1 = 1 \rightarrow X_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

يمكن تركها x_1 او فرضها t

Q_4 find eigen values and eigen vectors for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

Q_5 find eigen values and eigen vectors for $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$\rightarrow \lambda I - A = \begin{pmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 4 \end{pmatrix}$

$$\det(\lambda I - A) = (\lambda - 2)(\lambda - 4) + 1 = 0$$
$$\lambda^2 - 6\lambda + 8 + 1 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$
$$\lambda = 3, 3$$

Note!!

If $\lambda_1 = \lambda_2$ then:

1. Solve $(\lambda_1 I - A) X_{\lambda_1} = 0 \rightarrow$ to find X_{λ_1}
2. Solve $(\lambda_2 I - A) X_{\lambda_2} = X_{\lambda_1} \rightarrow$ to find X_{λ_2}

$\rightarrow$ for $\lambda = 3$, $X_{\lambda_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$a - b = 0$

$a = b$

$X_{\lambda_1} = \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\rightarrow X_{\lambda_2} = \begin{pmatrix} c \\ d \end{pmatrix}$

هنا الاختلاف

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$c - d = 1$

$d = c - 1$

$X_{\lambda_2} = \begin{pmatrix} c \\ c-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Q8 find eigen values and eigen vectors for $A=\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}$

$\rightarrow \lambda I - A = \begin{pmatrix}\lambda&0&2\\-1&\lambda-2&-1\\-1&0&\lambda-3\end{pmatrix}$

$\rightarrow \det(\lambda I - A) = \lambda(\lambda-2)(\lambda-3) - 0 + 2(\lambda-2) = 0$

باستخدام الصف الاول

$\lambda(\lambda^2-5\lambda+6) + 2\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 6\lambda + 2\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1, 2, 2$

$\det(\lambda I - A) = (\lambda-2)\begin{vmatrix}\lambda&2\\-1&\lambda-3\end{vmatrix}$

باستخدام العمود الثاني

$= (\lambda-2)(\lambda^2-3\lambda+2) = 0$

$= (\lambda-2)(\lambda-2)(\lambda-1) = 0$

$\lambda = 2, 2, 1$

$\rightarrow$ for $\lambda = 1$, $X_{\lambda_1} = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\left(\begin{array}{ccc|c}1&0&2&0\\-1&-1&-1&0\\-1&0&-2&0\end{array}\right) \rightarrow \begin{array}{c}R_1+R_2\\R_1+R_3\end{array}\left(\begin{array}{ccc|c}1&0&2&0\\0&-1&1&0\\0&0&0&0\end{array}\right)$$

$a + 2c = 0 \rightarrow a = -2c$

$-b + c = 0 \rightarrow b = c$

$X_{\lambda_1} = \begin{pmatrix}-2c\\c\\c\end{pmatrix} = \begin{pmatrix}-2\\1\\1\end{pmatrix}$

$\rightarrow$ for $\lambda = 2$, $X_{\lambda_2} = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\left(\begin{array}{ccc|c}2&0&2&0\\-1&0&-1&0\\-1&0&-1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\-1&0&-1&0\\-1&0&-1&0\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$a + c = 0$

$\boxed{a = -c}$

$X_{\lambda_2} = \begin{pmatrix}-c\\b\\c\end{pmatrix} = c\begin{pmatrix}-1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix}$

at $c=1$ $X_{\lambda_2} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

at $b=1$ $X_{\lambda_3} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Q_7 $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ find eigen values and eigen vectors

Q8 let $P(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$ be the characterstic equation of Matrix A

1. find det(A)

since $\lambda^3 - 2\lambda^2 + \lambda + 5$ is the char. eq.

$\rightarrow$ so $\det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda + 5$

$\rightarrow$ for $\lambda = 0$ $\det(-A) = 5$

$(-1)^3 \det(A) = 5$

$\det(A) = -5$

Remember

$\det(KA) = K^n \det(A)$, n: num of rows

2. Is 0 eigen value of A

$P(0) = 5 \neq 0$ so its not eigen value

Theorm: A square Matrix A is invertible iff $\lambda = 0$ is not eigen Value of A

Theorm: If A is diagonal, upper or lower, the the eigen values of A are theentries on the main diagonal

Q9 $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 6 & -1 & 2 & 7 \end{pmatrix}$ Find eigen Values of A

$\lambda = 1, 3, 5, 7$ since A is lower

Notes :

1. If λ eigen values of A, then λ^k is eigen values of A^k, $(k>0)$

2. If λ eigen values of A, then $\lambda+c$ is eigen values of $A+cI$ constant

3. If A is invertible and λ is eigen value of A, then $\frac{1}{\lambda}$ is eigen value of A^{-1}

Q10 If 2,5,-3 are the eigen values of A, find the eigen values of B when $B = A^{-3} - 3I$

$A^{-3} = (A^{-1})^3$

eigen values of $B = \left(\frac{1}{2}\right)^3 - 3 \quad , \left(\frac{1}{5}\right)^3 - 3 \quad , \left(\frac{-1}{3}\right)^3 - 3$

$= \frac{-23}{8}, \frac{-374}{125}, \frac{-82}{27}$

5.2 Diagonalization

If A and B are square matrices, then we say B is similar to A if there is an invertible matrix P such that

$$B = P^{-1}AP$$

→ Properities for similar matrices

If A and B are similar, then:

1. A and B have same determinant.
2. A and B have same trace.
3. A and B have same eigen values.
4. A and B have same rank and nullity.
5. A and B have same characterstic equation.

Proof :

1. If B similar to $A \rightarrow B = P^{-1}AP$ for some matrix P

$$\det(B) = \det(P^{-1}AP) = \cancel{\det}(P^{-1}) \cdot \det(A) \cdot \cancel{\det}(P) = \det(A)$$

2. $B = P^{-1}AP$

$$tr(B) = tr(P^{-1}AP) = tr(PP^{-1}A) = tr(IA) = tr(A)$$

Remember

$tr(BA) = tr(AB)$

Definition :

The square Matrix A is said to be diagonalizable if it Similar to diagonal Matrix

→ There exist D, P such that $D = P^{-1}AP$, where D is diagonal

Theorm :

If A is $n \times n$ matrix, then A is diagonalizable if A has n linearly independent eigen vectors

← اي بمعنى آخر ان يكون عدد الـ eigen vectors مساوي لعدد صفوف المصفوفة ودون تكرار

Q_1 let $A = \begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, Is A diagonalizable? If yes find D and P such that $D = P^{-1}AP$

$\lambda I - A = \begin{bmatrix} \lambda-3 & -5 \\ 0 & \lambda-4 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-3)(\lambda-4) = 0$

$\lambda = 3 \quad \lambda = 4$

for $\lambda = 3$

$\left(\begin{array}{cc|c} 0 & -5 & 0 \\ 0 & -1 & 0 \end{array}\right)$

$X_{\lambda_1} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

for $\lambda = 4$

$\left(\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$a - 5b = 0 \rightarrow a = 5b$

$X_{\lambda_2} = \begin{pmatrix} 5b \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$\rightarrow$ A is 2x2 and have 2 eigen values and 2 eigen vectors so A is diagonalizable

$\rightarrow$ To find D, P

$$P = \begin{array}{cc} X_{\lambda_1} & X_{\lambda_2} \\ \end{array} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{array}{l} \lambda_1 \\ \lambda_2 \end{array}$$

Q_2 $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\lambda I - A = \begin{pmatrix} \lambda-2 & -1 \\ 0 & \lambda-2 \end{pmatrix}$

$\det(\lambda I - A) = (\lambda-2)(\lambda-2)$

$\lambda = 2, 2$

$\rightarrow$ at $\lambda = 2$

$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$X_\lambda = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Since A has 1 eigen vector

$\rightarrow$ A is not diagonalizable

Q_3 $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

from sec 5.1

$\lambda = 2, 2, 1$

for $\lambda = 1 \rightarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ eigen vector

for $\lambda = 2 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors

$\rightarrow$ Since A is 3x3, and A have 3 eigen vectors

A is diagonalizable

$P = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Theorm:

If A is nxn matrix and A have n distinct eigen valve,

Then A is diagonalizable

Q_4 $A = \begin{pmatrix} 1 & 3 & 5 & 1 \\ 0 & 2 & 7 & -1 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ Is A diagonalizable?

Yes

$\hookrightarrow \lambda = 1, 2, 8, 7$ (4 eigen values and A is 4x4)

Notes

1. If $V_1, V_2, \dots V_n$ are eigen vectors corrosponding to λ_0 then

   $W_0 = \text{span}\{V_1, V_2, \dots V_n\}$ is called eigenspace

   $\dim(W_0) = n$

   عدد eigen vectors

2. $\dim(W_0)$ is called geometric multiplicity for λ_0 (G.m)

3. The num of times that $(\lambda - \lambda_0)$ appears as a factor in $\det(\lambda I - A)$ is called Algebric multiplicity for λ_0 (A.m)

4. If G.m = A.m for all λs of A, then A is diagonalizable

Chapter 6 : Inner product spaces

6.1 Inner products

Definition:

An inner product on a real vector space V is a function that associates a real number $\langle u,v \rangle$ with each pair of vectors in V in such way the following conditions hold:

1. $\langle u,v \rangle = \langle v,u \rangle$ for all $u,v \in V$
2. $\langle u+v,w \rangle = \langle u,w \rangle + \langle v,w \rangle$ for u,v,w in V
3. $\langle ku,v \rangle = k\langle u,v \rangle$, $u,v \in V$
4. $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0$ iff $v = 0$

← أي ان inner product هو function بربط كل عنصرين من عناصر V ببعضهم بعلاقة معينة تعطى في السؤال بحيث يكون الناتج عبارة عن constant موجود في real num

$$f(u,v) = \langle u,v \rangle$$

← $\langle u,v \rangle$ تقرأ ← inner u,v وهي نفسها $f(u,v)$

Q_1 let $V = R^2$, u, v in V such that $u = (u_1, u_2)$, $v = (v_1, v_2)$ and

$\langle u, v \rangle = u_1 v_1 + u_2 v_2$, show that $\langle u, v \rangle$ is inner product

$f(u,v)$

1. $\langle u, v \rangle = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = \langle v, u \rangle$

2. let $w \in R^2 \rightarrow w = (w_1, w_2)$

$$u + v = (u_1 + v_1, u_2 + v_2)$$

$$\langle u + v, w \rangle = u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$$

$$\langle u, w \rangle + \langle v, w \rangle$$

3. $ku = (ku_1, ku_2)$

$$\langle ku, v \rangle = k u_1 v_1 + k u_2 v_2 = k (u_1 v_1 + u_2 v_2) = k \langle u, v \rangle$$

4. $\langle v, v \rangle = v_1^2 + v_2^2 \geq 0$

if $\langle v, v \rangle = 0 \rightarrow v_1^2 + v_2^2 = 0$

$v_1 = 0$, $v_2 = 0$

so $v = (0, 0)$

$\rightarrow \langle u, v \rangle$ is inner product

Q2 let $V = R^3$, $U = (U_1, U_2, U_3)$, $V = (V_1, V_2, V_3)$, if

$\langle U,V \rangle = 2U_1V_1 + U_2V_2 + 5U_3V_3$, Is $\langle U,V \rangle$ I.P

1. $\langle U,V \rangle = \langle V,U \rangle$

2. let $W \in R^3 \rightarrow W = (W_1, W_2, W_3)$

$U+V = (U_1+V_1, U_2+V_2, U_3+V_3)$

$\langle U+V, W \rangle = \underline{2U_1W_1} + 2W_1V_1, \underline{U_2W_2} + V_2W_2 + \underline{5U_3W_3} + 5V_3W_3$

$= \langle U,W \rangle + \langle V,W \rangle$

3. $KU = (KU_1, KU_2, KU_3)$

$\langle KU, V \rangle = 2KU_1V_1 + KU_2V_2 + 5KU_3V_3 = K(2U_1V_1 + U_2V_2 + 5U_3V_3) = K\langle U,V \rangle$

4. $\langle V,V \rangle = 2V_1^2 + V_2^2 + 5V_3^2 \geq 0$

if $\langle V,V \rangle = 0 \rightarrow 2V_1^2 + V_2^2 + 5V_3^2 = 0$

$V_1 = 0, V_2 = 0, V_3 = 0$

so $V = (0,0,0)$

$\rightarrow \langle U,V \rangle$ is inner product

Q_3 Let $V = R^3$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, If $\langle u, v\rangle = u_1v_1 + u_3v_3$

Is $\langle u, v\rangle$ I.P?

If $u = (0,1,0)$ then $\langle u, u\rangle = 0$

but $u \neq 0_V$

so Its not I.P

Q_4 let $V = P_2$, $p, q \in V$, $p = a_0 + a_1x + a_2x^2$, $q = b_0 + b_1x + b_2x^2$

$\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$, show that $\langle p, q\rangle$ is Inner product

1. $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 = b_0a_0 + b_1a_1 + b_2a_2 = \langle q, p\rangle$

2. $r = c_0 + c_1x + c_2x^2$

$p + q = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2$

$\langle p+q, r\rangle = \underline{a_0c_0} + b_0c_0 + \underline{a_1c_1\cancel{x}} + b_1c_1\cancel{x} + \underline{a_2c_2\cancel{x^2}} + b_2c_2\cancel{x^2}$

$\langle p, r\rangle + \langle q, r\rangle$

3. $\langle kp, q\rangle = ka_0b_0 + ka_1b_1 + ka_2b_2 = k(a_0b_0 + a_1b_1 + a_2b_2) = k\langle p, q\rangle$

4. $\langle p, p\rangle = a_0^2 + a_1^2 + a_2^2 \geq 0$

if $a_0^2 + a_1^2 + a_2^2 = 0$

$a_0 = 0$, $a_1 = 0$, $a_2 = 0$

So $\langle p, q\rangle$ is I.P

Q_5 let $V = P_2$, $p, q \in V$, If $\int_0^1 p(x)q(x)dx = \langle p,q \rangle$

Is $\langle p,q \rangle$ Inner Product

1- $\langle p,q \rangle = \int_0^1 pq\,dx = \int_0^1 qp\,dx = \langle q,p \rangle$

2- $\langle p+q,r \rangle = \int_0^1 pr+qr\,dx$

$= \langle p,r \rangle + \langle q,r \rangle$

3- $\langle kp,q \rangle = \int_0^1 kpq\,dx = k\int_0^1 pq\,dx$

4- $\langle p,p \rangle = \int_0^1 p^2 dx \geq 0$

if $\langle p,p \rangle = 0 \rightarrow a_0 = 0$, $a_1 = 0$, $a_2 = 0$

so $\langle p,q \rangle$ is I.P

Notes!! let V inner product space , u, v, w in V

1. $\langle 0,v \rangle = \langle v,0 \rangle = 0$
2. $\langle u, v+w \rangle = \langle u,v \rangle + \langle u,w \rangle$
3. $\langle u, kv \rangle = k\langle u,v \rangle$
4. $\langle u-v, w \rangle = \langle u,w \rangle - \langle v,w \rangle$
5. $\langle u, v-w \rangle = \langle u,v \rangle - \langle u,w \rangle$

Definition:

Let V be inner product space, let v, u in V, then:

1. $\|u\|^2 = \langle u, u \rangle$

   $\|u\|$ is called the norm of u (length)

2. $d(u,v) = \|u - v\|$

   $d(u,v)$ is called the distance between u and v

Notes!!

1. $\|u\| \geq 0$ for all $u \in V$
2. $\|ku\| = |k| \, \|u\|$, k scalar

Q₁ let $\langle u,v \rangle$ be the Euclidean inner product on R^2, and let $u = (1,1)$, $v = (3,2)$, $w = (-1,0)$, $k = 5$.

compute $\langle v, w \rangle$, $\|u\|$, $d(u,v)$, $\langle ku, v \rangle$

Note!!

Euclidean means

$\langle u,v \rangle = u_1 v_1 + u_2 v_2$

اي dot product الطبيعي

1. $\langle v, w \rangle = -3 + 0 = -3$

2. $\|u\|^2 = \langle u, u \rangle = u_1^2 + u_2^2 = 1 + 1 = 2$

   $\|u\| = \sqrt{2}$

3. $d(u,v) = \|u - v\|$

   $u - v = (-2,-1) \rightarrow \|u - v\|^2 = \langle u-v, u-v \rangle = 4 + 1 = 5$

   $\|u - v\| = \sqrt{5}$

4. $\langle ku, v \rangle = k \langle u, v \rangle \rightarrow 3 + 2 = 5$

   $= 5 \cdot 5 = 25$

Q_2 If $V = R^2$, $u, v \in R$, $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$. find $\|w\|$ where $w = (4, -2)$

$\|w\|^2 = \langle w, w \rangle = 4.4.2 + 3(-2)(-2) = 44$

$\|w\| = \sqrt{44}$

Q_3 let $V = P_2$, if $\langle p, q \rangle$ is inner product such that $\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2c_2$, find $\|p\|$ and $\langle p, q \rangle$ where $p = 3 - 2x + 5x^2$, $q = 5 - 2x^2$

1. $\|p\|^2 = \langle p, p \rangle = (3)(3) + (-2)(-2) + (5)(5) = 38$

$\|p\| = \sqrt{38}$

2. $\langle p, q \rangle = (3)(5) + (-2)(0) + (5)(-2) = 5$

Q_4 If $\|u\| = 4$, $\|v\| = 3$, $\langle u, v \rangle = 5$, find $\|2u + 5v\|$

$\|2u+5v\|^2 = \langle 2u+5v, 2u+5v \rangle$

$= \langle 2u, 2u \rangle + \langle 2u, 5v \rangle + \langle 5v, 2u \rangle + \langle 5v, 5v \rangle$

$= 4\langle u, u \rangle + 10\langle u, v \rangle + 10\langle v, u \rangle + 25\langle v, v \rangle$

$= 4\|u\|^2 + 20\langle u, v \rangle + 25\|v\|^2$

$= 4(16) + 20(5) + 25(9)$

$= 239$

$\|2u+5v\| = \sqrt{235}$

6.2 Angle and orthogonally in inner product space

Theorm : cauchy - schwarz inequality

let V be inner product space, let u, v be vectors in V then

$$|\langle u,v\rangle| \leq \|u\|\,\|v\|$$

Q₁ Can you find an inner product on vector space V such that $\langle u,v\rangle = 8$, $\|u\| = 2$ and $\|v\| = 3$

$8 > 2.3$

So ans is No

Definition : $\cos\theta = \dfrac{\langle u,v\rangle}{\|u\|\,\|v\|}$

Q₂ let $\langle u,v\rangle$ be the Euclidean inner product on $\mathbb{R}^3$, If $u = (2,-1,1)$ $v = (3,2,4)$, find the angle between u and v

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\,\|v\|} = \frac{6-2+4}{\sqrt{6}\,\sqrt{29}} = \frac{8}{\sqrt{6}\,\sqrt{29}}$$

Theorm: If u, v and w are vectors in real inner product space V then

1. $\|u+v\| \leq \|u\| + \|v\|$

2. $d(u,v) \leq d(u,w) + d(w,v)$

Definition let V be real inner product space, let u, v be vectors in V, we say u and v are orthogonal iff $\langle u, v\rangle = 0$

Q3 let $\langle u, v\rangle$ be Euclidean inner product on R^2

If $u = (3, -2)$, $v = (4, k)$, find k such that u, v are orthogonal

$\langle u, v\rangle = 12 - 2k = 0$

$2k = 12$

$\boxed{k = 6}$

Q4 let $p = a_0 + a_1x + a_2x^2$, $q = b_0 + b_1x + b_2x^2$ in P_2, if $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$ is inner product on P_2, let $P_1 = 2 - 3x + x^2$, $P_2 = 5 + 6x + 8x^2$, are P_1 and P_2 orthogonal?

$\langle p, q\rangle = 10 - 18 + 8 = 0$ so P_1, P_2 are orthogonal

Q5 If $\|u\| = 3$, $\|v\| = 4$ and u, v orthogonal, find $\|u+v\|$

$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$

$= 9 + 0 + 16 = 25$

$\|u+v\| = \sqrt{25} = 5$

Definition let $S = \{v_1, v_2, \dots, v_n\}$ be subset of inner product space V, we say S is orthogonal set if $\langle v_i, v_j \rangle = 0$ for $i \neq j$

Q_6 let $S = \{\overset{\vec{v_1}}{(1,0,0)}, \overset{\vec{v_2}}{(0,1,0)}, \overset{\vec{v_3}}{(0,0,1)}\}$ Is S orthogonal set?

$\langle v_1, v_2 \rangle = 0$

$\langle v_1, v_3 \rangle = 0$ $\quad$ so S is orthogonal set

$\langle v_2, v_3 \rangle = 0$

Q_7 let $S = \{\overset{\vec{v_1}}{(1,1,0)}, \overset{\vec{v_2}}{(1,0,1)}, \overset{\vec{v_3}}{(1,-1,0)}\}$ IS orthogonal set?

$\langle v_1, v_2 \rangle = 1 + 0 + 0 = 1$ $\quad$ so S not orthogonal set

Definition let $S = \{v_1, v_2, \dots, v_n\}$ be subset of inner product space V we say S is orthonormal set if:

1. $\langle v_i, v_j \rangle = 0$ for $i \neq j$
2. $\langle v_i, v_i \rangle = 1$

Q_8 let $S = \{ (\underbrace{\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}}_{V_1}), (\underbrace{\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}}_{V_2}) \}$, Is S orthonormal set?

$\langle V_1, V_2 \rangle = \frac{1}{2} - \frac{1}{2} = 0$

$\langle V_1, V_1 \rangle = \frac{1}{2} + \frac{1}{2} = 1$ So S is orthonormal set

$\langle V_2, V_2 \rangle = \frac{1}{2} + \frac{1}{2} = 1$

Definition let V be inner product space, let W be subspace of V, we define orthogonal complement of W as

$$W^{\perp} = \{ v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W \}$$

Q_9 let $W = \{(a,b,c,d) : c = 2a+b, d = 4a\}$ find $W^{\perp}$ and basis for $W^{\perp}$

→ Step1 find basis for W

$w \in W = (a, b, 2a+b, 4a)$

$= a(1,0,2,4) + b(0,1,1,0)$

$\{(1,0,2,4), (0,1,1,0)\}$ basis for W

→ step 2 find solution of the basis

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_3 + x_4 = 0$

$x_2 + x_3 = 0$

$x_4 = t$, $x_3 = s$

$x_2 = -s$

$x_1 = -2s - 4t$

$\rightarrow W^{\perp} = \{(-2s-4t, -s, s, t)\}$

→ Step3 find basis for $W^{\perp}$

$(-2s-4t, -s, s, t) = s(-2,-1,1,0) + t(-4,0,0,1)$

basis for $W^{\perp} = \{(-2,-1,1,0), (-4,0,0,1)\}$

Q_{10} Find a basis for orthogonal complement of the subspace spanned by the vectors

$V_1 = (2,1,3)$, $V_2 = (-1,-4,2)$, $V_3 = (4,-5,13)$

→ Step 1 ✓

→ Step 2 $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ -1 & -4 & 2 & 0 \\ 4 & -5 & 13 & 0 \end{array}\right)$

solution $\{(-2t, t, t)\} = W^\perp$

→ Step 3 $t(-2,1,1)$

$\{(-2,1,1)\}$ basis for $W^\perp$

Ch8: linear Transformation (linear Map)

Definition:

If $T: V \to W$ is a function from vector space V to vector space W, Then we say T is linear Transformation iff :-

1. $T(u+v) = T(u) + T(v)$ for all u, v in V
2. $T(ku) = kT(u)$, k is scalar

Note!!

If $V = W$, then T is called linear operator

Q_1 Let $T: M_n \to \mathbb{R}$ such that $T(A) = tr(A)$, show that T is linear Transformation

let $A, B \in M_n$

1. $T(A+B) = tr(A+B) = tr(A) + tr(B) = T(A) + T(B)$ ✔
2. $T(KA) = tr(KA) = k\,tr(A) = kT(A)$ ✔

$\to T$ is linear transformation

Q2 If $T: M_n \to \mathbb{R}$ such that $T(A) = \det(A)$. Is T linear transformation

let $A, B \in M_n$

1. $T(A+B) = \det(A+B) \neq \det(A) + \det(B)$ ✗

2. $T(KA) = \det(KA) \neq K\det(A)$ ✗

Note!!

$\rightarrow$ T is not linear Transformation

يكفي اختلال شرط واحد لعدم

اعتبار T linear Transformation

Notes :

1. let V be any vector space, let $T: V \to V$ such that $T(v) = v$ for all $v \in V$, then T is linear Transformation, and we call T the Identity operator on V.

2. If $T: V \to W$ such that $T(v) = 0_W$, Then T is called Zero transformation

Q3 Let $T: R^2 \to R^2$ such that $T(x,y) = (2x+y, y-x)$, show That T is linear Transformation

let $u, v \in R^2 \rightarrow u = (x_1, y_1), v = (x_2, y_2)$

$u+v = (x_1+x_2, y_1+y_2)$

1. $T(u+v) = (2(x_1+x_2) + y_1+y_2, y_1+y_2 - x_1 - x_2)$

$= (2x_1+y_1 + 2x_2+y_2, y_1-x_1 + y_2-x_2)$

$= T(u) + T(v)$

2. $Ku = (kx_1, ky_1)$

$T(ku) = (2kx_1+ky_1, ky_1-kx_1) = K(2x_1+y_1, y_1-x_1) = K\,T(u)$

$\rightarrow$ T is linear transformation

Q_4 If $T: R^3 \to R^2$ such that $T(x,y,z) = (x+2z, y-z)$, show that T is linear transformation

let $U, V \in R^3 \to U = (x_1, y_1, z_1)$, $V = (x_2, y_2, z_2)$

$U+V = (x_1+x_2, y_1+y_2, z_1+z_2)$

1. $T(U+V) = (x_1+x_2+2z_1+2z_2, y_1+y_2-z_1-z_2)$
$= (x_1+2z_1+x_2+2z_2, y_1-z_1+y_2-z_2)$
$= T(U) + T(V)$

2. $T(kU) = (kx_1+2kz_1, ky_1-kz_1) = kT(U)$
$\to$ T is linear Transformation

Q_5 let $T: P_2 \to P_3$ such that $T(P) = xP$, show that T is linear Transformation

let $P_1, P_2 \in P_2$

1. $T(P_1+P_2) = xP_1 + xP_2 = x(P_1+P_2) =$
$= T(P_1) + T(P_2)$

2. $T(kP_1) = xkP_1 = k(xP_1) = kT(P_1)$
$\to$ T is linear Transformation

Q_6 If $T: P_n \to P_{n-1}$ such that $T(p) = p'$, show that T is linear transformation

let $P_1, P_2 \in P_n$

1. $T(P_1+P_2) = (P_1+P_2)' = P_1' + P_2' = T(P_1) + T(P_2)$

2. $T(KP_1) = (KP_1)' = KP_1' = KT(P_1)$

$\rightarrow$ T is linear transformation

Q_7 If $T: P_n \to \mathbb{R}$ such that $T(p) = \int_0^1 p(x)\,dx$, show that T is linear transformation

let $P_1, P_2 \in P_n$

1. $T(P_1+P_2) = \int_0^1 (P_1+P_2)\,dx = \int_0^1 P_1\,dx + \int_0^1 P_2\,dx = T(P_1) + T(P_2)$

2. $T(KP_1) = \int_0^1 KP_1\,dx = K\int_0^1 P_1\,dx = KT(P_1)$

$\rightarrow$ T is linear transformation

Definition:

let $T: V \to W$ be a linear transformation

1. $Range(T) = \{Tv : v \in V\}$

اي جميع صور عناصر المجال

2. $Kernel(T) = \{v \in V : Tv = 0_w\}$

اي جميع عناصر المجال التي تكون صورتها تساوي 0_w

Q_1 let $T: R^2 \to R^3$ such that $T(x,y) = (x+y, x, 2x-y)$ be a linear transformation, Is $U=(5,2,1)$, $V=(3,1,4)$ in range T?

1. $U = (5,2,1)$

نبحث عن قيم x,y التي تحقق المعادلة

$(x+y, x, 2x-y) = (5,2,1)$

اذا كان هناك حل للنظام تكون النقطة في range

$$\left.\begin{aligned} x+y&=5 \\ x&=2 \\ 2x-y&=1 \end{aligned}\right\rangle \; x=2,\ y=3$$

$\to (5,2,1)$ in Range

2. $V = (3,1,4)$

$$\left.\begin{aligned} x+y&=3 \\ x&=1 \end{aligned}\right\rangle \; x=1,\ y=2$$

$2x-y=4$ test this equation $2-2 \neq 0$ (means no solution)

so $\to (3,1,4)$ not in range

Q_2 let $T: R^3 \to R^2$ such that $T(x,y,z) = (x-3y, z+2y)$ be a linear transformation, find Kernel(T)

we need to find (x,y,z) such that $T(x,y,z) = (0,0)$

$(x-3y, z+2y) = (0,0)$

$$\left.\begin{array}{l} x-3y=0 \\ z+2y=0 \end{array}\right\rangle \text{let } y=t,\ x=3t,\ z=-2t$$

$\rightarrow$ Kernel$(T) = \{(3t, t, -2t) : t \in R\}$

* To check let $t=2 \rightarrow (6,2,-4)$

$T(6,2,-4) = (0,0)$

Notes!! If $T: V \to W$ is a linear transformation, then

1. Ker(T) is a subspace of V
2. range(T) is a subspace of W
3. dim(range(T)) = rank(T)
4. dim(Ker(T)) = nullity(T)
5. rank(T) + nullity(T) = dim V

Q_3 If $T: R^3 \to R^3$ such that $T(x,y,z) = (z-x, y+2x, 2z+y)$ be a linear map, find rank(T)

$\to$ find Ker(T)

$z - x = 0 \quad \to z = x$

$y + 2x = 0 \quad \to y = -2x$

$2z + y = 0 \qquad x = t,\ y = -2t,\ z = t$

$Ker(T) = \{(t, -2t, t) : t \in R\}$

$nullity(T) = dim(Ker(T)) = 1$

$rank(T) + nullity(T) = 3$

$rank(T) = 2$

Q_4 let $T: P_2 \to P_3$ such that $T(p) = xp$

1. Is $x+1$ in Range T
2. find Ker(T)
3. find rank(T)

1. $T(p) = x+1$

$xp = x+1 \quad \to p = 1 + \frac{1}{x} \notin P_2 \quad \to x+1$ not in Range T

2. $T(p) = 0$

$xp = 0 \quad \to \underset{p \in P_2}{p} = 0 \to 0 + 0x + 0x^2 = 0$

$Ker(T) = \{0\}$

3. $nullity(T) = 0$

$rank(T) + 0 = 3$

$rank(T) = 3$

Q_5 let $S = \{(1,1), (1,0)\}$ be basis for $\mathbb{R}^2$ and let $T: R^2 \to R^3$ be a linear transformation such that

$T(1,1) = (2,-1,3)$, $T(1,0) = (5,3,-2)$

find 1. $T(7,3)$

2. $T(a,b)$

$(7,3) = K_1(1,1) + K_2(1,0)$

$7 = K_1 + K_2$

$3 = K_1 \rightarrow K_2 = 4$

$(7,3) = 3(1,1) + 4(1,0)$

$T(7,3) = 3T(1,1) + 4T(1,0)$

$= 3(2,-1,3) + 4(5,3,-2)$

$= (26, 9, 1)$

Mid Exam/Fall 2021 by Anas AbuZahra

1. Assume that $\lim_{x\to 2} f(x)$ exist and $\frac{\sqrt{4x+1}-3}{x-2} \leq \frac{f(x)}{1-2x} \leq \frac{x}{3}$ then $\lim_{x\to 2} f(x) =$

A. 2 B. $\frac{2}{3}$ C. $\frac{-2}{3}$ d. -2 E. $F(2)$

2. Given $f(x)=\sqrt{3x-1}$ and $g(x)=\frac{1}{x-1}$, find Domain $f\circ g(x)$

3. The function $f(x)=x^3 \sin x$ is symmetric about

A. x-axis b. y-axis c. $y=x$ d. $y=-x$ e. origin

4. The vertical asymptote of $f(x)=\frac{3-x}{|2x-3|-x}$ is (are) :

5. If $x>0$, then $\sec(\tan^{-1}\frac{3}{x}) =$

6. If $\log_x (6-x) = 2$, then $x=$

A. -2 or 3 b. -3 or 2 c. -3 only d. 3 only e. 2 only

7. The function $f(x) = \frac{x+1}{x^3-x}$ has removable discontinuity at $x=$

8. $\lim_{x\to 1^-} \tan^{-1}\left(\frac{3x}{1-x}\right) =$

9. If $36^x - 6^x - 6 = 0$, then $x =$

10. find Domain $f(x) = \cos^{-1}(x+5) - \log_5(x+5)$

11. Given $f(x) = 3 - \ln\sqrt{x+1}$, then $f^{-1}(x) =$

12. $\lim\limits_{x \to \frac{1}{2}} \dfrac{\pi x + \sin^{-1}(x-1)}{\frac{2}{3} - x}$

13. find Range $f(x) = -3 + 2^{x+1}$

14. If $\csc\theta = \frac{3}{2}$, $\frac{\pi}{2} < \theta < \pi$, then $\sin 2\theta =$