Integral Calculus

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1. Parametric equations of plane curves.
2. Polar coordinates system.
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(1) Parametric Equations of Plane Curves

In this section, rather than considering only function $y = f(x)$, it is sometimes convenient to view both x and y as functions of a third variable t (called a parameter).

Definition

A plane curve is a set of ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I.

If we are given a curve C, we can express it in a parametric form $x(t) = f(t)$ and $y(t) = g(t)$. The resulting equations are called parametric equations. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Definition

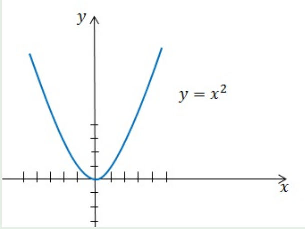
Let C be a curve consists of all ordered pairs $\big(f(t), g(t)\big)$, where f and g are continuous on an interval I. The equations

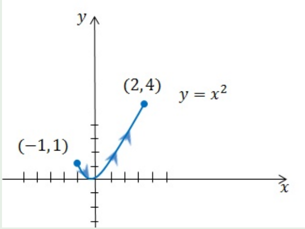
$$x = f(t), \;\; y = g(t) \;\; \text{for } t \in I$$

are parametric equations for C with parameter t.

Example

Consider the plane curve C given by $y = x^2$.





Consider the interval $-1 \leq x \leq 2$. Let $x = t$ and $y = t^2$ for $-1 \leq t \leq 2$. We have the same graph where the last equations are called parametric equations for the curve C.

Remark

1. *The parametric equations give the same graph of $y = f(x)$.*
2. *To find the parametric equations, we introduce a third variable t. Then, we rewrite x and y as functions of t.*
3. *The parametric equations give the orientation of the curve C indicated by arrows and determined by increasing values of the parameter as shown in the figure.*

Example

Write the curve given by $x(t) = 2t + 1$ and $y(t) = 4t^2 - 9$ as $y = f(x)$.

Solution:

Since $x = 2t + 1$, then $t = (x - 1)/2$. This implies

$$y = 4t^2 - 9 = 4\left(\frac{x-1}{2}\right)^2 - 9 \Rightarrow y = x^2 - 2x - 8.$$

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Example

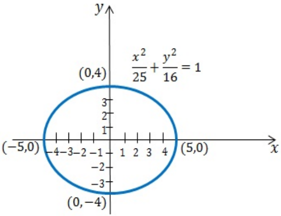
Sketch and identify the curve defined by the parametric equations

$$x = 5\cos\ t, \quad y = 2\sin\ t, \quad 0 \leq t \leq 2\pi.$$

By using the identity $\cos^2\ t + \sin^2\ t = 1$, we have

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Thus, the curve is an ellipse.



Example

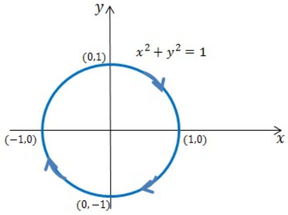
The curve C is given parametrically. Find an equation in x and y, then sketch the graph and indicate the orientation.

1. $x = \sin\ t,\ \ y = \cos\ t,\quad 0 \leq t \leq 2\pi.$
2. $x = t^2,\ \ y = 2\ln t,\quad t \geq 1.$

Solution:
1) By using the identity $\cos^2\ t + \sin^2\ t = 1$, we obtain

$$x^2 + y^2 = 1.$$

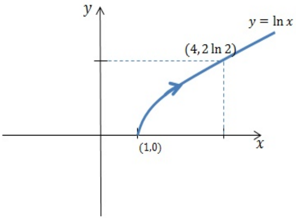
Therefore, the curve is a circle.



The orientation can be indicated as follows:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	1	0	-1	0
y	1	0	-1	0	1
(x,y)	$(0,1)$	$(1,0)$	$(0,-1)$	$(-1,0)$	$(0,1)$

2) Since $y = 2\ln t = \ln t^2$, then $y = \ln x$.



The orientation of the curve C for $t \geq 1$:

t	1	2	3
x	1	4	9
y	0	$2\ln 2$	$2\ln 3$
(x, y)	$(1, 0)$	$(4, 2\ln 2)$	$(9, 2\ln 3)$

The orientation of the curve C is determined by increasing values of the parameter t.

Tangent Lines

Suppose that f and g are differentiable functions. We want to find the tangent line to a smooth curve C given by the parametric equations $x = f(t)$ and $y = g(t)$ where y is a differentiable function of x. From the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}.$$

If $dx/dt \neq 0$, we can solve for dy/dx to have the tangent line to the curve C:

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ if } \frac{dx}{dt} \neq 0$$

Remark

- *If $dy/dt = 0$ such that $dx/dt \neq 0$, the curve has a horizontal tangent line.*
- *If $dx/dt = 0$ such that $dy/dt \neq 0$, the curve has a vertical tangent line.*

Example

Find the slope of the tangent line to the curve at the indicated value.

1. $x = t + 1,\ y = t^2 + 3t$; at $t = -1$
2. $x = t^3 - 3t,\ y = t^2 - 5t - 1$; at $t = 2$
3. $x = \sin\ t,\ y = \cos\ t$; at $t = \frac{\pi}{4}$

Solution:

1. The slope of the tangent line at $P(x, y)$ is
$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3}{1} = 2t + 3.$$
The slope of the tangent line at $t = -1$ is 1.
2. The slope of the tangent line is
$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-5}{3t^2-3}.$$
The slope of the tangent line at $t = 2$ is $\frac{-1}{9}$.
3. The slope of the tangent line is
$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin\ t}{\cos\ t} = -\tan\ t.$$
The slope of the tangent line at $t = \frac{\pi}{4}$ is -1.

Example

Find the equations of the tangent line and the vertical tangent line at $t = 2$ to the curve C given parametrically $x = 2t, \quad y = t^2 - 1$.

Solution:

The slope of the tangent line at $P(x, y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

The slope of the tangent line at $t = 2$ is $m = 2$. Thus, the slope of the vertical tangent line is $\frac{-1}{m} = \frac{-1}{2}$.

At $t = 2$, we have $(x_0, y_0) = (4, 3)$. Therefore, the tangent line is

$$y - 3 = 2(x - 4)$$

Point-Slope form: $y - y_0 = m(x - x_0)$

and the vertical tangent line is

$$y - 3 = -\frac{1}{2}(x - 4).$$

Example

Find the points on the curve C at which the tangent line is either horizontal or vertical.

1. $x = 1 - t,\ y = t^2.$
2. $x = t^3 - 4t,\ y = t^2 - 4.$

Example

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Solution:

(1) The slope of the tangent line is $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-1} = -2t.$

For the horizontal tangent line, the slope $m = 0$. This implies $-2t = 0$ and then, $t = 0$. At this value, we have $x = 1$ and $y = 0$. Thus, the graph of C has a horizontal tangent line at the point $(1, 0)$.

For the vertical tangent line, the slope $\frac{-1}{m} = 0$. This implies $\frac{1}{2t} = 0$, but this equation cannot be solved i.e., we cannot find values for t to satisfy $\frac{1}{2t} = 0$. Therefore, there are no vertical tangent lines.

(2) The slope of the tangent line is $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-4}$.

For the horizontal tangent line, the slope $m = 0$. This implies $\frac{2t}{3t^2-4} = 0$ and this is acquired if $t = 0$. At $t = 0$, we have $x = 0$ and $y = -4$. Thus, the graph of C has a horizontal tangent line at the point $(0, -4)$.

For the vertical tangent line, the slope $\frac{-1}{m} = 0$. This implies $\frac{-3t^2+4}{2t} = 0$ and this is acquired if $t = \pm\frac{2}{\sqrt{3}}$. At $t = \frac{2}{\sqrt{3}}$, we obtain $x = -\frac{16}{3\sqrt{3}}$ and $y = -\frac{8}{3}$. At $t = -\frac{2}{\sqrt{3}}$, we obtain $x = \frac{16}{3\sqrt{3}}$ and $y = -\frac{8}{3}$. Thus, the graph of C has vertical tangent lines at the points $(-\frac{16}{3\sqrt{3}}, -\frac{8}{3})$ and $(\frac{16}{3\sqrt{3}}, -\frac{8}{3})$.

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Let the curve C has the parametric equations $x = f(t)$, $y = g(t)$ where f and g are differentiable functions. To find the second derivative $\frac{d^2y}{dx^2}$, we use the formula:

$$\frac{d^2y}{dx^2} = \frac{d(y^{'})}{dx} = \frac{dy^{'}/dt}{dx/dt}$$

Note that $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$.

Example

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value.

1. $x = t$, $y = t^2 - 1$ at $t = 1$.
2. $x = \sin\ t$, $y = \cos\ t$ at $t = \frac{\pi}{3}$.

Solution:

1. $\frac{dy}{dt} = 2t$ and $\frac{dx}{dt} = 1$. Hence, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t$, then at $t = 1$, we have $\frac{dy}{dx} = 2(1) = 2$.

   The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = 2$.

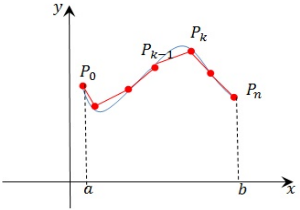
2. $\frac{dy}{dt} = -\sin\ t$ and $\frac{dx}{dt} = \cos\ t$. Thus, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\tan\ t$, then at $t = \frac{\pi}{3}$, we have $\frac{dy}{dx} = -\sqrt{3}$.

   The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\sec^2\ t}{\cos\ t} = -\sec^3\ t$. At $t = \frac{\pi}{3}$, we have $\frac{d^2y}{dx^2} = -8$

Arc Length and Surface Area of Revolution

Let C be a smooth curve has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$. Assume that the curve C does not intersect itself and f' and g' are continuous.

Let $P = \{t_0, t_1, t_2, ..., t_n\}$ is a partition of the interval $[a, b]$. Let $P_k = (x(t_k), y(t_k))$ be a point on C corresponding to t_k. If $d(P_{k-1}, P_k)$ is the length of the line segment $P_{k-1}P_k$, then the length of the line given in the figure is



$$L_p = \sum_{k=1}^{n} d(P_{k-1}, P_k)$$

In the previous chapter, we found that $L = \lim_{||P|| \to 0} L_p$. From the distance formula,

$$d(P_{k-1}, P_k) = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Therefore, the length of the arc from $t = a$ to $t = b$ is approximately

$$L \approx \lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{(\Delta x_k/\Delta t_k)^2 + (\Delta y_k/\Delta t_k)^2}\Delta t_k$$

From the mean value theorem, there exists numbers $w_k, z_k \in (t_{k-1}, t_k)$ such that

$$\frac{\Delta x_k}{\Delta t_k} = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} = f'(w_k), \quad \frac{\Delta y_k}{\Delta t_k} = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}} = g'(z_k)$$

By substitution, we obtain

$$L \approx \lim_{||P||\to 0} \sum_{k=1}^{n} \sqrt{\left[f'(w_k)\right]^2 + \left[g'(w_k)\right]^2}$$

If $w_k = z_k$ for every k, then we have Riemann sums for $\sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}$. The limit of these sums is

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\,.$$

In the following, we determine a formula to evaluate the surface area of revolution of parametric curves. Let the curve C has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$ and f' and g' are continuous. Let the curve C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$. If $g(t) \geq 0$ throughout $[a, b]$, then the area of the revolution surface generated by revolving C about the x-axis is

$$S.A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_a^b g(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Similarly, if the revolution is about the y-axis such that $f(t) \geq 0$ over $[a, b]$, the area of the revolution surface is

$$S.A = 2\pi \int_a^b f(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Theorem

Let C be a smooth curve has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$, and f' and g' are continuous. Assume that the curve C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$.

1. *The arc length of the curve is*

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

2. *If $y \geq 0$ over $[a, b]$, the surface area of revolution generated by revolving C about the x-axis is*

$$S.A = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \ ,$$

3. *If $x \geq 0$ over $[a, b]$, the surface area of revolution generated by revolving C about the y-axis is*

$$S.A = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example

Find the arc length of the curve $x = e^t \cos\ t$, $\ y = e^t \sin\ t$, $\ 0 \leq t \leq \frac{\pi}{2}$.

Solution:

First, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \cos\ t - e^t \sin\ t \Rightarrow (\frac{dx}{dt})^2 = (e^t \cos\ t - e^t \sin\ t)^2\ ,$$

$$\frac{dy}{dt} = e^t \sin\ t + e^t \cos\ t \Rightarrow (\frac{dy}{dt})^2 = (e^t \sin\ t + e^t \cos\ t)^2.$$

Thus,

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = e^{2t} \cos^2\ t - 2e^{2t} \cos\ t\ \sin\ t + e^{2t} \sin^2\ t + e^{2t} \sin^2\ t + 2e^{2t} \sin\ t\ \cos\ t$$

$$= e^{2t} + e^{2t} = 2e^{2t}.$$

Therefore, the arc length of the curve is

$$L = \sqrt{2} \int_0^{\frac{\pi}{2}} e^t\ dt = \sqrt{2}\ \Big[\ e^t\ \Big]_0^{\frac{\pi}{2}} = \sqrt{2}\ (e^{\frac{\pi}{2}} - 1).$$

Example

Find the surface area of the solid obtained by revolving the curve $x = 3\cos\ t,\ y = 3\sin\ t,\ 0 \leq t \leq \frac{\pi}{3}$ about the x-axis.

Solution: Since the revolution is about the x-axis, we apply the formula

$$S.A = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = -3\sin\ t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\sin^2\ t \quad \text{and} \quad \frac{dy}{dt} = 3\cos\ t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\cos^2\ t.$$

Thus,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9(\sin^2\ t + \cos^2\ t) = 9.$$

This implies

$$S.A = 18\pi \int_0^{\frac{\pi}{3}} \sin\ t\ dt = -18\pi \left[\cos\ t\right]_0^{\frac{\pi}{3}} = -18\pi \left[\frac{1}{2} - 1\right] = 9\pi.$$

Example

Find the surface area of the solid obtained by revolving the curve
$x = t^3,\ y = t,\ 0 \le t \le 1$ about the y-axis.

Solution: Since the revolution is about the y-axis, we apply the formula

$$S.A = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = 3t^2 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9t^4 \quad \text{and} \quad \frac{dy}{dt} = 1 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 1.$$

Thus,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 1.$$

This implies

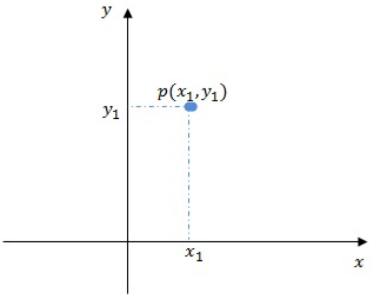
$$S.A = 2\pi \int_0^1 t^3 \sqrt{9t^4 + 1}\, dt = \frac{\pi}{18} \left[\left(9t^4 + 1\right)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{18} \left[10\sqrt{10} - 1 \right].$$

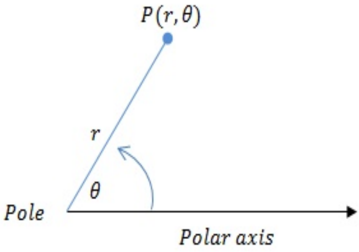
(2) Polar Coordinates System

Previously, we used Cartesian (or Rectangular) coordinates to determine points (x, y). In this section, we are going to study a new coordinate system called polar coordinate system. The figure shows the Cartesian and polar coordinates system.

Definition

The polar coordinate system is a two-dimensional system consisted of a pole and a polar axis (half line). Each point P on a plane is determined by a distance r from a fixed point O called the pole (or origin) and an angle θ from a fixed direction.





Remark

1. *From the definition, the point P in the polar coordinate system is represented by the ordered pair (r, θ) where r, θ are called polar coordinates.*
2. *The angle θ is positive if it is measured counterclockwise from the axis, but if it is measured clockwise the angle is negative.*
3. *In the polar coordinates, if $r > 0$, the point $P(r, \theta)$ will be in the same quadrant as θ; if $r < 0$, it will be in the quadrant on the opposite side of the pole with the half line. That is, the points $P(r, \theta)$ and $P(-r, \theta)$ lie in the same line through the pole O, but on opposite sides of O. The point $P(r, \theta)$ with the distance $|r|$ from O and the point $P(-r, \theta)$ with the half distance from O.*
4. *In the Cartesian coordinate system, every point has only one representation while in a polar coordinate system each point has many representations. The following formula gives all representations of a point $P(r, \theta)$ in the polar coordinate system*

$$P(r, \theta + 2n\pi) = P(r, \theta) = P(-r, \theta + (2n+1)\pi), \quad n \in \mathbb{Z}.$$

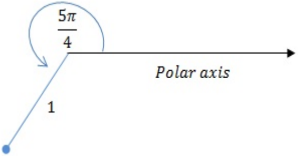
Example

Plot the points whose polar coordinates are given.

1. $(1, 5\pi/4)$
2. $(1, -3\pi/4)$
3. $(1, 13\pi/4)$
4. $(-1, \pi/4)$

Solution:

(1)



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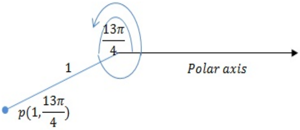

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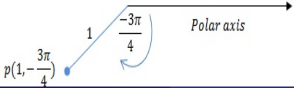
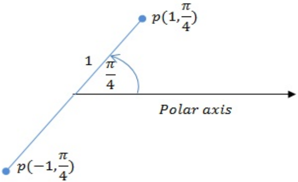


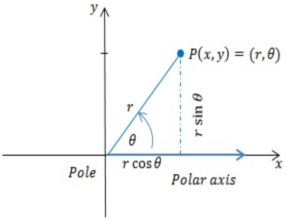
(4)



Let (x, y) be the rectangular coordinates and (r, θ) be the polar coordinates of the same point P. Let the pole be at the origin of the Cartesian coordinates system, and let the polar axis be the positive x-axis and the line $\theta = \frac{\pi}{2}$ be the positive y-axis as shown in Figure 1.
In the triangle, we have

$$\cos\ \theta = \frac{x}{r} \Rightarrow x = r\cos\ \theta\ ,$$

$$\sin\ \theta = \frac{y}{r} \Rightarrow y = r\sin\ \theta.$$



Hence,

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2,$$
$$= r^2(\cos^2\theta + \sin^2\theta).$$

This implies, $x^2 + y^2 = r^2$ and $\tan\ \theta = \frac{y}{x}$ for $x \neq 0$.

$$x = r\cos\ \theta, \quad y = r\sin\ \theta$$

$$\tan\ \theta = \frac{y}{x} \quad \text{for} \quad x \neq 0$$

$$x^2 + y^2 = r^2$$

Example

Convert from polar coordinates to rectangular coordinates.

1. $(1, \pi/4)$
2. $(2, \pi)$
3. $(2, -2\pi/3)$
4. $(4, 3\pi/4)$

Solution:

1) $r = 1$ and $\theta = \frac{\pi}{4}$.

$$x = r\cos\ \theta = (1)\cos\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}\ ,$$

$$y = r\sin\ \theta = (1)\sin\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence, $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

2) $r = 2$ and $\theta = \pi$.

$$x = r\cos\ \theta = 2\cos\ \pi = -2\ ,$$

$$y = r\sin\ \theta = 2\sin\ \pi = 0.$$

Hence, $(x, y) = (-2, 0)$.

3) $r = 2$ and $\theta = \frac{-2\pi}{3}$.

$$x = r\cos\ \theta = 2\cos\ \frac{-2\pi}{3} = -1\ ,$$

$$y = r\sin\theta = 2\sin\ \frac{-2\pi}{3} = -\sqrt{3}.$$

Hence, $(x, y) = (-1, -\sqrt{3})$.
4) $r = 4$ and $\theta = \frac{3\pi}{4}$.

$$x = r\cos\ \theta = 4\cos\ \frac{3\pi}{4} = -2\sqrt{2}\ ,$$

$$y = r\sin\ \theta = 4\sin\ \frac{3\pi}{4} = 2\sqrt{2}.$$

This implies $(x, y) = (-2\sqrt{2}, 2\sqrt{2})$.

Example

Convert from rectangular coordinates to polar coordinates for $r \geq 0$ and $0 \leq \theta \leq \pi$.

1. $(5, 0)$
2. $(2\sqrt{3}, -2)$
3. $(-2, 2)$
4. $(1, 1)$

Solution:

1. We have $x = 5$ and $y = 0$. By using $x^2 + y^2 = r^2$, we obtain $r = 5$. Also, we have $\tan\theta = \frac{y}{x} = \frac{0}{5} = 0$, then $\theta = 0$. This implies $(r, \theta) = (5, 0)$.
2. We have $x = 2\sqrt{3}$ and $y = -2$. Use $x^2 + y^2 = r^2$ to have $r = 4$. Also, since $\tan\ \theta = \frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}}$, then $\theta = \frac{5\pi}{6}$. Hence, $(r, \theta) = (4, \frac{5\pi}{6})$.
3. We have $x = -2$ and $y = 2$. Then, $r^2 = x^2 + y^2 = (-2)^2 + 2^2$ and this implies $r = 2\sqrt{2}$. Also, $\tan\ \theta = \frac{y}{x} = \frac{2}{-2} = -1$, then $\theta = \frac{3\pi}{4}$. This implies $(r, \theta) = (2\sqrt{2}, \frac{3\pi}{4})$.
4. We have $x = 1$ and $y = 1$. By using $x^2 + y^2 = r^2$, we have $r = \sqrt{2}$. Also, by using $\tan\ \theta = \frac{y}{x} = 1$, we obtain $\theta = \frac{\pi}{4}$. This implies, $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$.

A polar equation is an equation in r and θ, $r = f(\theta)$. A solution of the polar equation is an ordered pair (r_0, θ_0) satisfies the equation i.e., $r_0 = f(\theta_0)$. For example, $r = 2\cos\ \theta$ is a polar equation and $(1, \frac{\pi}{3})$, and $(\sqrt{2}, \frac{\pi}{4})$ are solutions of that equation.

Example

Find a polar equation that has the same graph as the equation in x and y.

1. $x = 7$
2. $y = -3$
3. $x^2 + y^2 = 4$
4. $y^2 = 9x$

Solution:

1) $x = 7 \Rightarrow r\cos\ \theta = 7 \Rightarrow r = 7\sec\theta$.
2) $y = -3 \Rightarrow r\sin\ \theta = -3 \Rightarrow r = -3\csc\theta$.
3) $x^2 + y^2 = 4 \Rightarrow r^2\cos^2\ \theta + r^2\sin^2\ \theta = 4$

$$\Rightarrow r^2(\cos^2\ \theta + \sin^2\ \theta) = 4$$

$$\Rightarrow r^2 = 4\ .$$

4) $y^2 = 9x \Rightarrow r^2\sin^2\ \theta = 9r\cos\ \theta$

$$\Rightarrow r\sin^2\ \theta = 9\cos\ \theta$$

$$\Rightarrow r = 9\cot\ \theta\csc\ \theta.$$

Example

Find an equation in x and y that has the same graph as the polar equation.

1. $r = 3$
2. $r = \sin\ \theta$
3. $r = 6\cos\ \theta$
4. $r = \sec\ \theta$

Solution:

1. $r = 3 \Rightarrow \sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9.$
2. $r = \sin\theta \Rightarrow r = \frac{y}{r} \Rightarrow r^2 = y \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y = 0.$
3. $r = 6\cos\ \theta \Rightarrow r = 6\ \frac{x}{r} \Rightarrow r^2 = 6x \Rightarrow x^2 + y^2 - 6x = 0.$
4. $r = \sec\ \theta \Rightarrow r = \frac{1}{\cos\ \theta} \Rightarrow r\cos\ \theta = 1 \Rightarrow x = 1.$

Tangent Line to Polar Curves

Theorem

Let $r = f(\theta)$ be a polar curve where f' is continuous. The slope of the tangent line to the graph of $r = f(\theta)$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\ \theta + \sin\ \theta(dr/d\theta)}{-r\sin\ \theta + \cos\ \theta(dr/d\theta)}.$$

Remark

1. *If* $\frac{dy}{d\theta} = 0$ *such that* $\frac{dx}{d\theta} \neq 0$, *the curve has a horizontal tangent line.*
2. *If* $\frac{dx}{d\theta} = 0$ *such that* $\frac{dy}{d\theta} \neq 0$, *the curve has a vertical tangent line.*
3. *If* $\frac{dx}{d\theta} \neq 0$ *at* $\theta = \theta_0$, *the slope of the tangent line to the graph of* $r = f(\theta)$ *is*

$$\frac{r_0 \cos\ \theta_0 + \sin\ \theta_0 (dr/d\theta)_{\theta=\theta_0}}{-r_0 \sin\ \theta_0 + \cos\ \theta_0 (dr/d\theta)_{\theta=\theta_0}}, \quad \textit{where} \quad r_0 = f(\theta_0)$$

Example

Find the slope of the tangent line to the graph of $r = \sin\ \theta$ at $\theta = \frac{\pi}{4}$.

Solution:

$$x = r\cos\ \theta \Rightarrow x = \sin\ \theta\cos\ \theta \Rightarrow \frac{dx}{d\theta} = \cos^2\ \theta - \sin^2\ \theta\ ,$$

$$y = r\sin\ \theta \Rightarrow y = \sin^2\ \theta \Rightarrow \frac{dy}{d\theta} = 2\sin\ \theta\cos\ \theta.$$

Hence,

$$\frac{dy}{dx} = \frac{2\sin\ \theta\cos\ \theta}{\cos^2\ \theta - \sin^2\ \theta}.$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 0$. Thus, the slope is undefined. In this case, the curve has a vertical tangent line.

Example

Find the points on the curve $r = 2 + 2\cos\ \theta$ for $0 \le \theta \le 2\pi$ at which tangent lines are either horizontal or vertical.

Solution:

$$x = r\cos\ \theta = 2\cos\ \theta + 2\cos^2\ \theta \Rightarrow \frac{dx}{d\theta} = -2\sin\ \theta - 4\cos\ \theta\sin\ \theta\ ,$$

$$y = r\sin\ \theta = 2\sin\ \theta + 2\cos\ \theta\sin\ \theta \Rightarrow \frac{dy}{d\theta} = 2\cos\ \theta - 2\sin^2\ \theta + 2\cos^2\ \theta.$$

For a horizontal tangent line,

$$\frac{dy}{d\theta} = 0 \Rightarrow 2\cos\ \theta - 2\sin^2\ \theta + 2\cos^2\ \theta = 0 \Rightarrow 2\cos^2\ \theta + \cos\ \theta - 1 = 0 \Rightarrow (2\cos\ \theta - 1)(\cos\ \theta +$$

This implies $\theta = \pi$, $\theta = \pi/3$, or $\theta = 5\pi/3$. Therefore, the tangent line is horizontal at $(0, \pi)$, $(3, \pi/3)$ or $(3, 5\pi/3)$.

For a vertical tangent line,

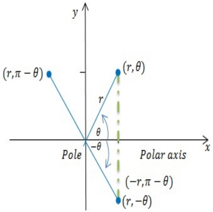
$$\frac{dx}{d\theta} = 0 \Rightarrow \sin\ \theta(2\cos\ \theta + 1) = 0.$$

This implies $\theta = 0$, $\theta = \pi$, $\theta = 2\pi/3$, or $\theta = 4\pi/3$. However, we have to ignore $\theta = \pi$ since at this value $dy/d\theta = 0$. Therefore, the tangent line is vertical at $(4, 0)$, $(1, 2\pi/3)$, or $(1, 4\pi/3)$.

Theorem

1. **Symmetry about the polar axis.**
   The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if replacing (r, θ) with $(r, -\theta)$ or with $(-r, \pi - \theta)$ does not change the equation.
2. **Symmetry about the vertical line $\theta = \frac{\pi}{2}$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if replacing (r, θ) with $(r, \pi - \theta)$ or with $(-r, -\theta)$ does not change the equation.
3. **Symmetry about the pole $\theta = 0$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the pole if replacing (r, θ) with $(-r, \theta)$ or with $(r, \theta + \pi)$ does not change the equation.

A



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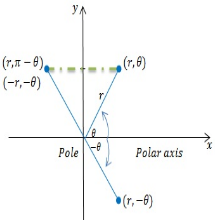

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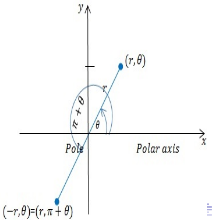


Example

1. The graph of $r = 4\cos\ \theta$ is symmetric about the polar axis since

$$4\cos\ (-\theta) = 4\cos\ \theta \quad \text{and} \quad -4\cos(\pi - \theta) = 4\cos\ \theta.$$

2. The graph of $r = 2\sin\ \theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since

$$2\sin\ (\pi - \theta) = 2\sin\ \theta \quad \text{and} \quad -2\sin\ (-\theta) = 2\sin\ \theta.$$

3. The graph of $r^2 = a^2 \sin\ 2\theta$ is symmetric about the pole since

$$(-r)^2 = a^2 \sin\ 2\theta,$$
$$\Rightarrow r^2 = a^2 \sin\ 2\theta.$$

and

$$r^2 = a^2 \sin\ \big(2(\pi + \theta)\big),$$
$$= a^2 \sin\ (2\pi + 2\theta),$$
$$r^2 = a^2 \sin\ 2\theta.$$

Some Special Polar Graphs
Lines in polar coordinates

1. The polar equation of a straight line $ax + by = c$ is $r = \frac{c}{a\cos\ \theta + b\sin\ \theta}$.
   Since $x = r\cos\ \theta$ and $y = r\sin\ \theta$, then

$$ax + by = c \Rightarrow r(a\cos\ \theta + b\sin\ \theta) = c \Rightarrow r = \frac{c}{(a\cos\ \theta + b\sin\ \theta)}$$

2. The polar equation of a vertical line $x = k$ is $r = k\sec\ \theta$.
   Let $x = k$, then $r\cos\ \theta = k$. This implies $r = \frac{k}{\cos\ \theta} = k\sec\ \theta$.
3. The polar equation of a horizontal line $y = k$ is $r = k\csc\ \theta$.
   Let $y = k$, then $r\sin\ \theta = k$. This implies $r = \frac{k}{\sin\ \theta} = r\csc\ \theta$.
4. The polar equation of a line that passes the origin point and makes an angle θ_0 with the positive x-axis is $\theta = \theta_0$.

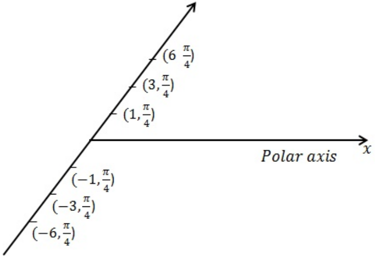
Example

Sketch the graph of $\theta = \frac{\pi}{4}$.

Solution:

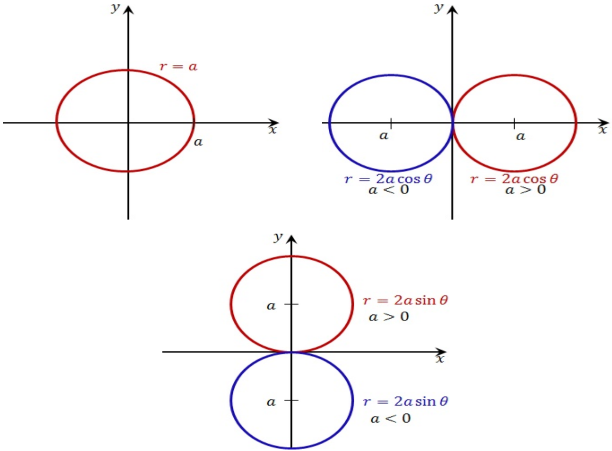
We are looking for a graph of the set of polar points

$$\{(r,\theta)\ |, r \in \mathbb{R}\}.$$



Circles in polar coordinates

1. The circle equation with center at the pole O and radius $|a|$ is $r = a$.
2. The circle equation with center at $(a, 0)$ and radius $|a|$ is $r = 2a\cos\ \theta$.
3. The circle equation with center at $(0, a)$ and radius $|a|$ is $r = 2a\sin\ \theta$.



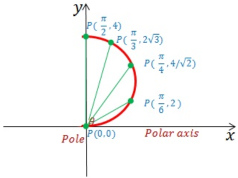
Example

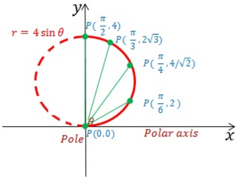
Sketch the graph of $r = 4\sin\ \theta$.

Solution:

Note that the graph of $r = 4\sin\ \theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $4\sin\ (\pi - \theta) = 4\sin\ \theta$. Therefore, we restrict our attention to the interval $[0, \pi/2]$ and by the symmetry, we complete the graph. The following table displays polar coordinates of some points on the curve:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	2	$4/\sqrt{2}$	$2\sqrt{3}$	4

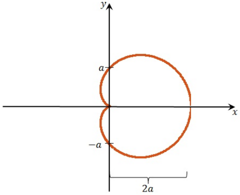




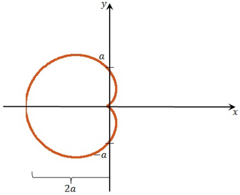
Cardioid curves

1. $r = a(1 \pm \cos\ \theta)$ **2.** $r = a(1 \pm \sin\ \theta)$

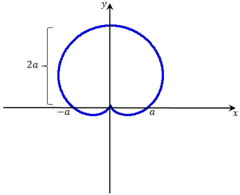
$r = a(1 + \cos\ \theta)$



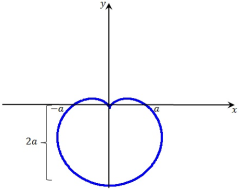
$r = a(1 - \cos\ \theta)$



$r = a(1 + \sin\ \theta)$



$r = a(1 - \sin\ \theta)$



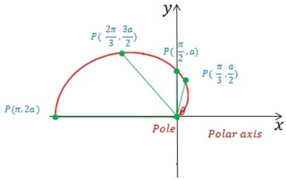
Example

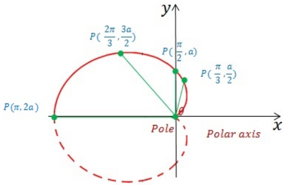
Sketch the graph of $r = a(1 - \cos\ \theta)$ where $a > 0$.

Solution:

The curve is symmetric about the polar axis since $\cos\ (-\theta) = \cos\ \theta$. Therefore, we restrict our attention to the interval $[0, \pi]$ and by the symmetry, we complete the graph. The following table displays some solutions of the equation $r = a(1 - \cos\ \theta)$:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a/2$	a	$3a/2$	$2a$



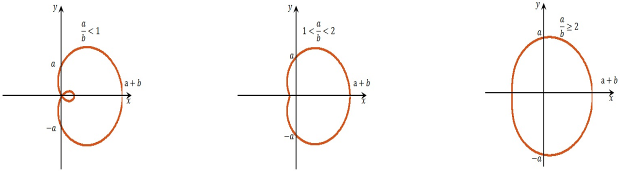


Limaçons curves

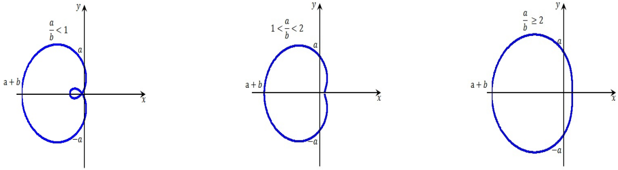
1. $r = a \pm b\cos\ \theta$ **2.** $r = a \pm b\sin\ \theta$

1. $r = a \pm b\cos\ \theta$

1. $r = a + b\cos\ \theta$

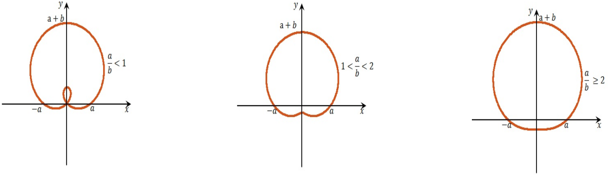


2. $r = a - b\cos\ \theta$

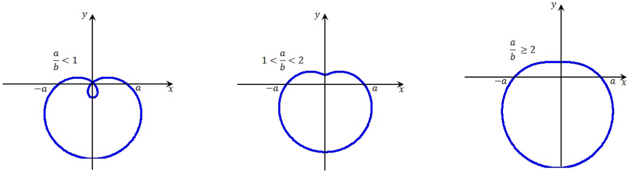


2. $r = a \pm b\sin\ \theta$

1. $r = a + b\sin\ \theta$



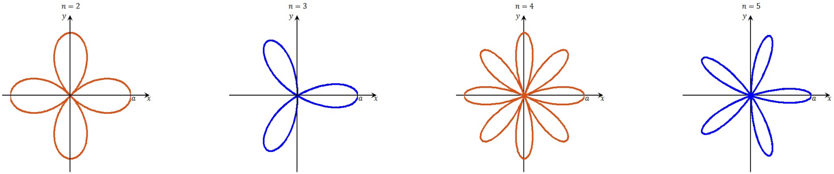
2. $r = a - b\sin\ \theta$



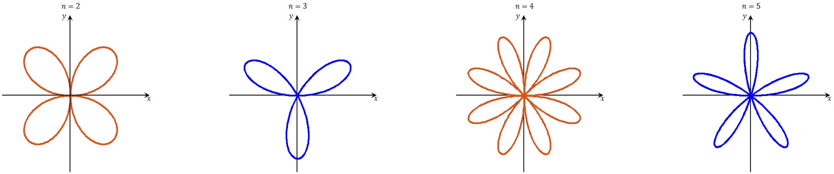
Roses

1. $r = a\ \cos\,(n\theta)$ **2.** $r = a\ \sin\,(n\theta)$ where $n \in \mathbb{N}$.

1. $r = a\ \cos\,(n\theta)$



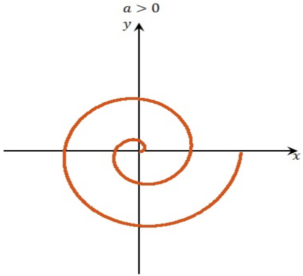
2. $r = a\ \sin\,(n\theta)$

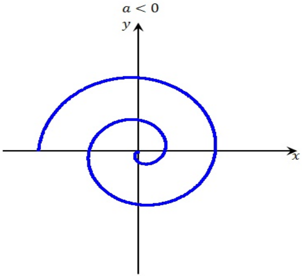


Note that if n is odd, there are n petals; however, if n is even, there are $2n$ petals.

Spiral of Archimedes

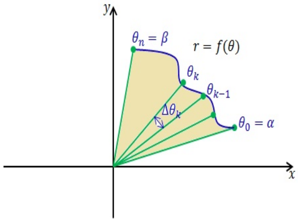
$r = a\,\theta$





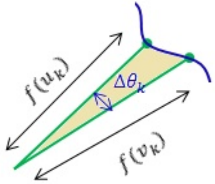
Area in Polar Coordinates

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \leq \alpha \leq \beta \leq 2\pi$. Let $f(\theta) \geq 0$ over that interval and R be a polar region bounded by the polar equations $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ as shown in Figure 44.



To find the area of R, we assume $P = \{\theta_1, \theta_2, ..., \theta_n\}$ is a regular partition of the interval $[\alpha, \beta]$. Consider the interval $[\theta_{k-1}, \theta_k]$ where $\Delta\theta_k = \theta_k - \theta_{k-1}$. By choosing $\omega_k \in [\theta_{k-1}, \theta_k]$, we have a circular sector where its angle and radius are $\Delta\theta_k$ and $f(\omega_k)$, respectively. The area between θ_{k-1} and θ_k can be approximated by the area of a circular sector.

Let $f(u_k)$ and $f(v_k)$ be maximum and minimum values of f on $[\theta_{k-1}, \theta_k]$. From the figure, we have



$$\underbrace{\frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k}_{\text{Area of the sector of radius} f(u_k)} \leq \Delta A_k \leq \underbrace{\frac{1}{2}\left[f(v_k)\right]^2 \Delta\theta_k}_{\text{Area of the sector of radius} f(v_k)}$$

By summing from $k = 1$ to $k = n$, we obtain

$$\sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(u_k) \leq \underbrace{\sum_{k=1}^{n} \Delta A_k}_{=A} \leq \sum_{k=1}^{n} \frac{1}{2}\left[f(v_k)\right]^2 \Delta\theta_k f(v_k)$$

The limit of the sums as the norm $||P||$ approaches zero,

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(u_k) = \lim_{||P|| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(v_k) = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta \, .$$

Therefore,

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) \geq g(\theta)$. The area of the polar region bounded by the graphs of f and g on the interval $[\alpha, \beta]$ is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2 \right] \, d\theta$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) \geq g(\theta)$. The area of the polar region bounded by the graphs of f and g on the interval $[\alpha, \beta]$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right]\, d\theta$$

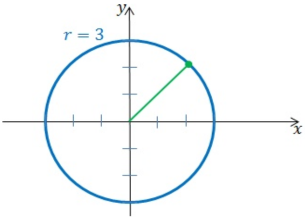
Example

Find the area of the region bounded by the graph of the polar equation.

1. $r = 3$
2. $r = 2\cos\ \theta$
3. $r = 4\sin\ \theta$
4. $r = 6 - 6\sin\ \theta$

Solution:
(1) The area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2\, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\Big[\,\theta\,\Big]_0^{2\pi} = 9\pi.$$

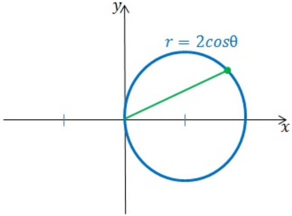


Note that one can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.,

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2\, d\theta\Big) = 2\int_0^{\frac{\pi}{2}} 9\, d\theta = 18\Big[\,\theta\,\Big]_0^{\frac{\pi}{2}} = 9\pi.$$

(2) We find the area of the upper half circle and multiply the result by 2 as follows:

$$\begin{aligned} A = 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos\ \theta)^2\ d\theta\Big) &= \int_0^{\frac{\pi}{2}} 4\cos^2\ \theta\ d\theta \\ &= 2\int_0^{\frac{\pi}{2}} (1+\cos\ 2\theta)\ d\theta \\ &= 2\Big[\theta + \frac{\sin\ 2\theta}{2}\Big]_0^{\frac{\pi}{2}} \\ &= 2\Big[\frac{\pi}{2} - 0\Big] \\ &= \pi. \end{aligned}$$



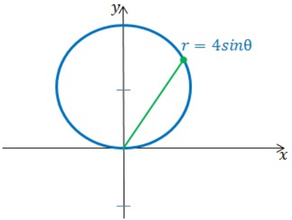
(3) The area of the region is

$$A = \frac{1}{2}\int_0^{\pi} (4\sin\ \theta)^2\ d\theta = \frac{16}{4}\int_0^{\pi} (1 - \cos\ 2\theta)\ d\theta$$

$$= 4\left[\theta - \frac{\sin\ 2\theta}{2}\right]_0^{\pi}$$

$$= 4\left[\pi - 0\right]$$

$$= 4\pi.$$



(4) The area of the region is

$$A = \frac{1}{2}\int_0^{2\pi} 36(1 - \sin\ \theta)^2\ d\theta$$

$$= 18\int_0^{2\pi} (1 - 2\sin\ \theta + \sin^2\ \theta)\ d\theta$$

$$= 18\left[\theta + 2\cos\ \theta + \frac{\theta}{2} - \frac{\sin\ 2\theta}{4}\right]_0^{2\pi}$$

$$= 18\left[(2\pi + 2 + \pi) - 2\right]$$

$$= 54\pi.$$