Integral Calculus

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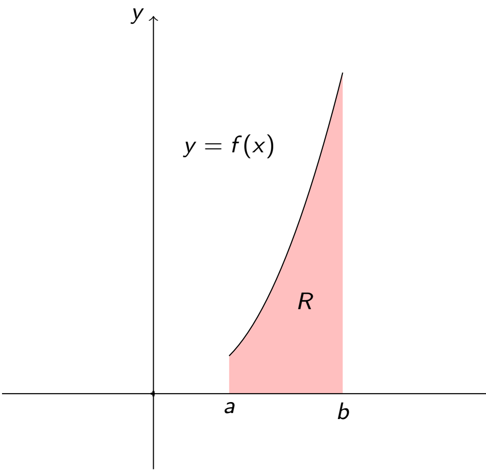
Main Contents

1. Areas.
2. Volumes of revolution solids:
    - Disk method.
    - Washer method.
    - Method of cylindrical shells.
3. Lengths of arcs.
4. Surfaces of revolution.

(1) Areas

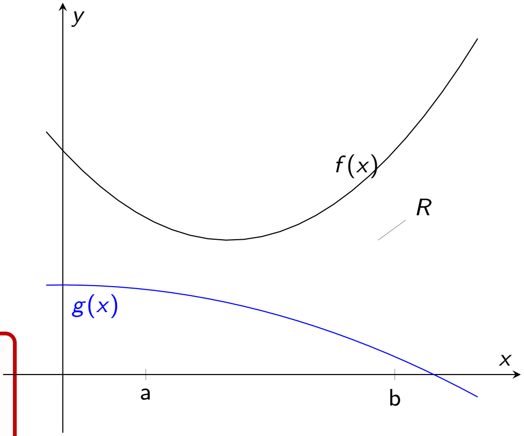
■ If $y = f(x)$ is a continuous function on $[a, b]$ and $f(x) \geq 0$ for every $x \in [a, b]$, the area of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$A = \int_a^b f(x)\ dx$$



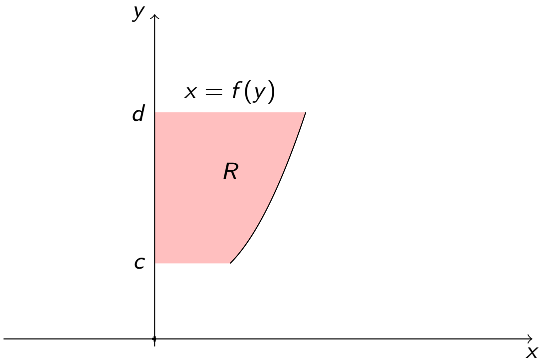
■ If f and g are continuous functions and $f(x) \geq g(x)$ $\forall x \in [a, b]$, then the area A of the region bounded by the graphs of f (the upper boundary of R) and g (the lower boundary of R) from $x = a$ to $x = b$ is subtracting the area of the region under $g(x)$ from the area of the region under $f(x)$. This can be stated as follows:

$$A = \int_a^b \left(f(x) - g(x)\right)\, dx$$

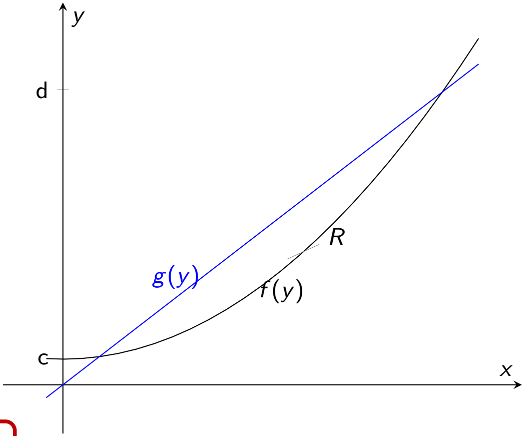


■ If $x = f(y)$ is a continuous function on $[c, d]$ and $f(y) \geq 0 \; \forall y \in [c, d]$, the area of the region bounded by the graph of $f(y)$ from $y = c$ to $y = d$ is given by the integral:

$$A = \int_c^d f(y)\, dy$$



■ If f and g are continuous functions and $f(y) \geq g(y)$ $\forall y \in [c, d]$, then the area A of the region bounded by the graphs of f (the right boundary of R) and g (the left boundary of R) from $y = c$ to $y = d$ is subtracting the area of the region bounded by $g(x)$ from the area of the region bounded by $f(x)$. This can be stated as follows:

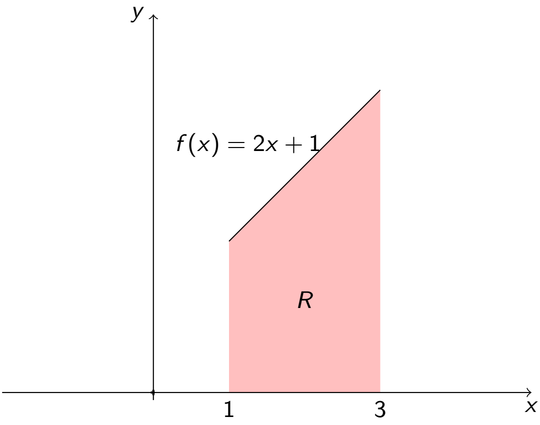


$$A = \int_{c}^{d} \left(f(y) - g(y) \right) \, dy$$

Example

Express the area of the shaded region as a definite integral then find the area.

(1)

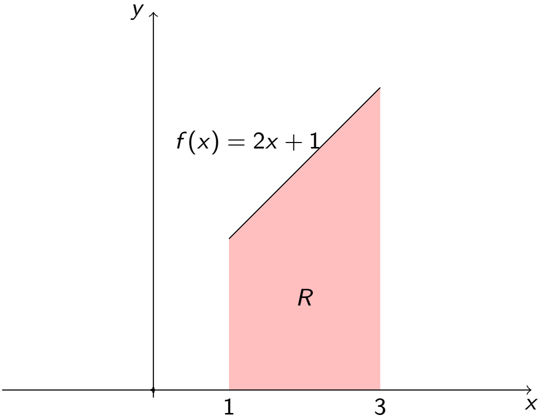


Solution:

Example

Express the area of the shaded region as a definite integral then find the area.

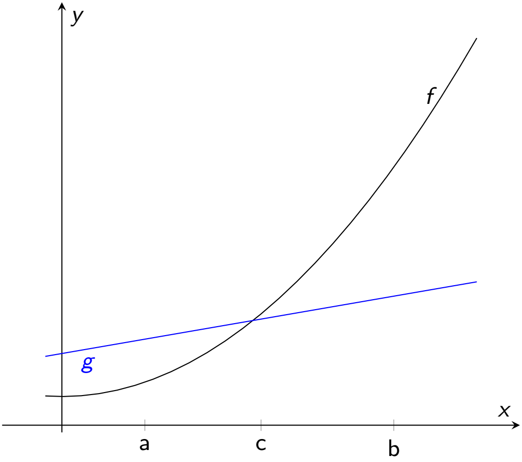
(1)



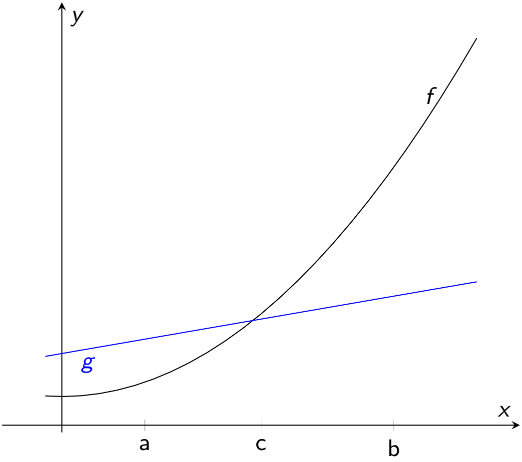
Solution:

Area : $A = \int_1^3 (2x+1)\, dx = \Big[x^2 + x\Big]_1^3 = \Big[(3^2+3) - (1^2+1)\Big] = 12 - 2 = 10.$

(2)



(2)



We have two regions:

Region (1) : in the interval $[a, c]$.

Upper graph: $y = g(x)$

Lower graph: $y = f(x)$

Area: $A_1 = \displaystyle\int_a^c \big(g(x) - f(x)\big)\, dx.$

Region (2) : in the interval $[c, b]$.

Upper graph: $y = f(x)$

Lower graph: $y = g(x)$

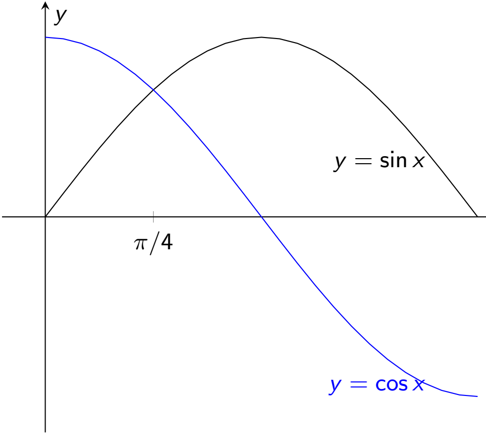
Area: $A_2 = \displaystyle\int_c^b \big(f(x) - g(x)\big)\, dx.$

The total area is $A = A_1 + A_2$.

Example

Sketch the region determined by the graphs of $y = \sin\ x$, $y = \cos\ x$, $x = 0$ and $x = \frac{\pi}{4}$, then find its area.

Solution: The figure on the right shows the region bounded by the two functions. Note that over the period $[0, \frac{\pi}{4}]$, the two curves intersect at $\frac{\pi}{4}$.



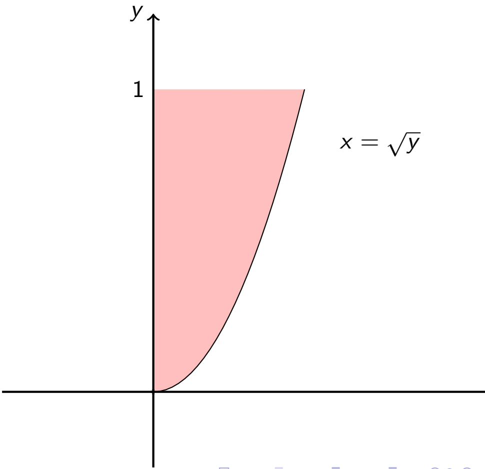
Hence,

$$\text{Area: } A = \int_0^{\frac{\pi}{4}} (\cos\ x - \sin\ x)\ dx$$
$$= \Big[\sin\ x + \cos\ x\Big]_0^{\frac{\pi}{4}}$$
$$= \Big[(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (1)\Big]$$
$$= \sqrt{2} - 1.$$

Example

Sketch the region bounded by the graph of $x = \sqrt{y}$ from $y = 0$ to $y = 1$, then find its area.

Solution: The region bounded by the function $x = \sqrt{y}$ from $y = 0$ to $y = 1$ is shown in the figure.



The area of the region is

$$A = \int_0^1 \sqrt{y}\, dy = \frac{2}{3}\left[\, y^{3/2}\, \right]_0^1 = \frac{2}{3}.$$

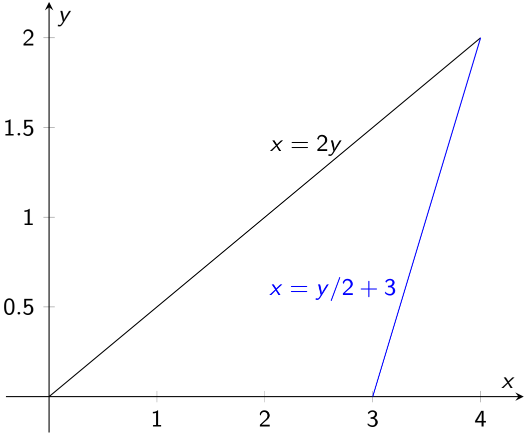
Example

Sketch the region bounded by the graphs of $x = 2y$ and $x = \frac{y}{2} + 3$, then find its area.

Solution:
First, we find the intersection points:

$$2y = \frac{y}{2}+3 \Rightarrow 4y = y+6 \Rightarrow y = 2.$$

The two curves intersect at $(4, 2)$.



$$\text{Area: } A = \int_0^2 \left(\frac{y}{2} + 3 - 2y\right)\, dy$$
$$= \int_0^2 \left(-\frac{3}{2}y + 3\right)\, dy$$
$$= \left[-\frac{3}{4}y^2 + 3y\right]_0^2$$
$$= -3 + 6 = 3.$$

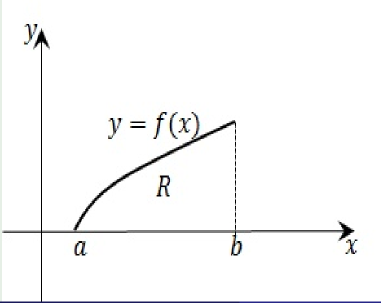
(2) Solids of Revolution and Volumes
(A) Solids of Revolution

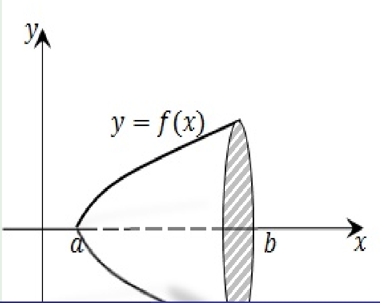
Definition

If R is a plane region, the solid of revolution S is a solid generated from revolving R about a line in the same plane where the line is called the axis of revolution.

Example

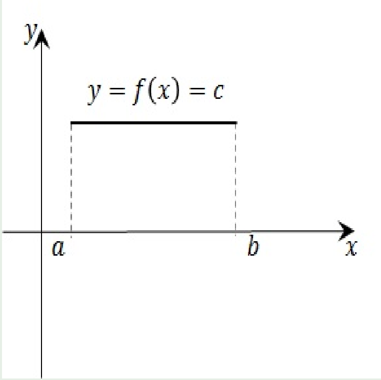
Let $y = f(x) \geq 0$ be continuous for every $x \in [a, b]$. Let R be a region bounded by the graph of f and the x-axis from $x = a$ to $x = b$. Revolution of the region R about the x-axis generates a solid given in the figure (right).

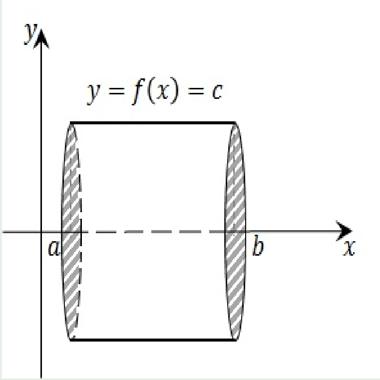




Example

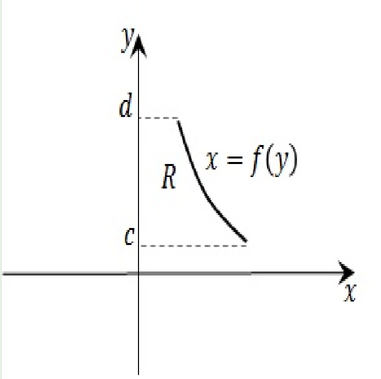
Let $y = f(x)$ be a constant function from $x = a$ to $x = b$, as in the figure. The region R is a rectangle and by revolving it about the x-axis, we obtain a circular cylinder.

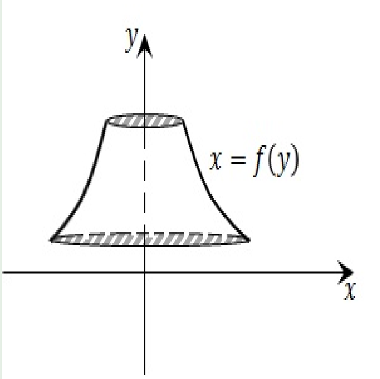




Example

Consider the region R bounded by the graph of $x = f(y)$ from $y = c$ to $y = d$. Revolution of R about the y-axis generates a solid given in the figure.





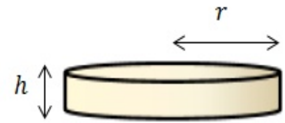
In this section, we study three methods to evaluate the volumes of the revolution solids known as disk method, washer method and method of cylindrical shells.

(i) Disk Method

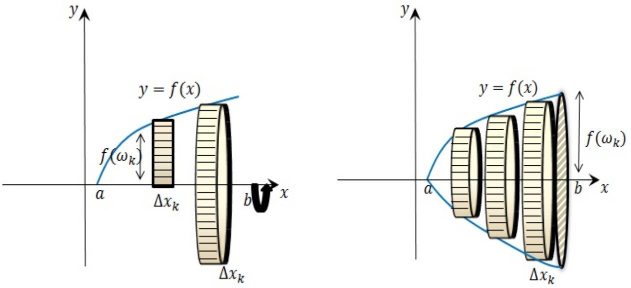
Let f be continuous on $[a, b]$ and let R be the region bounded by the graph of f and the x-axis form $x = a$ to $x = b$. Let S be the solid generated by revolving R about the x-axis. Assume that P is a partition of $[a, b]$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark where $\omega_k \in [x_{k-1}, x_k]$. From each subinterval $[x_{k-1}, x_k]$, we form a rectangle, its high and width are $f(\omega_k)$ and Δx_k, respectively.

The revolution of the vertical rectangle about the x-axis generates a circular disk as shown in Figure 16. Its radius and high are

$$r = f(\omega_k) \ , \quad h = \Delta x_k.$$



$$V = \pi r^2 h$$



From Figure 16, the volume of each circular disk is

$$V_k = \pi(f(\omega_k))^2 \Delta x_k, \quad k = 1, 2, ..., n$$

The sum of volumes of the circular disks approximates the volume of the solid of revolution:

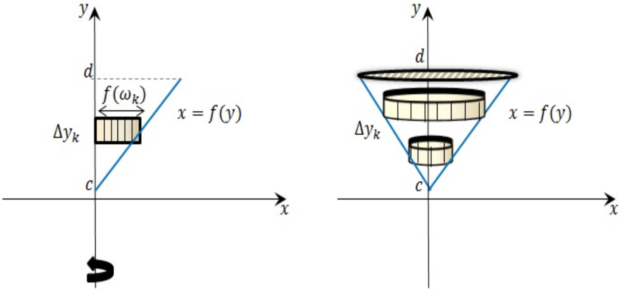
$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi \ (f(\omega_k))^2 \ \Delta x_k = \pi \ \int_a^b \left[\ f(x) \ \right]^2 dx.$$

Similarly, we can find the volume of the solid of revolution generated by revolving the region about the y-axis. Let f be continuous on $[c, d]$ and let R be the region bounded by the graph of f and the y-axis from $y = c$ to $y = d$. Let S be the solid generated by revolving R about the y-axis. Assume that P is a partition of $[c, d]$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark where $\omega_k \in [y_{k-1}, y_k]$. From each $[y_{k-1}, y_k]$, we form a rectangle, its high and width are $f(\omega_k)$ and Δy_k, respectively.
The revolution of each horizontal rectangle about the y-axis generates a circular disk as shown in Figure 17. Its radius and high are

$$r = f(\omega_k)\ , \quad h = \Delta y_k.$$

Therefore, the volume of each circular disk is

$$V_k = \pi(f(\omega_k))^2 \Delta y_k, \quad k = 1, 2, ..., n$$



The volume of the solid of revolution given in Figure 17 (right) is approximately the sum of the volumes of circular disks:

$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \pi(f(\omega_k))^2 \Delta y_k$$
$$= \pi \int_c^d \left[f(y)\right]^2 \, dy.$$

Theorem

1. *If R is a region bounded by the graph of f on the interval* $[a, b]$*, the volume of the solid of revolution determined by revolving R about the x-axis is*

$$V = \pi \int_a^b \left[\, f(x) \,\right]^2 dx.$$

2. *If R is a region bounded by the graph of f on the interval* $[c, d]$*, the volume of the solid of revolution determined by revolving R about the y-axis is*

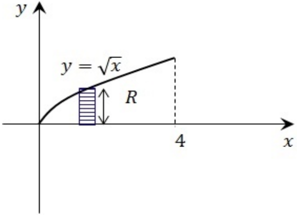
$$V = \pi \int_c^d \left[\, f(y) \,\right]^2 dy.$$

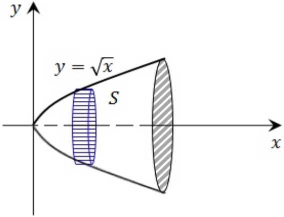
Example

Sketch the region R bounded by the graphs of equations $y = \sqrt{x}$, $x = 4$ and $y = 0$. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:

The figure shows the solid generated by revolving the region R about the x-axis.





Since the revolution is about the x-axis, we have a vertical disk with radius $y = \sqrt{x}$ and thickness dx.

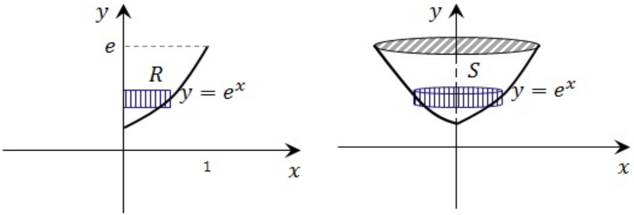
Thus, the volume of the solid S is

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{\pi}{2} \left[x^2 \right]_0^4 = \frac{\pi}{2} \left[16 - 0 \right] = 8\pi.$$

Example

Sketch the region R bounded by the graphs of equations $y = e^x$, $y = e$ and $x = 0$. Then, find the volume of the solid generated by revolving R about the y-axis.

Solution:



The figure shows the region R and the solid S generated by revolving the region about the y-axis. Since the revolution is about the y-axis, then we need to rewrite the function to become $x = f(y)$.

$$y = e^x \Rightarrow \ln y = \ln e^x \Rightarrow x = \ln y = f(y).$$

Now, we have a horizontal disk with radius $x = \ln y$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_1^e (\ln y)^2 \, dy = \pi \left[2y + y\,(\ln y)^2 - 2y\,\ln y \right]_1^e = \pi(e - 2).$$

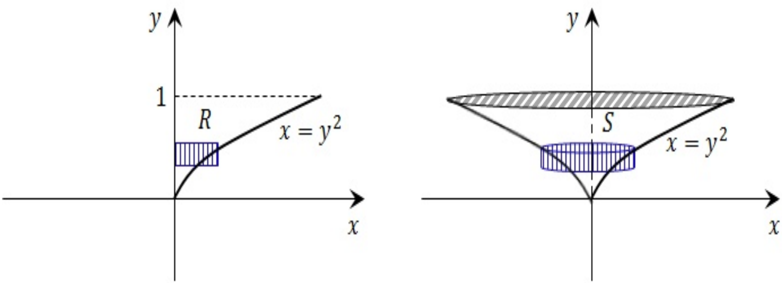
Example

Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0, 1]$. Then, find the volume of the solid generated by revolving R about the y-axis.

Example

Sketch the region R bounded by the graph of the equation $x = y^2$ on the interval $[0, 1]$. Then, find the volume of the solid generated by revolving R about the y-axis.

Solution:



Since the revolution of R is about the y-axis, we have a horizontal disk with radius $x = y^2$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_0^1 (y^2)^2 \, dy = \frac{\pi}{5}\Big[\, y^5 \,\Big]_0^1 = \frac{\pi}{5}\Big[1 - 0\Big] = \frac{\pi}{5}.$$

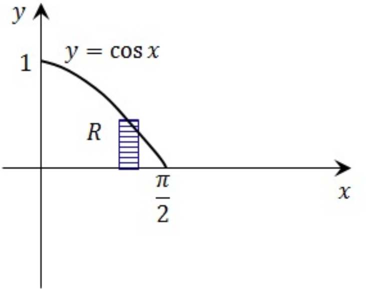
Example

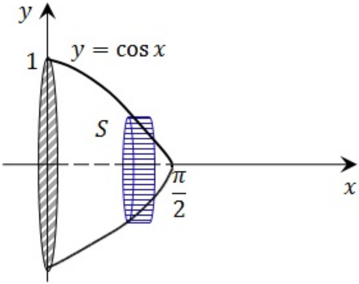
Sketch the region R bounded by the graph of the equation $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{2}$. Then, find the volume of the solid generated by revolving R about the x-axis.

Example

Sketch the region R bounded by the graph of the equation $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{2}$. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:





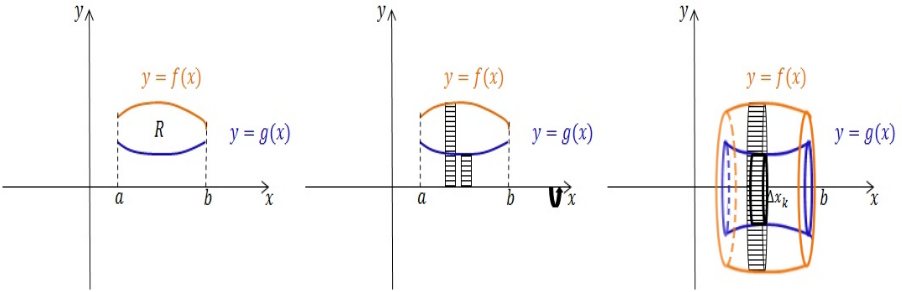
The figure shows the region R and the solid S generated by revolving the region about the x-axis. Thus, the disk to evaluate the volume of the generated solid S is vertical where the radius is $y = \cos\ x$ and the thickness is dx. Hence,

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2\ x\ dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos\ 2x)\ dx = \frac{\pi}{2} \left[x + \frac{\sin\ 2x}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{4}.$$

(ii) Washer Method

The washer method is a generalization of the disk method for a region between two functions f and g. Let R be a region bounded by the graphs of f and g from $x = a$ to $x = b$ such that $f \geq g$ on $[a, b]$ as shown in Figure 23). The volume of the solid S generated by revolving the region R about the x-axis can be found by calculating the difference between the volumes of the two solids generated by revolving the regions under f and g about the x-axis as follows:

the outer radius: $y_1 = f(x)$
the inner radius: $y_2 = g(x)$
the thickness: dx



The volume of a washer is $dV = \pi\left[(\text{the outer radius})^2 - (\text{the inner radius})^2\right]\cdot$ thickness.

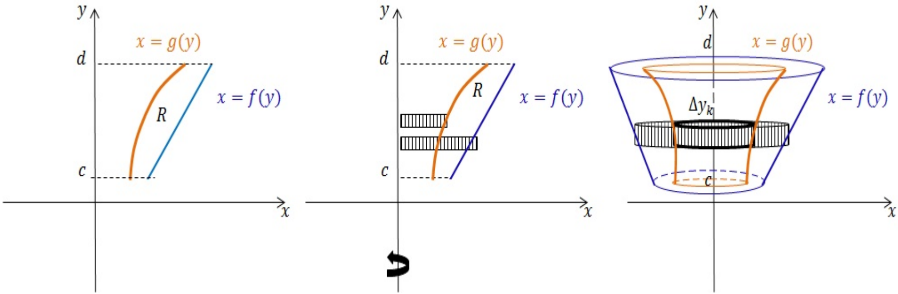
This implies $dV = \pi\left[(f(x))^2 - (g(x))^2\right]dx$.

Hence, the volume of the solid over the period $[a, b]$ is

$$V = \pi \int_a^b \left[\left(f(x)\right)^2 - \left(g(x)\right)^2\right]\, dx.$$

Similarly, let R be a region bounded by the graphs of f and g such that $f \geq g$ on $[c, d]$ as shown in Figure 24. The volume of the solid S generated by revolving R about the y-axis is

$$V = \pi \int_c^d \left[\left(f(y)\right)^2 - \left(g(y)\right)^2\right]\, dy.$$



Theorem

1. *If R is a region bounded by the graphs of f and g on the interval $[a, b]$ such that $f \geq g$, the volume of the solid of revolution determined by revolving R about the x-axis is*
$$V = \pi \int_a^b \left[\left(f(x)\right)^2 - \left(g(x)\right)^2 \right] dx.$$
2. *If R is a region bounded by the graphs of f and g on the interval $[c, d]$ such that $f \geq g$, the volume of the solid of revolution determined by revolving R about the y-axis is*
$$V = \pi \int_c^d \left[\left(f(y)\right)^2 - \left(g(y)\right)^2 \right] dy.$$

Example

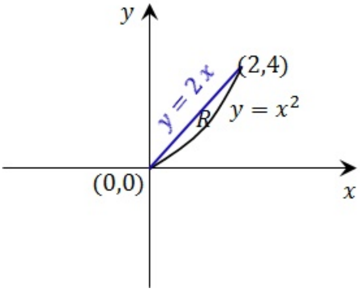
Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about the x-axis.

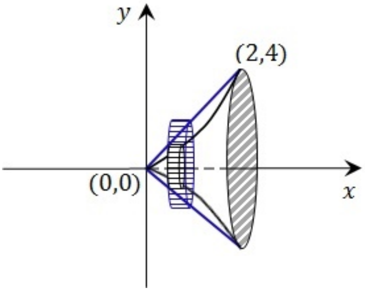
Solution:

Let $f(x) = x^2$ and $g(x) = 2x$. First, we find the intersection points:

$$\begin{aligned} f(x) = g(x) &\Rightarrow x^2 = 2x \\ &\Rightarrow x^2 - 2x = 0 \\ &\Rightarrow x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Substituting $x = 0$ into $f(x)$ or $g(x)$ gives $y = 0$. Similarly, if we substitute $x = 2$ into the two functions, we have $y = 2$. Thus, the two curves intersect in two points $(0, 0)$ and $(2, 4)$.





The figure shows the region R and the solid generated by revolving R about the x-axis.
A vertical rectangle generates a washer where
the outer radius: $y_1 = 2x$,
the inner radius: $y_2 = x^2$ and
the thickness: dx.
The volume of the washer is

$$dV = \pi\left[(2x)^2 - (x^2)^2\right]\ dx.$$

Thus, the volume of the solid over the interval $[0, 2]$ is

$$\begin{aligned} V = \pi \int_0^2 \left((2x)^2 - (x^2)^2\right)\ dx &= \pi \int_0^2 (4x^2 - x^4)\ dx \\ &= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\ &= \pi\left[\frac{32}{3} - \frac{32}{5}\right] \\ &= \frac{64}{15}\pi. \end{aligned}$$

The figure shows the region R and the solid generated by revolving R about the x-axis.
A vertical rectangle generates a washer where
the outer radius: $y_1 = 2x$,
the inner radius: $y_2 = x^2$ and
the thickness: dx.
The volume of the washer is

$$dV = \pi\left[(2x)^2 - (x^2)^2\right]\ dx.$$

Thus, the volume of the solid over the interval $[0, 2]$ is

$$\begin{aligned} V = \pi \int_0^2 \left((2x)^2 - (x^2)^2\right)\ dx &= \pi \int_0^2 (4x^2 - x^4)\ dx \\ &= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\ &= \pi\left[\frac{32}{3} - \frac{32}{5}\right] \\ &= \frac{64}{15}\pi. \end{aligned}$$

Example

Consider a region R bounded by the graphs of the functions $y = \sqrt{x}$, $y = 6 - x$ and the x-axis. Revolve this region about the y-axis and find the volume of the generated solid.

Solution:

Since the revolution is about the y-axis, we need to rewrite the functions in terms of y i.e., $x = f(y)$ and $x = g(y)$.

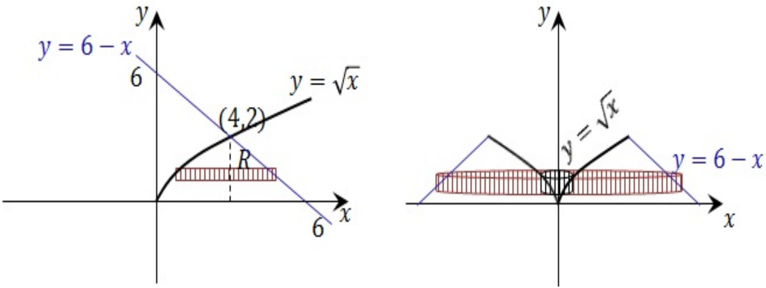
$$y = \sqrt{x} \Rightarrow x = y^2 = f(y)$$

$$y = 6 - x \Rightarrow x = 6 - y = g(y).$$

Now, we find the intersection points:

$$f(y) = g(y) \Rightarrow y^2 = 6 - y \Rightarrow y^2 + y - 6 = 0 \Rightarrow y = -3 \text{ or } y = 2.$$

Since $y = \sqrt{x}$, we ignore the value $y = -3$. By substituting $y = 2$ into the two functions, we have $x = 4$. Thus, the two curves intersect in one point $(4, 2)$. The solid S generated by revolving R about the y-axis is shown in the figure.



Since the revolution is about the y-axis, then we have a horizontal rectangle that generates a washer where
the outer radius: $x_1 = 6 - y$,
the inner radius: $x_2 = y^2$ and
the thickness: dy.
The volume of the washer is

$$dV = \pi\left[(6-y)^2 - (y^2)^2\right]\, dy.$$

The volume of the solid over the interval $[0, 2]$ is

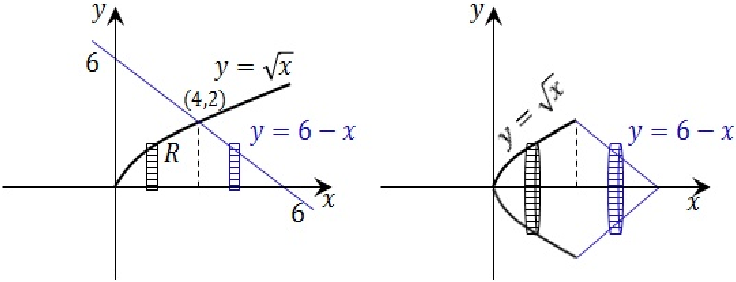
$$\begin{aligned} V = \pi \int_0^2 \left[(6-y)^2 - (y^2)^2\right]\, dy &= \pi\left[-\frac{(6-y)^3}{3} - \frac{y^5}{5}\right]_0^2 \\ &= \pi\left[\left(-\frac{64}{3} - \frac{32}{5}\right) - \left(-\frac{216}{3} - 0\right)\right] \\ &= \frac{664}{15}\pi. \end{aligned}$$

Example

Consider the same region as in Example 13 enclosed by the graphs of $y = \sqrt{x}$, $y = 6 - x$ and the x-axis. Revolve this region about the x-axis instead and find the volume of the generated solid.

Solution:

From the figure, we find that the solid is made up of two separate regions and each requires its own integral. Meaning that, we use the disk method to evaluate the volume of the solid generated by revolving each curve.



$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx + \pi \int_4^6 (6-x)^2 \, dx$$

$$= \pi \int_0^4 x \, dx + \pi \int_4^6 (6-x)^2 \, dx$$

$$= \frac{\pi}{2} \left[x^2 \right]_0^4 - \frac{\pi}{3} \left[(6-x)^3 \right]_4^6$$

32

(we used the substitution method to do the second integral with $u = 6 - x$ and $du = dx$)

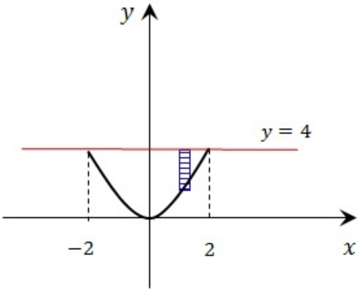
The revolution of a region is not always about the x-axis or the y-axis. It could be about a line paralleled to the x-axis or the y-axis. If the axis of revolution is a line $y = y_0$, evaluating the volume of the generated solid is similar to the case when the region revolves about the x-axis. Whereas, if the axis of revolution is a line $x = x_0$, evaluating the volume of the generated solid is similar to the case when the region revolves about the y-axis.

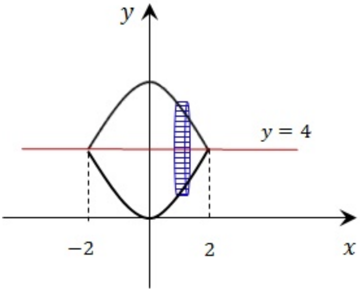
Example

Let R is a region bounded by graphs of the functions $y = x^2$ and $y = 4$. Evaluate the volume of the solid generated by revolving R about the given line.
(a) $y = 4$ **(b)** $x = 2$

Solution:

(a) We have a vertical circular disk:
the radius of the disk: $4 - y = 4 - x^2$, and
the thickness: dx.





The volume of the disk is

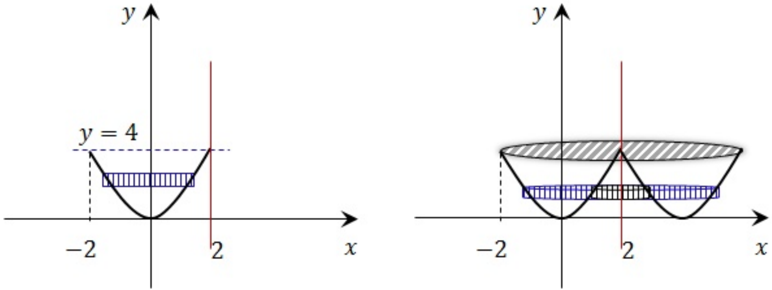
$$dV = \pi(4 - x^2)^2 \; dx.$$

The volume of the solid over the interval $[-2, 2]$ is

$$V = \pi \int_{-2}^{2} (4 - x^2)^2 \; dx = \pi \int_{-2}^{2} (16 - 8x^2 + x^4) \; dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2} = \frac{512}{15}\pi.$$

(b) In this case, a horizontal rectangle will generate a washer where
the outer radius: $2 + \sqrt{y}$,
the inner radius: $2 - \sqrt{y}$ and
the thickness: dy.



The volume of the washer is

$$dV = \pi\left[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\right]\, dy = 8\pi\sqrt{y}\, dy.$$

The volume of the solid over the interval $[0, 4]$ is

$$V = 8\pi \int_0^4 \sqrt{y}\, dx = \frac{16\pi}{3}\left[\, y^{\frac{3}{2}}\,\right]_0^4 = \frac{128}{3}\pi.$$

Example

Sketch the region R bounded by graphs of the equations $x = (y-1)^2$ and $x = y+1$. Then, find the volume of the solid generated by revolving R about $x = 4$.

Example

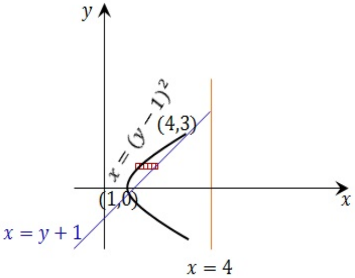
Sketch the region R bounded by graphs of the equations $x = (y-1)^2$ and $x = y + 1$. Then, find the volume of the solid generated by revolving R about $x = 4$.

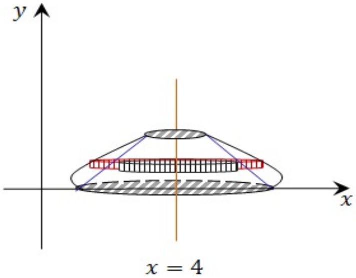
Solution:

First, we find the intersection points:

$$\begin{aligned}(y-1)^2 = y+1 &\Rightarrow y^2 - 2y + 1 = y + 1\\ &\Rightarrow y^2 - 3y = 0\\ &\Rightarrow y = 0 \text{ or } y = 3.\end{aligned}$$

Thus, the two curves intersect in two points $(1,0)$ and $(4,3)$.





The figure shows the region R and the solid S. A horizontal rectangle generates a washer where
the outer radius: $4-(y-1)^2$,
the inner radius: $4-(y+1)=3-y$ and
the thickness: dy.
The volume of the washer is

$$dV=\pi\Big[(4-(y-1)^2)^2-(3-y)^2\Big]\ dy=\pi\Big[16-8(y-1)^2+(y-1)^4-(3-y)^2\Big]\ dy.$$

Thus, the volume of the solid over the interval $[0,3]$ is

$$\begin{aligned} V&=\pi\Big(\int_0^3 16\ dy-8\int_0^3 (y-1)^2\ dy+\int_0^3 (y-1)^4\ dy-\int_0^3 (3-y)^2\ dy\Big)\\ &=\pi\Big[16y-\frac{8(y-1)^3}{3}+\frac{(y-1)^5}{5}+\frac{(3-y)^3}{3}\Big]_0^3\\ &=\frac{108}{5}\pi. \end{aligned}$$

(iii) Method of Cylindrical Shells

In the washer method, we assume that the rectangle from each subinterval is vertical to the axis of the revolution while in the method of cylindrical shells, the rectangle is parallel to the axis of the revolution.

As shown in figure, let

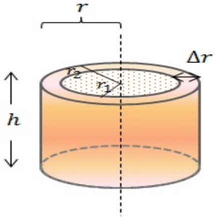
r_1 be the inner radius of the shell,

r_2 be the outer radius of the shell,

h be high of the shell,

$\Delta r = r_2 - r_1$ be the thickness of the shell,

$r = \frac{r_1 + r_2}{2}$ be the average radius of the shell.



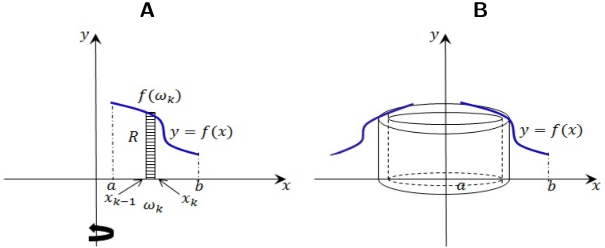
The volume of the cylindrical shell is

$$V = \underbrace{V_2}_{\text{the outer cylinder}} - \underbrace{V_1}_{\text{the inner cylinder}}$$

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\frac{r_2 + r_1}{2})h(r_2 - r_1) = 2\pi r h \Delta r \end{aligned}$$

Now, consider the graph shown in the figure. The revolution of the region R about the y-axis generates a solid given in (B) of the same figure. Let P be a partition of the interval $[a, b]$ and let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be a mark on P where ω_k is the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about the y-axis generates a cylindrical shell where
the high $= f(\omega_k)$,
the average radius $= \omega_k$ and
the thickness $= \Delta x_k$.



Hence, the volume of the cylindrical shell is $V_k = 2\pi\omega_k f(\omega_k)\Delta x_k$. To evaluate the volume of the whole solid, we sum the volumes of all cylindrical shells. This implies

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k.$$

From the Riemann sum

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k = \int_a^b xf(x)\,dx$$

Similarly, if the revolution of the region is about the x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d yf(y)\ dy.$$

Theorem

1. *If R is a region bounded by the graph of f on the interval $[a, b]$, the volume of the solid of revolution determined by revolving R about the y-axis is*

$$V = 2\pi \int_a^b xf(x)\ dx.$$

2. *If R is a region bounded by the graph of f on the interval $[a, b]$, the volume of the solid of revolution determined by revolving R about the x-axis is*

$$V = 2\pi \int_c^d yf(y)\ dy.$$

The method of cylindrical shells is sometimes easier than the washer method. This is because solving equations for one variable in terms of another is not always simple (i.e., solving x in terms of y). For example, for the volume of the solid obtained by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis, by the washer method, we would have to solve the cubic equation for x in terms of y, but this is not simple.

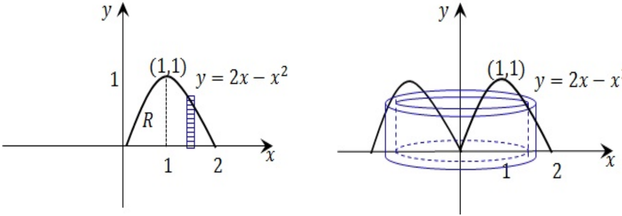
Example

Sketch the region R bounded by graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of the cylindrical shells, find the volume of the solid generated by revolving R about the y-axis.

Example

Sketch the region R bounded by graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of the cylindrical shells, find the volume of the solid generated by revolving R about the y-axis.

Solution: The figure shows the region R and the solid S generated by revolving R about the y-axis.



Since the revolution is about the y-axis, the rectangle is vertical and by revolving it, we obtain a cylindrical shell where
the high: $y = 2x - x^2$,
the average radius: x,
the thickness: dx.
The volume of the cylindrical shell is $dV = 2\pi x(2x - x^2)\, dx = 2\pi(2x^2 - x^3)\, dx$.

Thus, the volume of the solid over the interval $[0, 2]$ is

$$\begin{aligned} V &= 2\pi \int_0^2 (2x^2 - x^3)\ dx \\ &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left(\frac{16}{3} - \frac{16}{4} \right) = \frac{8\pi}{3}. \end{aligned}$$

Thus, the volume of the solid over the interval $[0, 2]$ is

$$
\begin{aligned}
V &= 2\pi \int_0^2 (2x^2 - x^3)\ dx \\
&= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 \\
&= 2\pi \left(\frac{16}{3} - \frac{16}{4}\right) = \frac{8\pi}{3}.
\end{aligned}
$$

Example

Sketch the region R bounded by graphs of the equations $x = \sqrt{y}$ and $x = 2$, and the y-axis. Then, find the volume of the solid generated by revolving R about the x-axis.

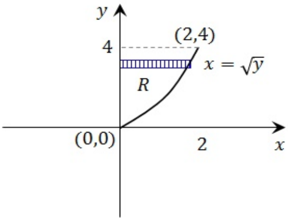
Thus, the volume of the solid over the interval [0, 2] is

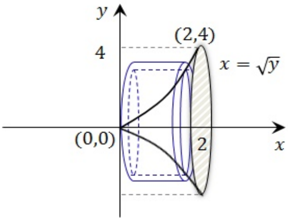
$$V = 2\pi \int_0^2 (2x^2 - x^3)\ dx$$
$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$
$$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{8\pi}{3}.$$

Example

Sketch the region R bounded by graphs of the equations $x = \sqrt{y}$ and $x = 2$, and the y-axis. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:





Since the revolution is about the x-axis, the rectangle is horizontal and by revolving it, we have a cylindrical shell where
the high: $x = \sqrt{y}$,
the average radius: y
the thickness: dy.
The volume of the cylindrical shell is $dV = 2\pi\ y\ \sqrt{y}\ dy$.
Thus, the volume of the solid over the interval $[0, 4]$ is

$$
\begin{aligned}
V = 2\pi \int_0^4 y\sqrt{y}\ dy &= 2\pi \int_0^4 y^{\frac{3}{2}}\ dy \\
&= \frac{4\pi}{5}\left[\ y^{\frac{5}{2}}\ \right]_0^4 \\
&= \frac{4\pi}{5}\Big[32 - 0\Big] = \frac{128\pi}{5}.
\end{aligned}
$$

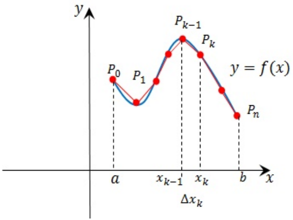
(3) Arc Length and Surfaces of Revolution

(A) Arc Length

Let $y = f(x)$ be a smooth function on $[a, b]$. Assume that $P = \{x_0, x_1, ..., x_n\}$ is a regular partition of the interval $[a, b]$ and let $P_0, P_1, ..., P_n$ be points on the curve as shown in the figure.

The distance between any two points of the curve is

$$\begin{aligned} d(P_{k-1}, P_k) &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{(\Delta x_k)^2 + \big(f(x_k) - f(x_{k-1})\big)^2} \\ &= \Delta x_k \sqrt{1 + \frac{\big(f(x_k) - f(x_{k-1})\big)^2}{(\Delta x_k)^2}} \\ &= \frac{b-a}{n}\sqrt{1 + \left[\frac{f(x_k) - f(x_{k-1})}{\Delta x_k}\right]^2} \end{aligned}$$



From the mean value theorem of differential calculus for the function f on $[x_{k-1}, x_k]$, we have

$$f'(c_i) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

for some $c_i \in (x_{k-1}, x_k)$. Thus, the distance between P_{k-1} and P_k is

$$d(P_{k-1}, P_k) = \frac{b-a}{n}\sqrt{1 + [f'(c_i)]^2}.$$

The sum of all these distances is

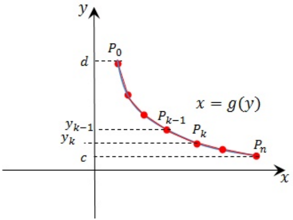
$$\frac{b-a}{n}\left[\sqrt{1+\left[f'(c_1)\right]^2}+\sqrt{1+\left[f'(c_2)\right]^2}+\ldots+\sqrt{1+\left[f'(c_n)\right]^2}\right].$$

The previous sum is a Riemann sum for the function $\sqrt{1+\left[f'(x_k)\right]^2}$ from a to b where for a better approximation, we let n be large enough. Thus, the arc length of the function f is

$$L(f)=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

Similarly, let $x=g(y)$ be a smooth function on $[c,d]$. The length of the arc of the function g from $(g(c),c)$ to $(g(d),d)$ is

$$L(g)=\int_c^d \sqrt{1+\left[g'(y)\right]^2}\,dy.$$



Theorem

1. *Let $y = f(x)$ be a smooth function on $[a, b]$. The length of the arc of f from $(a, f(a))$ to $(b, f(b))$ is*

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

2. *Let $x = g(y)$ be a smooth function on $[c, d]$. The length of the arc of g from $(g(c), c)$ to $(g(d), d)$ is*

$$L(g) = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy.$$

Example

Find the arc length of the graph of the given equation from A to B.

1. $y = 5 - \sqrt{x^3}$; $A(0,5)$, $B(4,-3)$
2. $x = 4y$; $A(0,0)$, $B(4,1)$

Solution:

(1) $y = f(x) = 5 - \sqrt{x^3} \Rightarrow f'(x) = -\frac{3}{2}x^{\frac{1}{2}}$

$$\Rightarrow (f'(x))^2 = \frac{9}{4}x$$

$$\Rightarrow 1 + (f'(x))^2 = \frac{4+9x}{4}$$

$$\Rightarrow \sqrt{1 + (f'(x))^2} = \frac{\sqrt{4+9x}}{2}.$$

The length of the curve is

$$L(f) = \frac{1}{2}\int_0^4 \sqrt{4+9x}\,dx = \frac{1}{27}\left[(4+9x)^{\frac{3}{2}}\right]_0^4$$

$$= \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

$$= \frac{8}{27}\left[10\sqrt{10} - 1\right].$$

(2) $x = g(y) = 4y \Rightarrow g'(y) = 4$

$$\begin{aligned}&\Rightarrow (g'(y))^2 = 16\\&\Rightarrow 1 + (g'(y))^2 = 17\\&\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{17}.\end{aligned}$$

The length of the curve is

$$\begin{aligned}L(g) = \sqrt{17}\int_0^1 dy &= \sqrt{17}\Big[\ y\ \Big]_0^1\\&= \sqrt{17}\Big[1 - 0\Big] = \sqrt{17}.\end{aligned}$$

$$
\begin{aligned}
(2)\ x = g(y) = 4y &\Rightarrow g'(y) = 4 \\
&\Rightarrow (g'(y))^2 = 16 \\
&\Rightarrow 1 + (g'(y))^2 = 17 \\
&\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{17}.
\end{aligned}
$$

The length of the curve is

$$
\begin{aligned}
L(g) = \sqrt{17}\int_0^1 dy &= \sqrt{17}\Big[\ y\ \Big]_0^1 \\
&= \sqrt{17}\Big[1 - 0\Big] = \sqrt{17}.
\end{aligned}
$$

Example

Find the arc length of the graph of the given equation over the indicated interval.

1. $y = \cosh\ x;\qquad 0 \leq x \leq 2$
2. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2};\qquad -2 \leq y \leq -1$

Solution:

(1) $y = f(x) = \cosh x \Rightarrow f'(x) = \sinh\ x$

$$\Rightarrow (f'(x))^2 = \sinh^2\ x$$
$$\Rightarrow 1 + (f'(x))^2 = 1 + \sinh^2\ x = \cosh^2\ x$$
$$\Rightarrow \sqrt{1 + (f'(x))^2} = \cosh\ x.$$

The length of the curve is

$$L(f) = \int_0^2 \cosh\ x\, dx = \Big[\sinh\ x\Big]_0^2 = \sinh 2 - \sinh 0 = \sinh 2.$$

$(\sinh\ 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0)$

(2) $x = g(y) = \frac{1}{8}y^4 + \frac{1}{4}y^{-2} \Rightarrow g'(y) = \frac{1}{2}(y^3 - \frac{1}{y^3})$

$$\Rightarrow (g'(y))^2 = \frac{(y^6 - 1)^2}{4y^6}$$
$$\Rightarrow 1 + (g'(y))^2 = \frac{4y^6 + y^{12} - 2y^6 + 1}{4y^6}$$
$$\Rightarrow 1 + (g'(y))^2 = \frac{y^{12} + 2y^6 + 1}{4y^6}$$
$$\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{\frac{(y^6 + 1)^2}{4y^6}} = \frac{y^6 + 1}{2y^3}.$$

Since $y < 0$ over $[-2, -1]$, the length of the curve is

$$L(g) = -\frac{1}{2}\int_{-2}^{-1} (y^3 + y^{-3})\ dy = -\frac{1}{2}\left[\frac{y^4}{4} - \frac{1}{2y^2}\right]_{-2}^{-1} = \frac{33}{16}.$$

(B) Surfaces of Revolution

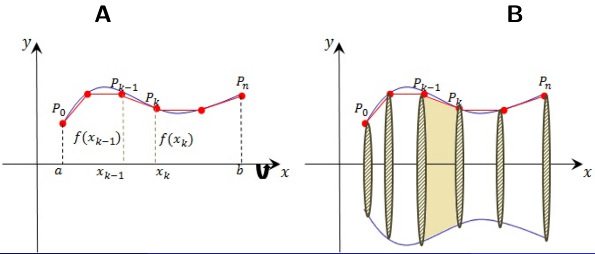
Definition

Let f is a continuous function on $[a, b]$. The surface of revolution is generated by revolving the graph of the function f about an axis.

Let $y = f(x) \geq 0$ be a smooth function on the interval $[a, b]$. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of the interval $[a, b]$ and $P_0, P_1, ..., P_n$ be the points on the curve as shown in the figure. Let D_k be a frustum of a cone generated by revolving the line segment $P_{k-1}P_k$ about the x-axis with radii $f(x_{k-1})$ and $f(x_k)$. Since area of the frustum of a cone with radii r_1 and r_2 and slant length ℓ is $S.A = \pi(r_1 + r_2)\ell$, then

$$S.A(D_k) = \pi[f(x_k) + f(x_{k-1})]\Delta\ell_k$$

where $\Delta\ell_k$ is the distance between P_{k-1} and P_k i.e.,
$\Delta\ell_k = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$.



From the intermediate value theorem, there exists $\omega_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) = f'(\omega_k)\Delta x_k.$$

This implies $\Delta \ell_k = \Delta x_k \sqrt{1 + [f'(\omega_k)]^2}$. For n large, $f(x_k) \approx f(x_{k-1}) \approx f(\omega_k)$ and this implies $S.A = \sum_{k=1}^{n} 2\pi f(\omega_k)\sqrt{1 + [f'(\omega_k)]^2}\Delta x_k$.

From the Riemann sum,

$$S.A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi f(\omega_k)\sqrt{1 + [f'(\omega_k)]^2}\Delta x_k = 2\pi \int_a^b | f(x) | \sqrt{1 + [f'(x)]^2}\ dx$$

$$= 2\pi \int_a^b | y | \sqrt{1 + (\frac{dy}{dx})^2}\ dx.$$

If the revolution is about the y-axis, then

$$S.A = 2\pi \int_a^b | x | \sqrt{1 + [f'(x)]^2}\ dx = 2\pi \int_a^b | x | \sqrt{1 + (\frac{dy}{dx})^2}\ dx.$$

Similarly, if $x = g(y)$ is a smooth function on $[c, d]$, then the surface area $S.A$ generated by revolving the graph of g about the y-axis from $y = c$ to $y = d$ is

$$S.A = 2\pi \int_c^d | g(y) | \sqrt{1 + \left[g'(y)\right]^2}\ dy = 2\pi \int_c^d | x | \sqrt{1 + (\frac{dx}{dy})^2}\ dy.$$

If the revolution is about the x-axis, then

$$S.A = 2\pi \int_c^d | y | \sqrt{1 + \left[g'(y)\right]^2}\ dy = 2\pi \int_c^d | y | \sqrt{1 + (\frac{dx}{dy})^2}\ dy.$$

Theorem

1. *Let* $y = f(x)$ *be a smooth function on* $[a, b]$.
   - *If the revolution is about the* x*-axis, the surface area of revolution is*

$$S.A = 2\pi \int_a^b |\, y \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

   - *If the revolution is about the* y*-axis, the surface are of revolution is*

$$S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

2. *Let* $x = g(y)$ *be a smooth function on* $[c, d]$.
   - *If the revolution is about the* y*-axis, the surface area of revolution is*

$$S.A = 2\pi \int_c^d |\, x \,| \sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

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Note that the absolute value is for the case when the function is negative for some values in the closed interval.

Example

Find the surface area generated by revolving the graph of the function $\sqrt{4-x^2}$, $-2 \leq x \leq 2$ about the x-axis.

Note that the absolute value is for the case when the function is negative for some values in the closed interval.

Example

Find the surface area generated by revolving the graph of the function $\sqrt{4-x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution:

We apply the formula $S.A = 2\pi \int_a^b \mid y \mid \sqrt{1+(f'(x))^2}\; dx$.

$$\begin{aligned} y = \sqrt{4-x^2} \Rightarrow f'(x) &= \frac{-2x}{2\sqrt{4-x^2}} \\ \Rightarrow (f'(x))^2 &= \frac{x^2}{4-x^2} \\ \Rightarrow 1+(f'(x))^2 &= \frac{4}{4-x^2} \\ \Rightarrow \sqrt{1+(f'(x))^2} &= \frac{2}{\sqrt{4-x^2}}. \end{aligned}$$

The area of the revolution surface is
$S.A = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\; \frac{2}{\sqrt{4-x^2}}\; dx = 4\pi\left[2+2\right] = 16\pi.$

Example

Find the surface area generated by revolving the graph of the function $y = 2x$, $0 \leq x \leq 3$ about the y-axis.

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Find the surface area generated by revolving the graph of the function $y = 2x$, $0 \leq x \leq 3$ about the y-axis.

Solution:

We apply the formula $S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + (f'(x))^2}\, dx$.

$$\begin{aligned} y = 2x &\Rightarrow f'(x) = 2 \\ &\Rightarrow (f'(x))^2 = 4 \\ &\Rightarrow 1 + (f'(x))^2 = 5 \\ &\Rightarrow \sqrt{1 + (f'(x))^2} = \sqrt{5}. \end{aligned}$$

The area of the revolution surface is $S.A = 2\pi \int_0^3 |\, x \,| \ \sqrt{5}\, dx = \sqrt{5}\pi \Big[\, x^2 \,\Big]_0^3 = 9\sqrt{5}\pi$.

Example

Find the surface area generated by revolving the graph of the function $x = y^3$ on the interval $[0, 1]$ about the y-axis.

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Find the surface area generated by revolving the graph of the function $x = y^3$ on the interval $[0, 1]$ about the y-axis.

Solution:

We apply the formula $S.A = 2\pi \int_c^d |\, x \,| \sqrt{1 + (g'(y))^2}\, dy$.

$$\begin{aligned} x = y^3 &\Rightarrow g'(y) = 3y^2 \\ &\Rightarrow (g'(y))^2 = 9y^4 \\ &\Rightarrow 1 + (g'(y))^2 = 1 + 9y^4 \\ &\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{1 + 9y^4}. \end{aligned}$$

The area of the revolution surface is

$S.A = 2\pi \int_0^1 y^3 \sqrt{1 + 9y^4}\, dy = \frac{\pi}{27} \Big[(1 + 9y^4)^{\frac{3}{2}} \Big]_0^1 = \frac{\pi}{27} \Big[10\sqrt{10} - 1 \Big].$