Integral Calculus

Dr. M. Alghamdi

Department of Mathematics

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Integration by Parts

Integration by parts is a method to transfer the original integral to an easier one that can be evaluated. Practically, the integration by parts divides the original integral into two parts u and dv, then we find the du by deriving u and v by integrating dv.

Theorem

If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u \, dv = uv - \int v \, du.$$

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Theorem

If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u\ dv = uv - \int v\ du.$$

Proof: We know that $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$. Thus, $f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x)$.
By integrating both sides, we obtain

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int \frac{d}{dx}(f(x)g(x))\ dx - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx.\end{aligned}$$

Since $u = f(x)$ and $v = g(x)$, then $du = f'(x)\ dx$ and $dv = g'(x)\ dx$. Therefore,

$$\int u\ dv = uv - \int v\ du. \ \blacksquare$$

Example

Evaluate the integral $\int x \cos x \, dx$.

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Solution:

Let $I = \int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Hence,

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Let $I = \int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Hence,

$$\begin{aligned} u &= x \Rightarrow du = dx \,, \\ dv = \cos x \, dx &\Rightarrow v = \int \cos x \, dx = \sin x. \end{aligned}$$

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution:

Let $I = \int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \, ,$$

$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

From the theorem, we have

$$I = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c.$$

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Evaluate the integral $\int x \cos x \, dx$.

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Let $I = \int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Hence,

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Try to choose

$u = \cos x$ and $dv = x \, dx$

Do you have the same result?

Example

Evaluate the integral $\int x\ e^x\ dx$.

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$$dv = e^x\ dx \Rightarrow v = \int e^x\ dx = e^x.$$

From Theorem .1, we have

$$I = x\ e^x - \int e^x\ dx = x\ e^x - e^x + c.$$

Example

Evaluate the integral $\int x\ e^x\ dx$.

Solution:

Let $I = \int x\ e^x\ dx$. Let $u = x$ and $dv = e^x\ dx$. Hence,

$$u = x \Rightarrow du = dx\ ,$$

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From Theorem .1, we have

$$I = x\ e^x - \int e^x\ dx = x\ e^x - e^x + c.$$

Try to choose
$u = e^x$ and $dv = x\ dx$

We will obtain

$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x\ dx.$$

However, the integral $\int \frac{x^2}{2} e^x\ dx$ is more difficult than the original one $\int xe^x\ dx$.

Remark

1. *Remember that when we consider the integration by parts, we want to obtain an easier integral. As we saw in the previous example, if we choose* $u = e^x$ *and* $dv = x\ dx$*, we have* $\int \frac{x^2}{2}\ e^x\ dx$ *which is more difficult than the original one.*
2. *When considering the integration by parts, we have to choose* dv *a function that can be integrated.*
3. *Sometimes we need to use the integration by parts twice as in the upcoming examples.*

Example

Evaluate the integral $\displaystyle\int \ln x\ dx$.

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Evaluate the integral $\displaystyle\int \ln x\ dx$.

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Example

Evaluate the integral $\displaystyle\int \ln x\ dx$.

Solution: Let $I = \displaystyle\int \ln x\ dx$. Let $u = \ln x$ and $dv = dx$. Hence,

$$u = \ln x \Rightarrow du = \frac{1}{x}\ dx\ ,$$

$$dv = dx \Rightarrow v = \int 1\ dx = x.$$

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1. *Remember that when we consider the integration by parts, we want to obtain an easier integral. As we saw in the previous example, if we choose $u = e^x$ and $dv = x\ dx$, we have $\int \frac{x^2}{2}\ e^x\ dx$ which is more difficult than the original one.*
2. *When considering the integration by parts, we have to choose dv a function that can be integrated.*
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Solution: Let $I = \displaystyle\int \ln x\ dx$. Let $u = \ln x$ and $dv = dx$. Hence,

$$u = \ln x \Rightarrow du = \frac{1}{x}\ dx\ ,$$

$$dv = dx \Rightarrow v = \int 1\ dx = x.$$

From Theorem .1, we obtain $I = x \ln x - \displaystyle\int x\frac{1}{x}\ dx = x \ln x - \int 1\ dx = x \ln x - x + c$.

Example

Evaluate the integral $\int e^x \cos\ x\ dx$.

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Example

Evaluate the integral $\int e^x \cos\ x\ dx$.

Solution: Let $I = \int e^x \cos\ x\ dx$. Let $u = e^x$ and $dv = \cos\ x\ dx$.

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \cos\ x\ dx \Rightarrow v = \int \cos\ x\ dx = \sin\ x.$$

Hence, $I = e^x \sin\ x - \int e^x \sin\ x\ dx$.

Example

Evaluate the integral $\int e^x \cos\ x\ dx$.

Solution: Let $I = \int e^x \cos\ x\ dx$. Let $u = e^x$ and $dv = \cos\ x\ dx$.

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \cos\ x\ dx \Rightarrow v = \int \cos\ x\ dx = \sin\ x.$$

Hence, $I = e^x \sin\ x - \int e^x \sin\ x\ dx$.

The integral $\int e^x \sin\ x\ dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin\ x\ dx$. Let $u = e^x$ and $dv = \sin\ x\ dx$. Hence,

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \sin\ x\ dx \Rightarrow v = \int \sin\ x\ dx = -\cos\ x.$$

Hence, $J = -e^x \cos\ x + \int e^x \cos\ x\ dx$. By substituting the result of J into I, we have

Example

Evaluate the integral $\int e^x \cos\ x\ dx$.

Solution: Let $I = \int e^x \cos\ x\ dx$. Let $u = e^x$ and $dv = \cos\ x\ dx$.

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \cos\ x\ dx \Rightarrow v = \int \cos\ x\ dx = \sin\ x.$$

Hence, $I = e^x \sin\ x - \int e^x \sin\ x\ dx$.

The integral $\int e^x \sin\ x\ dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin\ x\ dx$. Let $u = e^x$ and $dv = \sin\ x\ dx$. Hence,

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \sin\ x\ dx \Rightarrow v = \int \sin\ x\ dx = -\cos\ x.$$

Hence, $J = -e^x \cos\ x + \int e^x \cos\ x\ dx$. By substituting the result of J into I, we have

$$I = e^x \sin\ x - J = e^x \sin\ x + e^x \cos\ x - \int e^x \cos\ x\ dx \Rightarrow I = e^x \sin\ x + e^x \cos\ x - I.$$

Example

Evaluate the integral $\int e^x \cos\ x\ dx$.

Solution: Let $I = \int e^x \cos\ x\ dx$. Let $u = e^x$ and $dv = \cos\ x\ dx$.

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \cos\ x\ dx \Rightarrow v = \int \cos\ x\ dx = \sin\ x.$$

Hence, $I = e^x \sin\ x - \int e^x \sin\ x\ dx$.

The integral $\int e^x \sin\ x\ dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin\ x\ dx$. Let $u = e^x$ and $dv = \sin\ x\ dx$. Hence,

$$u = e^x \Rightarrow du = e^x\ dx\ ,$$

$$dv = \sin\ x\ dx \Rightarrow v = \int \sin\ x\ dx = -\cos\ x.$$

Hence, $J = -e^x \cos\ x + \int e^x \cos\ x\ dx$. By substituting the result of J into I, we have

$$I = e^x \sin\ x - J = e^x \sin\ x + e^x \cos\ x - \int e^x \cos\ x\ dx \Rightarrow I = e^x \sin\ x + e^x \cos\ x - I.$$

This implies $2I = e^x \sin\ x + e^x \cos\ x \Rightarrow I = \frac{1}{2}(e^x \sin\ x + e^x \cos\ x) + c$.

Example

Evaluate the integral $\int x^2 e^x \, dx$.

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Evaluate the integral $\int x^2 e^x \, dx$.

Solution: Let $I = \int x^2 e^x \, dx$. Let $u = x^2$ and $dv = e^x \, dx$. Hence,

$$u = x^2 \Rightarrow du = 2x \, dx \ ,$$

$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

This implies, $I = x^2 e^x - 2\int x e^x \, dx$.

Example

Evaluate the integral $\int x^2 e^x \, dx$.

Solution: Let $I = \int x^2 e^x \, dx$. Let $u = x^2$ and $dv = e^x \, dx$. Hence,

$$u = x^2 \Rightarrow du = 2x \, dx \; ,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

This implies, $I = x^2 e^x - 2 \int x e^x \, dx$.

We use the integration by parts again for the integral $\int x e^x \, dx$. Let $J = \int x e^x \, dx$. Let $u = x$ and $dv = e^x \, dx$. Hence,

$$u = x \Rightarrow du = \; dx \; ,$$
$$dv = e^x dx \Rightarrow v = \int e^x \, dx = e^x.$$

Therefore, $J = x e^x - \int e^x \, dx = x e^x - e^x + c$. By substituting the result into I, we have

$$I = x^2 e^x - 2(x e^x - e^x) + c = e^x (x^2 - 2x + 2) + c.$$

Example

Evaluate the integral $\int_0^1 \tan^{-1} x \, dx$.

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Evaluate the integral $\int_0^1 \tan^{-1} x \, dx$.

Solution:

Let $I = \int \tan^{-1} x \, dx$. Let $u = \tan^{-1} x$ and $dv = dx$. Hence,

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{x^2+1} \, dx \, ,$$
$$dv = dx \Rightarrow v = \int 1 \, dx = x.$$

By applying the theorem, we obtain

$$I = x \tan^{-1} x - \int \frac{x}{x^2+1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + c.$$

Therefore,

$$\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) \right]_0^1 = (\tan^{-1}(1) - \frac{1}{2} \ln 2) - (0 - \frac{1}{2} \ln 1) = \frac{\pi}{4} - \ln \sqrt{2}$$

Trigonometric Functions

(A) Integration of Powers of Trigonometric Functions

In this section, we evaluate integrals of forms $\int \sin^n x \cos^m x \, dx$, $\int \tan^n x \sec^m x \, dx$ and $\int \cot^n x \csc^m x \, dx$.

(A) Integration of Powers of Trigonometric Functions

In this section, we evaluate integrals of forms $\int \sin^n x \cos^m x \, dx$, $\int \tan^n x \sec^m x \, dx$ and $\int \cot^n x \csc^m x \, dx$.

Form 1: $\int \sin^n x \cos^m x \, dx$.

This form is treated as follows:

1. If n is an odd integer, write
$$\sin^n x \cos^m x = \sin^{n-1} x \cos^m x \sin x$$
Then, use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.
2. If m is an odd integer, write
$$\sin^n x \cos^m x = \sin^n x \cos^{m-1} x \cos x$$
Then, use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$.
3. If m and n are even, use the identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$.

Example

Evaluate the integral.

1. $\int \sin^3 x \, dx$
2. $\int \cos^4 x \, dx$
3. $\int \sin^5 x \cos^4 x \, dx$
4. $\int \sin^2 x \cos^2 x \, dx$

Example

Evaluate the integral.

1. $\int \sin^3 x \, dx$
2. $\int \cos^4 x \, dx$
3. $\int \sin^5 x \cos^4 x \, dx$
4. $\int \sin^2 x \cos^2 x \, dx$

Solution:

1) Write $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$. Hence,

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx.$$

Let $u = \cos x$, then $du = - \, sin \, x \, dx$. By substitution, we have

$$-\int (1 - u^2) \, du = -u + \frac{u^3}{3} + c.$$

This implies

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + c.$$

2) Write $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$. Hence,

$$\begin{aligned}
\int \cos^4 x \, dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4}\left(\int 1 \, dx + \int 2\cos 2x \, dx + \int \cos^2 2x \, dx\right) \\
&= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\int (1 + \cos 4x) \, dx\right) \\
&= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}(x + \frac{\sin 4x}{4})\right) + c.
\end{aligned}$$

2) Write $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$. Hence,

$$\begin{aligned}\int \cos^4 x\, dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx \\ &= \frac{1}{4}\left(\int 1\, dx + \int 2\cos 2x\, dx + \int \cos^2 2x\, dx\right) \\ &= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\int (1+\cos 4x)\, dx\right) \\ &= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}(x + \frac{\sin 4x}{4})\right) + c.\end{aligned}$$

3) Write $\sin^5 x \cos^4 x = \sin^4 x \cos^4 x \sin x = (1-\cos^2 x)^2 \cos^4 x \sin x$. Let $u = \cos x$, then $du = -\sin x\, dx$. Thus, the integral becomes

$$-\int (1-u^2)^2 u^4\, du = -\int (u^4 - 2u^6 + u^8)\, du = -(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}) + c.$$

This implies $\int \sin^5 x \cos^4 x\, dx = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c.$

4) The integrand
$\sin^2 x \cos^2 x = (\frac{1-\cos 2x}{2})(\frac{1+\cos 2x}{2}) = \frac{1-\cos^2 2x}{4} = \frac{\sin^2 2x}{4} = \frac{1}{4}(\frac{1-\cos 4x}{2})$. Hence,

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}\int (1 - \cos 4x) \, dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + c.$$

Form 2: $\int \tan^n x \;\sec^m x \, dx$.

This form is treated as follows:

1. If $n = 0$, write $\sec^m x = \sec^{m-2} x \, \sec^2 x$.
   1. If $m > 1$ is odd, use the integration by parts.
   2. If m is even, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.
2. If $m = 0$ and n is odd or even, write $\tan^n x = \tan^{n-2} x \, \tan^2 x$. Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \tan x$.
3. If n is even and m is odd, use the identity $\tan^2 x = \sec^2 x - 1$ to reduce the power m and then use the integration by parts.
4. If $m \geq 2$ is even, write $\tan^n x \, \sec^m x = \tan^n x \, \sec^{m-2} x \, \sec^2 x$. Then, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$. Alternatively, write

   $$\tan^n x \, \sec^m x = \tan^{n-1} x \, \sec^{m-1} x \, \tan x \, \sec x$$

   Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.
5. If n is odd and $m \geq 1$, write $\tan^n x \, \sec^m x = \tan^{n-1} x \, \sec^{m-1} x \, \tan x \, \sec x$. Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.

Example

Evaluate the integral.

1. $\int \tan^5 x \, dx$
2. $\int \tan^6 x \, dx$
3. $\int \sec^3 x \, dx$
4. $\int \tan^5 x \sec^4 x \, dx$
5. $\int \tan^4 x \sec^4 x \, dx$

Example

Evaluate the integral.

1. $\int \tan^5 x \, dx$
2. $\int \tan^6 x \, dx$
3. $\int \sec^3 x \, dx$
4. $\int \tan^5 x \sec^4 x \, dx$
5. $\int \tan^4 x \sec^4 x \, dx$

Solution:

1) Write $\tan^5 x = \tan^3 x \tan^2 x = \tan^3 x (\sec^2 x - 1)$. Thus,

$$\begin{aligned}
\int \tan^5 x \, dx &= \int \tan^3 x (\sec^2 x - 1) \, dx \\
&= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx \\
&= \frac{\tan^4 x}{4} - \int \tan x (\sec^2 x - 1) \, dx \\
&= \frac{\tan^4 x}{4} - \int \tan x \sec^2 x \, dx + \int \tan x \, dx \\
&= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c.
\end{aligned}$$

2) Write $\tan^6 x = \tan^4 x \tan^2 x = \tan^4 x (\sec^2 x - 1)$. The integral becomes

$$\begin{aligned}\int \tan^6 x\,dx &= \int \tan^4 x (\sec^2 x - 1)\,dx \\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx \\ &= \frac{\tan^5 x}{5} - \int \tan^2 x (\sec^2 x - 1)\,dx \\ &= \frac{\tan^5 x}{5} - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int (\sec^2 x - 1)\,dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c.\end{aligned}$$

2) Write $\tan^6 x = \tan^4 x \tan^2 x = \tan^4 x (\sec^2 x - 1)$. The integral becomes

$$\begin{aligned}
\int \tan^6 x\, dx &= \int \tan^4 x (\sec^2 x - 1)\, dx \\
&= \int \tan^4 x \sec^2 x\, dx - \int \tan^4 x\, dx \\
&= \frac{\tan^5 x}{5} - \int \tan^2 x (\sec^2 x - 1)\, dx \\
&= \frac{\tan^5 x}{5} - \int \tan^2 x \sec^2 x\, dx + \int \tan^2 x\, dx \\
&= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int (\sec^2 x - 1)\, dx \\
&= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c.
\end{aligned}$$

3) Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \displaystyle\int \sec x \sec^2 x\, dx$.
We use the integration by parts to evaluate the integral as follows:

$$u = \sec x \Rightarrow du = \sec x \tan x\, dx\ ,$$

$$dv = \sec^2 x\, dx \Rightarrow v = \int \sec^2 x\, dx = \tan x.$$

Hence,

$$\begin{aligned} I &= \sec\ x\ \tan\ x - \int \sec\ x\ \tan^2\ x\ dx \\ &= \sec\ x\ \tan\ x - \int (\sec^3\ x - \sec\ x)\ dx \\ &= \sec\ x\ \tan\ x - I + \ln|\sec\ x + \tan\ x| \\ I &= \frac{1}{2}(\sec\ x\ \tan\ x + \ln|\sec\ x + \tan\ x|) + c. \end{aligned}$$

Hence,

$$
\begin{aligned}
I &= \sec x \tan x - \int \sec x \tan^2 x \, dx \\
&= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx \\
&= \sec x \tan x - I + \ln|\sec x + \tan x| \\
I &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + c.
\end{aligned}
$$

4) Express the integrand $\tan^5 x \sec^4 x$ as follows

$$\tan^5 x \sec^4 x = \tan^5 x \sec^2 x \sec^2 x = \tan^5 x (\tan^2 x + 1) \sec^2 x.$$

This implies

$$
\begin{aligned}
\int \tan^5 x \sec^4 x \, dx &= \int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx \\
&= \int (\tan^7 x + \tan^5 x) \sec^2 x \, dx \\
&= \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + c.
\end{aligned}
$$

5) Write $\tan^4 x \sec^4 x = \tan^4 x (\tan^2 x + 1) \sec^2 x$. The integral becomes

$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int (\tan^6 x + \tan^4 x) \sec^2 x \, dx$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + c.$$

Form 3: $\int \cot^n x \csc^m x \, dx$.

The treatment of this form is similar to the integral $\int \tan^n x \sec^m x \, dx$, except we use the identity

$$\cot^2 x + 1 = \csc^2 x.$$

Example

Evaluate the integral.

1. $\int \cot^3 x \, dx$
2. $\int \cot^4 x \, dx$
3. $\int \cot^5 x \csc^4 x \, dx$

Solution:

1) Write $\cot^3 x = \cot x\,(\csc^2 x - 1)$. Then,

$$\begin{aligned}\int \cot^3 x\,dx &= \int \cot x\,(\csc^2 x - 1)\,dx\\ &= \int (\cot x \csc^2 x - \cot x)\,dx\\ &= \int \cot x \csc^2 x\,dx - \int \cot x\,dx\\ &= -\frac{1}{2}\cot^2 x - \ln|\sin x| + c.\end{aligned}$$

2) The integrand can be expressed as $\cot^4 x = \cot^2 x\,(\csc^2 x - 1)$. Thus,

$$\begin{aligned}\int \cot^4 x\,dx &= \int \cot^2 x\,(\csc^2 x - 1)\,dx\\ &= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx\\ &= -\frac{\cot^3 x}{3} - \int (\csc^2 x - 1)\,dx\\ &= -\frac{\cot^3 x}{3} + \cot x + x + c.\end{aligned}$$

3) Write $\cot^5 x \csc^4 x = \csc^3 x \cot^4 x \csc x \cot x$. This implies

$$\begin{aligned}
\int \cot^5 x \csc^4 x \, dx &= \int \csc^3 x \cot^4 x \csc x \cot x \, dx \\
&= \int \csc^3 x \, (\csc^2 x - 1)^2 \csc x \cot x \, dx \\
&= \int (\csc^7 x - 2\csc^5 x + \csc^3 x) \csc x \cot x \, dx \\
&= -\frac{\csc^8 x}{8} + \frac{\csc^6 x}{3} - \frac{\csc^4 x}{4} + c.
\end{aligned}$$

(B) Integration of Forms sin ux cos vx, sin ux sin vx and cos ux cos vx

We deal with these integrals by using the following formulas:

$$\sin\ ux\ \cos\ vx = \frac{1}{2}\left(\sin\ (u-v)\ x + \sin\ (u+v)\ x\right)$$
$$\sin\ ux\ \sin\ vx = \frac{1}{2}\left(\cos\ (u-v)\ x - \cos\ (u+v)\ x\right)$$
$$\cos\ ux\ \cos\ vx = \frac{1}{2}\left(\cos\ (u-v)\ x + \cos\ (u+v)\ x\right)$$

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$$\cos\ ux\ \cos\ vx = \frac{1}{2}\left(\cos\ (u-v)\ x + \cos\ (u+v)\ x\right)$$

Example

Evaluate the integral.

1. $\int \sin\ 5x\ \sin\ 3x\ dx$
2. $\int \sin\ 7x\ \cos\ 2x\ dx$
3. $\int \cos\ 5x\ \sin\ 2x\ dx$
4. $\int \cos\ 6x\ \cos\ 4x\ dx$

Solution:

1) From the previous formulas, we have $\sin\ 5x\ \sin\ 3x = \frac{1}{2}\big(\cos\ 2x - \cos\ 8x\big)$. Hence,

$$\int \sin\ 5x\ \sin\ 3x\ dx = \frac{1}{2}\int (\cos\ 2x - \cos\ 8x)\ dx$$
$$= \frac{1}{4}\sin\ 2x - \frac{1}{16}\sin\ 8x + c.$$

2) Since $\sin\ 7x\ \cos\ 2x = \frac{1}{2}\big(\sin\ 5x + \sin\ 9x\big)$, then

$$\int \sin\ 7x\ \cos\ 2x\ dx = \frac{1}{2}\int (\sin\ 5x + \sin\ 9x)\ dx$$
$$= -\frac{1}{10}\cos\ 5x - \frac{1}{18}\cos\ 9x + c.$$

3) Since $\cos\ 5x\ \sin\ 2x = \frac{1}{2}\big(\sin\ 3x + \sin\ 7x\big)$, then

$$\int \cos\ 5x\ \sin\ 2x\ dx = \frac{1}{2}\int (\sin\ 3x + \sin\ 7x)\ dx$$
$$= -\frac{1}{6}\cos\ 3x - \frac{1}{14}\cos\ 7x + c.$$

4) Since $\cos\ 6x\ \cos\ 4x = \frac{1}{2}\big(\cos\ 2x + \cos\ 10x\big)$, then

$$\int \cos\ 6x\ \cos\ 4x\ dx = \frac{1}{2}\int (\cos\ 2x + \cos\ 10x)\ dx$$
$$= \frac{1}{4}\sin\ 2x + \frac{1}{20}\sin\ 10x + c.$$

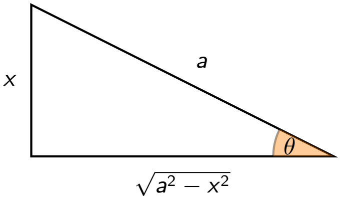
Trigonometric Substitutions

In this section, we are going to study integrals containing the following expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ where $a > 0$.

- $\sqrt{a^2 - x^2} = a \cos \theta$ if $x = a \sin \theta$.

If $x = a \sin \theta$ where $\theta \in [-\pi/2, \pi/2]$, then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2(1 - \sin^2 \theta)} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= a \cos \theta.\end{aligned}$$

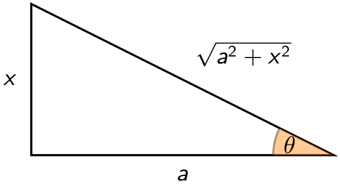


If the expression $\sqrt{a^2 - x^2}$ is in a denominator, then we assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

■ $\sqrt{a^2+x^2} = a \sec \theta$ if $x = a \tan \theta$.

If $x = a \tan \theta$ where $\theta \in (-\pi/2, \pi/2)$, then

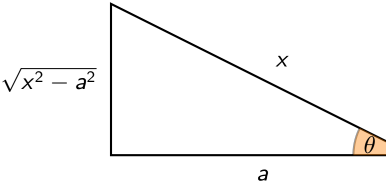
$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2(1+\tan^2 \theta)}\\ &= \sqrt{a^2\sec^2 \theta}\\ &= a \sec \theta.\end{aligned}$$



■ $\sqrt{x^2 - a^2} = a \tan\theta$ if $x = a\sec\theta$.

If $x = a\sec\theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2\sec^2\theta - a^2}\\ &= \sqrt{a^2(\sec^2\theta - 1)}\\ &= \sqrt{a^2\tan^2\theta}\\ &= a\tan\theta.\end{aligned}$$



Example

Evaluate the integral.

1. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$
2. $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$
3. $\displaystyle\int \sqrt{x^2+9}\,dx$

Example

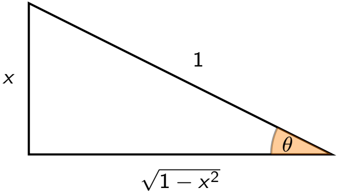
Evaluate the integral.

1. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$
2. $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$
3. $\displaystyle\int \sqrt{x^2+9}\,dx$

Solution:

1) Let $x=\sin\ \theta$ where $\theta \in (-\pi/2, \pi/2)$, thus $dx = \cos\ \theta\ d\theta$. By substitution, we have

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \sin^2\ \theta\ d\theta \\ &= \frac{1}{2}\int (1-\cos\ 2\theta)\ d\theta \\ &= \frac{1}{2}\left(\theta - \frac{1}{2}\sin\ 2\theta\right) + c \\ &= \frac{1}{2}\left(\theta - \sin\ \theta\ \cos\ \theta\right) + c.\end{aligned}$$

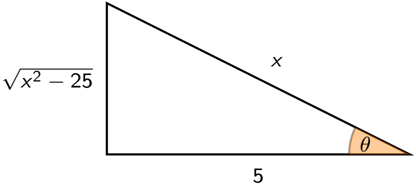


Now, we must return to the original variable x:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\sin^{-1}\ x - x\sqrt{1-x^2}) + c.$$

2) Let $x = 5\ \sec\ \theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5\ \sec\ \theta\ \tan\ \theta\ d\theta$. After substitution, the integral becomes

$$\begin{aligned}\int \frac{\sqrt{25\ \sec^2\ \theta - 25}}{625\ \sec^4\ \theta}\, 5 \sec\ \theta\ \tan\ \theta\ d\theta &= \frac{1}{25}\int \frac{\tan^2\ \theta}{\sec^3\ \theta}\ d\theta \\ &= \frac{1}{25}\int \sin^2\ \theta\ \cos\ \theta\ d\theta \\ &= \frac{1}{75}\sin^3\ \theta + c.\end{aligned}$$



We must return to the original variable x:

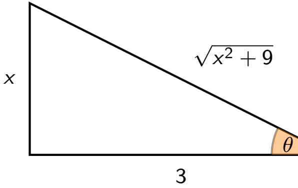
$$\int \frac{\sqrt{x^2 - 25}}{x^4}\ dx = \frac{(x^2 - 25)^{3/2}}{75x^3}$$

Hence,

$$\int_5^6 \frac{\sqrt{x^2 - 25}}{x^4}\ dx = \frac{1}{75}\left[\frac{(x^2 - 25)^{3/2}}{x^3}\right]_5^6 = \frac{1}{600}.$$

3) Let $x = 3\tan\ \theta$ where $\theta \in (-\pi/2, \pi/2)$. This implies $dx = 3\sec^2\ \theta\ d\theta$. By substitution, we have

$$\begin{aligned}\int \sqrt{x^2+9}\ dx &= \int \sqrt{9\tan^2\ \theta + 9}\ \ (3\sec^2\ \theta)\ d\theta \\ &= 9\int \sec^3\ \theta\ d\theta \\ &= \frac{9}{2}\Big(\sec\ \theta\ \tan\ \theta + \ln\big|\sec\ \theta + \tan\ \theta\big|\Big).\end{aligned}$$



This implies

$$\int \sqrt{x^2+9}\ dx = \frac{9}{2}\left(\frac{x\sqrt{x^2+9}}{9} + \ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right) + c.$$

Integrals of Rational Functions

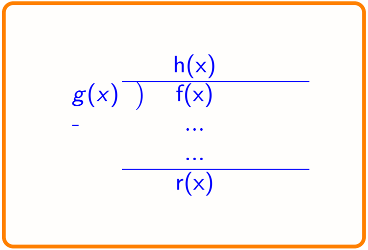
In this section, we study rational functions of form $q(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

Step 1: If the degree of $g(x)$ is less than the degree of $f(x)$, we do polynomial long-division; otherwise we move to step 2.

From the long division shown on the right side, we have

$$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)} ,$$

where $h(x)$ is the quotient and $r(x)$ is the remainder.



Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.
Step 3: Find the partial fraction decomposition. This step depends on the result of step 2 where the fraction $\frac{f(x)}{g(x)}$ or $\frac{r(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = P_1(x) + P_2(x) + P_3(x) + ... + P_n(x) \ ,$$

each $P_k(x) = \frac{A_k}{(ax+b)^n}$, $n \in \mathbb{N}$ or $P_k(x) = \frac{A_k x + B_k}{(ax^2+bx+c)^n}$ if $b^2 - 4ac < 0$. The constants A_k and B_k are real numbers and computed later.
Step 4: Integrate the result of step 3.

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of $f(x) = x + 1$ is less than the degree of $g(x) = x^2 - 2x - 8$.

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:

Step 1: This step can be skipped since the degree of $f(x) = x + 1$ is less than the degree of $g(x) = x^2 - 2x - 8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x+2)(x-4).$$

Example

Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\, dx$.

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Step 1: This step can be skipped since the degree of $f(x) = x+1$ is less than the degree of $g(x) = x^2 - 2x - 8$.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 - 2x - 8 = (x+2)(x-4).$$

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax - 4A + Bx + 2B}{(x+2)(x-4)}.$$

We need to find the constants A and B.

Coefficients of the numerators:

$$A + B = 1 \to 1$$
$$-4A + 2B = 1 \to 2$$

By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$.

Illustration

Multiply equation 1 by 4 and add the result to equation 2

$$4A + 4B = 4$$
$$-4A + 2B = 1$$

Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x^2-2x-8}\, dx = \int \frac{1/6}{x+2}\, dx + \int \frac{5/6}{x-4}\, dx = \frac{1}{6}\ln|\, x+2\,| + \frac{5}{6}\ln|\, x-4\,| + c.$$

Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x^2-2x-8}\,dx = \int \frac{1/6}{x+2}\,dx + \int \frac{5/6}{x-4}\,dx = \frac{1}{6}\ln|x+2| + \frac{5}{6}\ln|x-4| + c.$$

Example

Evaluate the integral $\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\,dx.$

Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x^2-2x-8}\, dx = \int \frac{1/6}{x+2}\, dx + \int \frac{5/6}{x-4}\, dx = \frac{1}{6}\ln|\,x+2\,| + \frac{5}{6}\ln|\,x-4\,| + c.$$

Example

Evaluate the integral $\displaystyle\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\, dx.$

Solution:

Step 1: Do the polynomial long-division.

Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do the polynomial long-division given on the right side. Then, we have

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2}.$$

$$\begin{array}{r|rrrr} & 2x & -10 & & \\ \hline x^2+3x+2) & 2x^3 & -4x^2 & -15x & +5 \\ & -(2x^3 & +6x^2 & +4x) & \\ \hline & & -10x^2 & -19x & +5 \\ & & -(-10x^2 & -30x & -20) \\ \hline & & & 11x & +25 \end{array}$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2 + 3x + 2 = (x + 1)(x + 2).$$

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Step 3: Find the partial fraction decomposition.

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}$$

We need to find the constants A and B.
Coefficients of the numerators:

$$A + B = 11 \rightarrow 1$$
$$2A + B = 25 \rightarrow 2$$

By doing some calculation, we have $A = 14$ and $B = -3$.

Illustration

$-2 \times 1 + 2$

$$-2A - 2B = -22$$
$$2A + B = 25$$

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$$-B = 3$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

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$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}$$

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Illustration

$-2 \times 1 + 2$

$$-2A - 2B = -22$$
$$2A + B = 25$$

$- - - - - - - - - -$

$$-B = 3$$

Step 4: Integrate the result of step 3.

$$\int q(x)\ dx = \int (2x-10)\ dx + \int \frac{14}{x+1}\ dx + \int \frac{-3}{x+2}\ dx$$
$$= x^2 - 10x + 14\ln|x+1| - 3\ln|x+2| + c.$$

Remark

1. *The number of constants* $A, B, C, etc.$ *is equal to the degree of the denominator* $g(x)$*. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of* $g(x)$*.*
2. *If the denominator* $g(x)$ *contains irreducible quadratic factors, the numerators of the partial fractions should be polynomials of degree one (see step 3 on page 59).*

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Example

Evaluate the integral $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx$.

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1. *The number of constants A, B, C, etc. is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.*
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Example

Evaluate the integral $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

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1. *The number of constants A, B, C, etc. is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.*
2. *If the denominator $g(x)$ contains irreducible quadratic factors, the numerators of the partial fractions should be polynomials of degree one (see step 3 on page 59).*

Example

Evaluate the integral $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Step 3: Find the partial fraction decomposition.

Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5} = \frac{A(x^2 - 4x - 5) + B(x-5) + C(x^2 + 2x +}{(x+1)^2(x-5)}$$

Coefficients of the numerators:

$$A + C = 2 \to 1$$
$$-4A + B + 2C = -25 \to 2$$
$$-5A - 5B + C = -33 \to 3$$

Illustration

$5 \times 2 + 3 =$
$-25A + 11C = -158 \to 4$
$25 \times 1 + 4 =$
$36C = -108 \Rightarrow C = -3$

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.
Step 4: Integrate the result of step 3.

$$\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\, dx = \int \frac{5}{x+1}\, dx + \int \frac{1}{(x+1)^2}\, dx + \int \frac{-3}{x-5}\, dx$$
$$= 5\ln|\, x+1\,| + \int (x+1)^{-2}\, dx - 3\ln|\, x-5\,|$$
$$= 5\ln|\, x+1\,| - \frac{1}{(x+1)} - 3\ln|\, x-5\,| + c.$$

Example

Evaluate the integral $\int \frac{x+1}{x(x^2+1)} dx$.

Example

Evaluate the integral $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Example

Evaluate the integral $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

$$A + B = 0 \rightarrow 1$$
$$C = 1 \rightarrow 2$$
$$A = 1 \rightarrow 3$$

We have $A = 1$, $B = -1$ and $C = 1$.

Example

Evaluate the integral $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:

Steps 1 and 2 can be skipped in this example.

Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

$$\begin{aligned} A+B &= 0 \to 1 \\ C &= 1 \to 2 \\ A &= 1 \to 3 \end{aligned}$$

We have $A = 1$, $B = -1$ and $C = 1$.

Step 4: Integrate the result of step 3.

$$\begin{aligned} \int \frac{x+1}{x(x^2+1)}\, dx &= \int \frac{1}{x}\, dx + \int \frac{-x+1}{x^2+1}\, dx \\ &= \ln|x| - \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + c. \end{aligned}$$

Integrals Involving Quadratic Forms

In this section, we provide a new technique for integrals that contain irreducible quadratic expressions $ax^2 + bx + c$ where $b \neq 0$. This technique is completing square method: $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Notes:

- If a quadratic polynomial has real roots, it is called reducible; otherwise it is called irreducible.
For the expression $ax^2 + bx + c$, if $b^2 - 4ac < 0$, then the quadratic expression is irreducible.
- To complete the square, we need to find $\left(\frac{b}{2a}\right)^2$, then add and subtract it.

Example

For the quadratic expression $x^2 - 6x + 13$, we have $a = 1, b = -6$ and $c = 13$. Since $b^2 - 4ac = -16 < 0$, then the quadratic expression is irreducible. To complete the square, we find $(\frac{b}{2a})^2 = 9$, then we add and substrate it as follows:

$$x^2 - 6x + 13 = \underbrace{x^2 - 6x + 9}_{=(x-3)^2} - \underbrace{9 + 13}_{=4}$$

Hence, $x^2 - 6x + 13 = (x - 3)^2 + 4$.

Example

Evaluate the integral $\int \frac{1}{x^2 - 6x + 13}\, dx$.

Example

Evaluate the integral $\int \frac{1}{x^2 - 6x + 13}\ dx$.

Solution:

The quadratic expression $x^2 - 6x + 13$ is irreducible. By completing the square, we have from the previous example

$$\int \frac{1}{x^2 - 6x + 13}\ dx = \int \frac{1}{(x-3)^2 + 4}\ dx.$$

Let $u = x - 3$, then $du = dx$. By substitution,

$$\int \frac{1}{u^2 + 4}\ du = \frac{1}{2}\tan^{-1}\ \frac{u}{2} + c = \frac{1}{2}\tan^{-1}\ (\frac{x-3}{2}) + c.$$

Example

Evaluate the integral $\int \frac{x}{x^2 - 4x + 8}\ dx$.

Example

Evaluate the integral $\int \frac{1}{x^2 - 6x + 13}\, dx$.

Solution:

The quadratic expression $x^2 - 6x + 13$ is irreducible. By completing the square, we have from the previous example

$$\int \frac{1}{x^2 - 6x + 13}\, dx = \int \frac{1}{(x-3)^2 + 4}\, dx.$$

Let $u = x - 3$, then $du = dx$. By substitution,

$$\int \frac{1}{u^2 + 4}\, du = \frac{1}{2} \tan^{-1} \frac{u}{2} + c = \frac{1}{2} \tan^{-1} \left(\frac{x-3}{2}\right) + c.$$

Example

Evaluate the integral $\int \frac{x}{x^2 - 4x + 8}\, dx$.

Solution:

For the quadratic expression $x^2 - 4x + 8$, we have $b^2 - 4ac < 0$. Therefore, the quadratic expression $x^2 - 4x + 8$ is irreducible. By completing the square, we obtain

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4$$
$$= (x-2)^2 + 4.$$

Hence
$$\int \frac{x}{x^2 - 4x + 8}\ dx = \int \frac{x}{(x-2)^2 + 4}\ dx.$$
Let $u = x - 2$, then $du = dx$. By substitution,
$$\int \frac{u+2}{u^2+4}\ du = \int \frac{u}{u^2+4}\ du + \int \frac{2}{u^2+4}\ du$$
$$= \frac{1}{2} \ln |\ u^2 + 4\ | + \tan^{-1}\ \frac{u}{2}$$
$$= \frac{1}{2} \ln \left((x-2)^2 + 4\right) + \tan^{-1}\ \left(\frac{x-2}{2}\right) + c$$
$$= \frac{1}{2} \ln \left(x^2 - 4x + 8\right) + \tan^{-1}\ \left(\frac{x-2}{2}\right) + c.$$

Example

Evaluate the integral $\int \frac{1}{\sqrt{2x - x^2}}\ dx.$

Solution:

By completing the square, we have
$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$. Hence

$$\int \frac{1}{\sqrt{2x - x^2}}\, dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\, dx.$$

Let $u = x - 1$, then $du = dx$. By substitution, the integral becomes

$$\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + c = \sin^{-1}(x-1) + c.$$

Solution:

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$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$. Hence

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Example

Evaluate the integral $\int \sqrt{x^2 + 2x - 1}\, dx$.

Solution:

By completing the square, we have
$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$. Hence

$$\int \frac{1}{\sqrt{2x - x^2}}\, dx = \int \frac{1}{\sqrt{1 - (x-1)^2}}\, dx.$$

Let $u = x - 1$, then $du = dx$. By substitution, the integral becomes

$$\int \frac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1} u + c = \sin^{-1}(x-1) + c.$$

Example

Evaluate the integral $\int \sqrt{x^2 + 2x - 1}\, dx$.

Solution:

By completing the square, we have $x^2 + 2x - 1 = (x^2 + 2x + 1) - 1 - 1 = (x+1)^2 - 2$.
Hence,

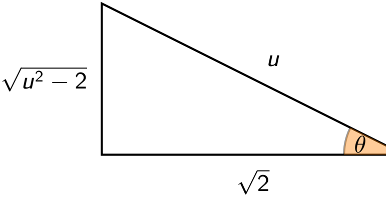
$$\int \sqrt{x^2 + 2x - 1}\, dx = \int \sqrt{(x+1)^2 - 2}\, dx.$$

Let $u = x + 1$, then $du = dx$. The integral becomes $\int \sqrt{u^2 - 2}\, du$.

Use the trigonometric substitutions, in particular let

$$u = \sqrt{2}\ \sec\ \theta \Rightarrow du = \sqrt{2}\ \sec\ \theta\ \tan\ \theta\ d\theta$$

where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$. By substitution, we have



$$2\int \tan^2\ \theta\ \sec\ \theta\ d\theta = 2\int (\sec^3\ \theta - \sec\ \theta)\ d\theta.$$

From Example 8, we have

$$2\int (\sec^3\ \theta - \sec\ \theta)\ d\theta = \sec\ \theta\ \tan\ \theta - \ln\left|\sec\ \theta + \tan\ \theta\right| + c.$$

By returning to the variable u and then to x,

$$\int \sqrt{u^2 - 2}\ du = \frac{u\sqrt{u^2 - 2}}{2} - \ln\left|\frac{u + \sqrt{u^2 - 2}}{\sqrt{2}}\right| + c = \frac{(x+1)\sqrt{(x+1)^2 - 2}}{2} - \ln\left|\frac{x + 1 + \sqrt{}}{}\right.$$

Miscellaneous Substitutions

(A) Fractional Functions in sin x and cos x

The integrals that consist of rational expressions in $\sin\ x$ and $\cos\ x$ are treated by using the substitution $u = \tan\ (x/2)$, $-\pi < x < \pi$. This implies that $du = \frac{\sec^2\ (x/2)}{2}\ dx$ and since $\sec^2\ x = \tan^2\ x + 1$, then $du = \frac{u^2+1}{2}\ dx$. Also,

$$\begin{aligned}
\sin\ x &= \sin\ 2(\frac{x}{2}) \\
&= 2\ \sin\ \frac{x}{2}\ \cos\ \frac{x}{2} && \text{(multiply and divide by } \cos\ \tfrac{x}{2}\text{)} \\
&= 2\frac{\sin\ \frac{x}{2}}{\cos\ \frac{x}{2}}\cos\ \frac{x}{2}\ \cos\ \frac{x}{2} \\
&= 2\tan\ \frac{x}{2}\ \cos^2\ \frac{x}{2} && (\cos\ x = \tfrac{1}{\sec\ x}) \\
&= \frac{2\tan\ \frac{x}{2}}{\sec^2\ \frac{x}{2}} \\
&= \frac{2u}{u^2+1}.
\end{aligned}$$

For $\cos\ x$, we have $\cos\ x = \cos\ 2(\frac{x}{2}) = \cos^2\ \frac{x}{2} - \sin^2\ \frac{x}{2}$.

We can find that $\cos \frac{x}{2} = \frac{1}{\sqrt{u^2+1}}$ and $\sin\ \frac{x}{2} = \frac{u}{\sqrt{u^2+1}}$. (use the identities $\sec^2\ \frac{x}{2} = \tan^2\ \frac{x}{2} + 1$ and $\cos^2\ \frac{x}{2} + \sin^2\ \frac{x}{2} = 1$)

This implies $\cos\ x = \frac{1-u^2}{1+u^2}$.

Theorem

For an integral that contains a rational expression in $\sin x$ *and* $\cos x$*, we assume*

$$\sin\ x = \frac{2u}{1+u^2}, \quad \textit{and} \quad \cos\ x = \frac{1-u^2}{1+u^2}.$$

to produce a rational expression in u *where* $u = \tan\ (x/2)$*, and* $du = \frac{1+u^2}{2}\ dx$*.*

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{1+\sin\ x}\ dx$
2. $\displaystyle\int \frac{1}{2+\cos\ x}\ dx$
3. $\displaystyle\int \frac{1}{1+\sin\ x+\cos\ x}\ dx$

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Example

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2. $\displaystyle\int \frac{1}{2+\cos\ x}\ dx$
3. $\displaystyle\int \frac{1}{1+\sin\ x+\cos\ x}\ dx$

Solution:

1) Let $u = \tan\ \frac{x}{2}$, then $du = \frac{1+u^2}{2}\ dx$ and $\sin x = \frac{2u}{1+u^2}$. By substituting that into the integral, we have

$$\int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\ du = 2\int \frac{1}{u^2+2u+1}\ du = 2\int (u+1)^{-2}\ du = \frac{-2}{u+1} + c$$

$$= \frac{-2}{\tan\ x/2+1} + c.$$

2) Let $u = \tan\ \frac{x}{2}$, then $du = \frac{1+u^2}{2}\ dx$ and $\cos\ x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\ du &= 2\int \frac{1}{u^2+3}\ du \\ &= \frac{2}{\sqrt{3}}\ \tan^{-1}\ \frac{u}{\sqrt{3}} + c \\ &= \frac{2}{\sqrt{3}}\ \tan^{-1}\ \left(\frac{\tan\ x/2}{\sqrt{3}}\right) + c.\end{aligned}$$

2) Let $u = \tan \ \frac{x}{2}$, then $du = \frac{1+u^2}{2} \ dx$ and $\cos \ x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} \ du &= 2 \int \frac{1}{u^2+3} \ du \\ &= \frac{2}{\sqrt{3}} \ \tan^{-1} \ \frac{u}{\sqrt{3}} + c \\ &= \frac{2}{\sqrt{3}} \ \tan^{-1} \ \left(\frac{\tan \ x/2}{\sqrt{3}}\right) + c.\end{aligned}$$

3) Let $u = \tan \ \frac{x}{2}$, this implies $du = \frac{1+u^2}{2} \ dx$, $\sin \ x = \frac{2u}{1+u^2}$ and $\cos \ x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} \ du &= \int \frac{2}{2+2u} \ du \\ &= \int \frac{1}{1+u} \ du \\ &= \ln \mid 1+u \mid + c \\ &= \ln \left|1+\tan \ \frac{x}{2}\right| + c.\end{aligned}$$

Integrals of Fractional Powers

In the case of an integrand that consists of fractional powers, it is better to use the substitution $u = x^{\frac{1}{n}}$ where n is the least common multiple of the denominators of the powers. In the following, we provide an example.

Example

Evaluate the integral $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}}\, dx$.

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In the case of an integrand that consists of fractional powers, it is better to use the substitution $u = x^{\frac{1}{n}}$ where n is the least common multiple of the denominators of the powers. In the following, we provide an example.

Example

Evaluate the integral $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}} \, dx$.

Solution:

Let $u = x^{\frac{1}{4}}$, we find $x = u^4$ and $dx = 4u^3 du$. Therefore, $x^{\frac{1}{2}} = (x^{\frac{1}{4}})^2 = u^2$.
By substitution, we have

$$\begin{aligned}\int \frac{1}{u^2 + u} 4u^3 \, du &= 4 \int \frac{u^2}{u+1} \, du \\ &= 4 \int (u-1) \, du + 4 \int \frac{1}{1+u} \, du \\ &= 2u^2 - 4u + 4 \ln |\, u + 1 \,| + c \\ &= 2\sqrt{x} - 4\sqrt[4]{x} + 4 \ln |\, \sqrt[4]{x} + 1 \,| + c.\end{aligned}$$

Integrals of Form $\sqrt[n]{f(x)}$

If the integrand is of from $\sqrt[n]{f(x)}$, it is useful to assume $u = \sqrt[n]{f(x)}$. This case differs from that given in the substitution method in Chapter **??** i.e., $\sqrt[n]{f(x)}\ f'(x)$ and the difference lies on the existence of the derivative of $f(x)$.

Example

Evaluate the integral $\int \sqrt{e^x + 1}\ dx$.

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Example

Evaluate the integral $\int \sqrt{e^x + 1}\ dx$.

Solution:

Let $u = \sqrt{e^x + 1}$, we obtain $du = \frac{e^x}{2\sqrt{e^x+1}}\ dx$ and $u^2 = e^x + 1$. By substitution, we have

$$\begin{aligned}
\int \frac{2u^2}{u^2-1}\ du &= \int 2\ du + 2\int \frac{1}{u^2-1}\ du \\
&= 2u + \int \frac{1}{u-1}\ du + \int \frac{1}{u+1}\ du \\
&= 2u + \ln \mid u-1 \mid - \ln \mid u+1 \mid + c \\
&= 2\sqrt{e^x+1} + \ln(\sqrt{e^x+1} - 1) - \ln(\sqrt{e^x+1} + 1) + c.
\end{aligned}$$