Integral Calculus

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Inverse Trigonometric Functions

The most common notations to name the inverse trigonometric functions are *arcsinx*, *arccosx*, *arctanx*, etc. However, the notations $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, etc. are often used as well. In this book, we use the latter notations to denote to the inverse trigonometric functions.

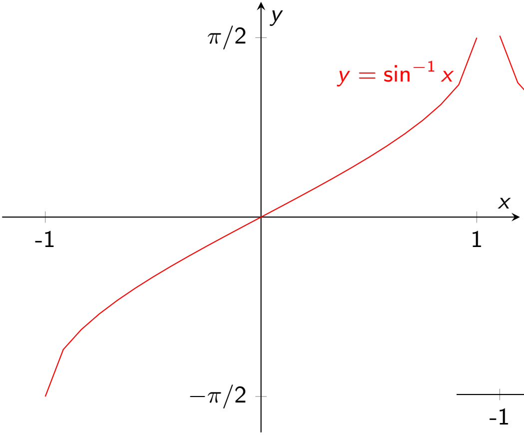
Common mistake: some students write $\sin^{-1}\ x = (\sin\ x)^{-1} = \frac{1}{\sin\ x}$ and this is not true.

To find the inverse of any function, we need to show that the function is bijective (i.e., is it one-to-one and onto?). From your knowledge, none of the six trigonometric functions are bijective. Therefore, in order to have inverse trigonometric functions, we should consider subsets of their domains. In the following, we show the graph of the inverse trigonometric functions, and their domains and ranges.

The inverse sine function

$\sin\ y = x \Leftrightarrow y = \sin^{-1}\ x$

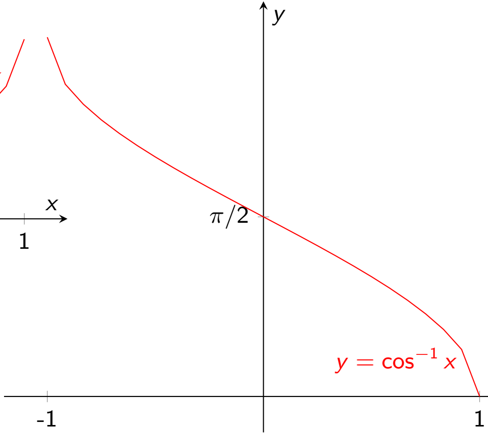
Domain: $[-1,1]$ Range: $[-\frac{\pi}{2},\frac{\pi}{2}]$



The inverse cosine function

$\cos\ y = x \Leftrightarrow y = \cos^{-1}\ x$

Domain: $[-1,1]$ Range: $[0,\pi]$

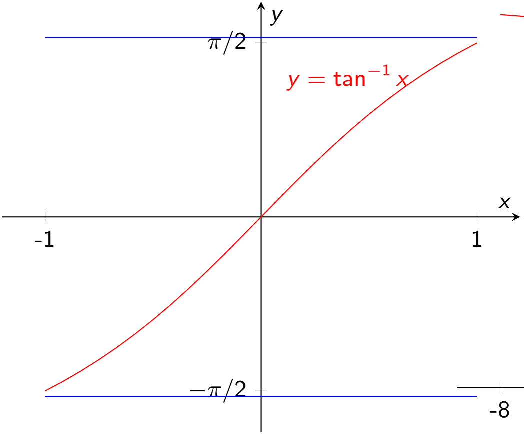


The inverse tangent function

$\tan\ y = x \Leftrightarrow y = \tan^{-1}\ x$

Domain: $\mathbb{R}$

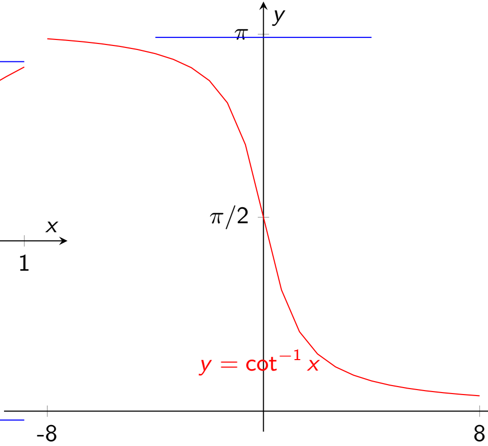
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



The inverse cotangent function

$\cot\ y = x \Leftrightarrow y = \cot^{-1}\ x$

Domain: $\mathbb{R}$ Range: $(0, \pi)$



The inverse secant function

$\sec y = x \Leftrightarrow y = \sec^{-1} x$

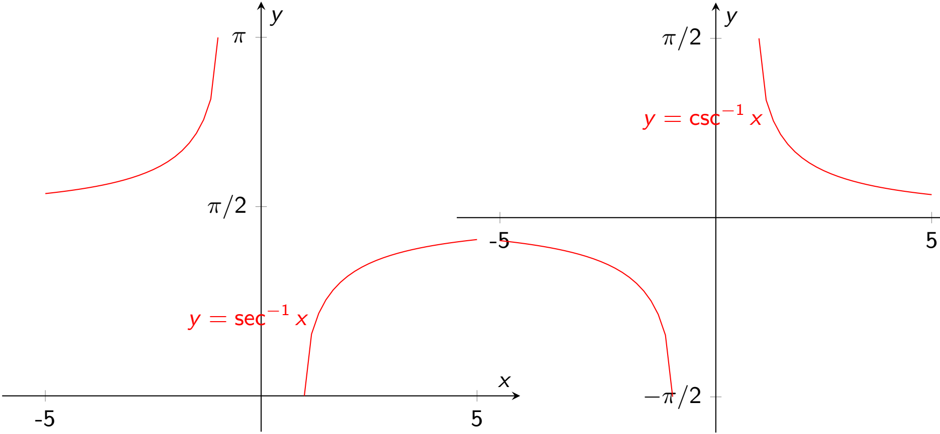
Domain: $\mathbb{R} \setminus (-1, 1)$ Range: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

The inverse cosecant function

$\csc y = x \Leftrightarrow y = \csc^{-1} x$

Domain: $\mathbb{R} \setminus (-1, 1)$

Range: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



Differentiating and Integrating the Inverse Trigonometric Functions

Theorem

If $u = g(x)$ is a differentiable function, then

1. $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} u'$
2. $\frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} u'$
3. $\frac{d}{dx} \tan^{-1} u = \frac{1}{u^2+1} u'$
4. $\frac{d}{dx} \cot^{-1} u = \frac{-1}{u^2+1} u'$
5. $\frac{d}{dx} \sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} u'$
6. $\frac{d}{dx} \csc^{-1} u = \frac{-1}{u\sqrt{u^2-1}} u'$

Example

Find the derivative of the function.

1. $y = \sin^{-1} 5x$
2. $y = \tan^{-1} e^x$
3. $y = \sec^{-1} 2x$
4. $y = \sin^{-1} (x-1)$

Solution:

1. $y' = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}.$

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2. $y' = \frac{e^x}{(e^x)^2+1} = \frac{e^x}{e^{2x}+1}$.

Solution:

1. $y' = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$.
2. $y' = \frac{e^x}{(e^x)^2+1} = \frac{e^x}{e^{2x}+1}$.
3. $y' = \frac{2}{2x\sqrt{4x^2-1}} = \frac{1}{x\sqrt{4x^2-1}}$.

Solution:

1. $y' = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}.$
2. $y' = \frac{e^x}{(e^x)^2+1} = \frac{e^x}{e^{2x}+1}.$
3. $y' = \frac{2}{2x\sqrt{4x^2-1}} = \frac{1}{x\sqrt{4x^2-1}}.$
4. $y' = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}.$

Solution:

1. $y' = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$.
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4. $y' = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}$.

From the list of the derivatives of the inverse trigonometric functions, we have the following integration rules:

1. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$
2. $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$
3. $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + c$

Theorem

For $a > 0$,

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{4 - 25x^2}}\, dx.$
2. $\displaystyle\int \frac{1}{x\sqrt{x^6 - 4}}\, dx.$
3. $\displaystyle\int \frac{1}{9x^2 + 5}\, dx.$
4. $\displaystyle\int \frac{1}{\sqrt{e^{2x} - 1}}\, dx.$

Solution:

1. $$\int \frac{1}{\sqrt{4-25x^2}}\,dx = \int \frac{1}{\sqrt{4-(5x)^2}}\,dx.$$
Let $u = 5x$, then $du = 5dx \Rightarrow dx = \frac{du}{5}$. By substitution, we have
$$\frac{1}{5}\int \frac{1}{\sqrt{4-u^2}}\,du = \frac{1}{5}\sin^{-1}\frac{u}{2} + c = \frac{1}{5}\sin^{-1}\frac{5x}{2} + c.$$

2. $$\int \frac{1}{x\sqrt{x^6-4}}\,dx = \int \frac{1}{x\sqrt{(x^3)^2-4}}\,dx.$$
Let $u = x^3$, then $du = 3x^2dx$. By substitution, we obtain
$$\frac{1}{3}\int \frac{1}{u\sqrt{u^2-4}}\,du = \frac{1}{3}\frac{1}{2}\sec^{-1}\frac{u}{2} + c = \frac{1}{6}\sec^{-1}\frac{x^3}{2} + c.$$

3. $$\int \frac{1}{9x^2+5}\,dx = \int \frac{1}{(3x)^2+5}\,dx.$$
Let $u = 3x$, then $du = 3dx$. By substitution, we have
$$\frac{1}{3}\int \frac{1}{u^2+5}\,du = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + c = \frac{1}{3\sqrt{5}}\tan^{-1}\frac{3x}{\sqrt{5}} + c.$$

4. $$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{\sqrt{(e^x)^2-1}}\,dx.$$
Let $u = e^x$, $du = e^x dx$. After substitution, we have
$$\int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}u + c = \sec^{-1}e^x + c.$$

Definition

The hyperbolic sine function (sinh) *and the hyperbolic cosine function* (cosh) *are defined as follows:*

$$\sinh\ x = \frac{e^x - e^{-x}}{2},\ \forall x \in \mathbb{R}\ ,$$

$$\cosh\ x = \frac{e^x + e^{-x}}{2},\ \forall x \in \mathbb{R}.$$

Other hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

$$\tanh\ x = \frac{\sinh\ x}{\cosh\ x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},\quad \forall x \in \mathbb{R}$$

$$\coth\ x = \frac{\cosh\ x}{\sinh\ x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},\quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\text{sech}\ x = \frac{1}{\cosh\ x} = \frac{2}{e^x + e^{-x}},\quad \forall x \in \mathbb{R}$$

$$\text{csch}\ x = \frac{1}{\sinh\ x} = \frac{2}{e^x - e^{-x}},\quad \forall x \in \mathbb{R} \setminus \{0\}$$

Properties of the Hyperbolic Functions

1) The graph of the hyperbolic functions depends on the natural exponential functions e^x and e^{-x} (as shown in Figure **??**).

2) The hyperbolic sine function is an odd function (i.e., $\sinh(-x) = -\sinh x$); whereas the hyperbolic cosine is an even function (i.e., $\cosh(-x) = \cosh x$). Therefore, the functions tanh, coth and csch are odd functions and the function sech is an even function. This in turn indicates that the graphs of the functions sinh, tanh, coth and csch are symmetric with respect to the original point; whereas the graph of the functions cosh and sech are symmetric around the y-axis.

3) $\cosh^2 x - \sinh^2 x = 1, \ \forall x \in \mathbb{R}$.

To verify this item, we have from Definition .1 that

$$\cosh x - \sinh x = e^{-x} \quad \text{and} \quad \cosh x + \sinh x = e^x.$$

Hence,

$$(\cosh x - \sinh x)(\cosh x + \sinh x) = \cosh^2 x - \sinh^2 x = e^{-x}e^x = e^0 = 1.$$

4) Since $\cos^2 t + \sin^2 t = 1$ for any $t \in \mathbb{R}$, then the point $P(\cos t, \sin t)$ is located on the unit circle $x^2 + y^2 = 1$. However, for any $t \in \mathbb{R}$, the point $P(\cosh t, \sinh t)$ is located on the hyperbola $x^2 - y^2 = 1$. Figure 1 illustrates this item.

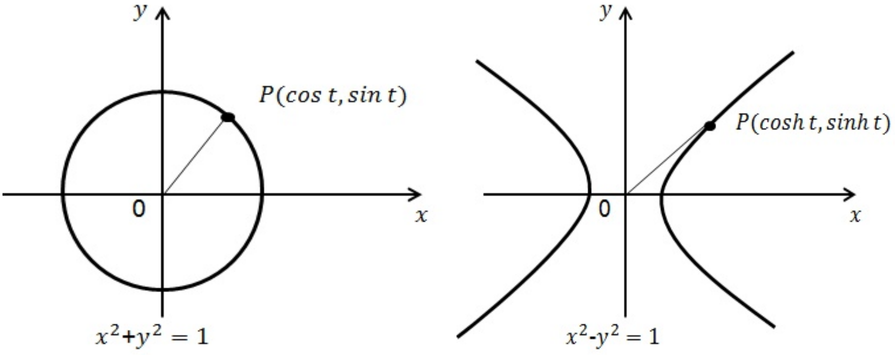
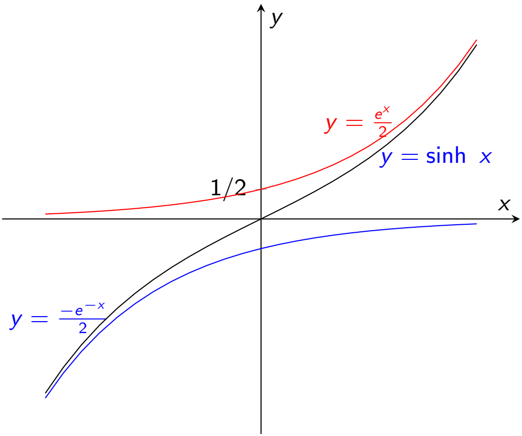
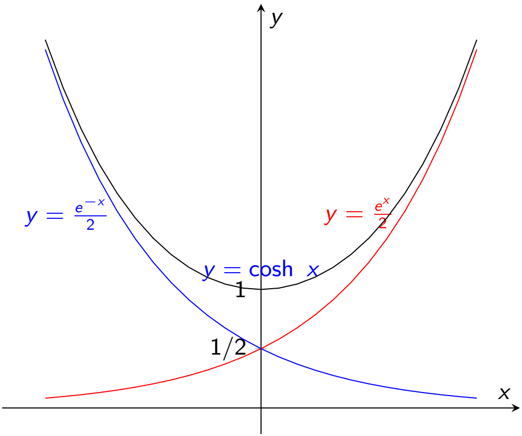
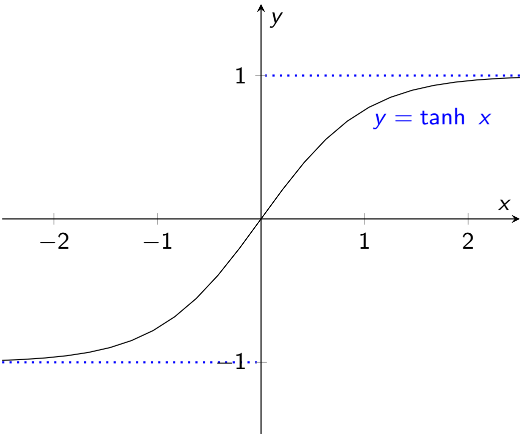
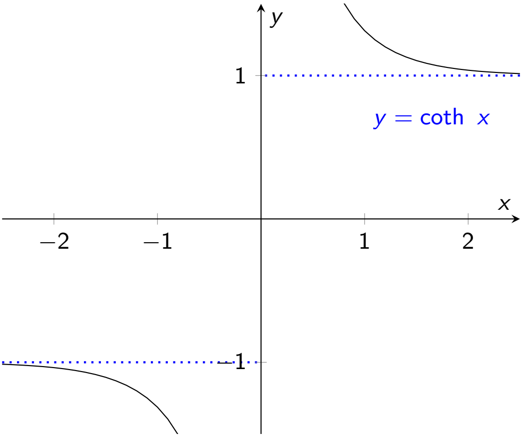


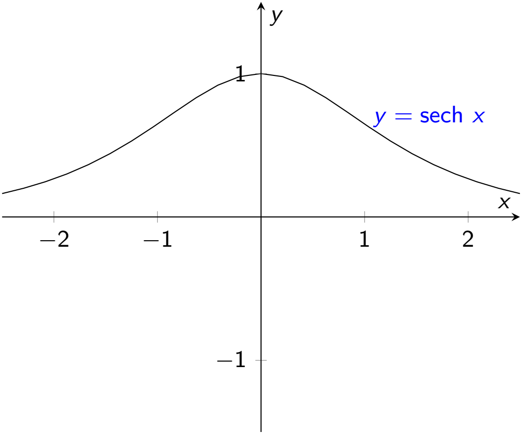
Figure: sinh x and cosh x versus sin x and cos x.

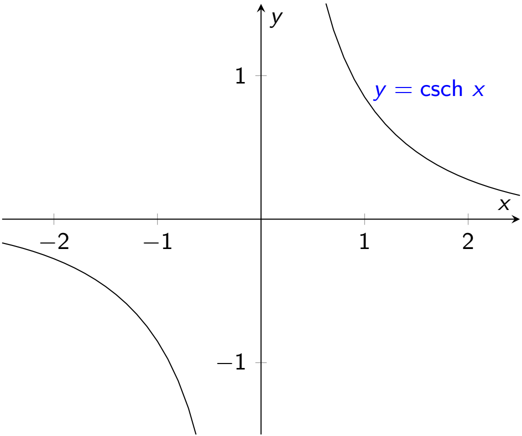

y
x
1/2
y = e^x/2
y = sinh x
y = −e^{−x}/2


y
$y = \frac{e^{-x}}{2}$
$y = \frac{e^{x}}{2}$
$y = \cosh\ x$
1
1/2
x


y
1
y = tanh x
x
−2
−1
1
2
−1


y
1
y = coth x
x
−2
−1
1
2
−1


y
1
y = sech x
x
−2
−1
1
2
−1


y
1
y = csch x
x
−2
−1
1
2
−1

Theorem

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
3. $\sinh 2x = 2 \sinh x \cosh x$
4. $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$
5. $1 - \tanh^2 x = sech^2\, x$
6. $\coth^2 x - 1 = csch^2\, x$
7. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
8. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

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Differentiating and Integrating the Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx} \sinh u = \cosh u \; u'$
2. $\frac{d}{dx} \cosh u = \sinh u \; u'$
3. $\frac{d}{dx} \tanh u = sech^2 u \; u'$
4. $\frac{d}{dx} \coth u = -csch^2 u \; u'$
5. $\frac{d}{dx} sech\, u = -sech\, u \tanh u \; u'$
6. $\frac{d}{dx} csch\, u = -csch\, u \coth u \; u'$

Example

Find the derivative of the functions.

1. $y = \sinh\,(x^2)$
2. $y = \sqrt{x}\ \cosh\ x$
3. $y = e^{\sinh\ x}$
4. $y = (x+1)\ \tanh^2\,(x^3)$

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Solution:

1. $y' = 2x\ \cosh(x^2)$.
2. $y' = \frac{1}{2\sqrt{x}}\ \cosh\ x + \sqrt{x}\ \sinh\ x$.

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3. $y' = e^{\sinh\ x}\ \cosh\ x$.
4. $y' = \tanh^2(x^3) + 6x^2(x+1)\ \tanh(x^3)\ \text{sech}^2(x^3)$.

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Example

Find $\frac{dy}{dx}$ if $y = x^{\cosh\ x}$.

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Example

Find $\frac{dy}{dx}$ if $y = x^{\cosh\ x}$.

Solution: Take the natural logarithm of each side to have

$$\ln y = \cosh\ x\ \ln x.$$

Example

Find the derivative of the functions.

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3. $y' = e^{\sinh\,x}\,\cosh\,x$.
4. $y' = \tanh^2\,(x^3) + 6x^2(x+1)\,\tanh\,(x^3)\,\text{sech}^2\,(x^3)$.

Example

Find $\frac{dy}{dx}$ if $y = x^{\cosh\,x}$.

Solution: Take the natural logarithm of each side to have

$$\ln y = \cosh\,x\,\ln x.$$

By differentiating both sides, we obtain $\frac{y'}{y} = \sinh\,x\,\ln x + \frac{\cosh\,x}{x}$. Therefore,

$$y' = \left[\sinh\,x\,\ln x + \frac{\cosh\,x}{x}\right] x^{\cosh\,x}.$$

Theorem

- $\int \sinh\ x\ dx = \cosh\ x + c$
- $\int \cosh\ x\ dx = \sinh\ x + c$
- $\int sech^2\ x\ dx = \tanh\ x + c$
- $\int csch^2\ x\ dx = -\coth\ x + c$
- $\int sech\ x\ \tanh\ x\ dx = -sech\ x + c$
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Example

Evaluate the integral.

1. $\int \sinh^2\ x\ \cosh\ x\ dx$
2. $\int e^{\cosh\ x}\ \sinh\ x\ dx$
3. $\int \tanh\ x\ dx$
4. $\int e^x\ \text{sech}\ x\ dx$

Solution:

1. Let $u = \sinh\ x$, then $du = \cosh\ x\ dx$. By substitution, we have $\int u^2\ du = u^3/3 + c$. Hence,

$$\int \sinh^2\ x\ \cosh\ x\ dx = \frac{\sinh^3\ x}{3} + c.$$

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$$\int e^{\cosh\ x} \sinh\ x\ dx = e^{\cosh\ x} + c.$$

3. We know that $\tanh\ x = \frac{\sinh\ x}{\cosh\ x}$, so $\int \tanh\ x\ dx = \int \frac{\sinh\ x}{\cosh\ x}\ dx$.
Let $u = \cosh\ x$, then $du = \sinh\ x\ dx$. By substitution, we have $\int \frac{1}{u}\ du = \ln \mid u \mid + c$.
This implies
$$\int \tanh\ x\ dx = \ln \cosh\ x + c.$$

Solution:

1. Let $u = \sinh\ x$, then $du = \cosh\ x\ dx$. By substitution, we have $\int u^2\ du = u^3/3 + c$. Hence,
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2. Let $u = \cosh\ x$, then $du = \sinh\ x\ dx$. By substitution, we have $\int e^u\ du = e^u + c$. Hence,
$$\int e^{\cosh\ x} \sinh\ x\ dx = e^{\cosh\ x} + c.$$

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This implies
$$\int \tanh\ x\ dx = \ln \cosh\ x + c.$$

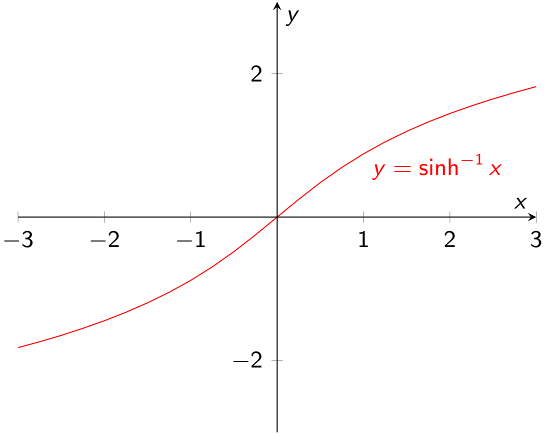
4. $\int e^x\ \text{sech}\ x\ dx = \int \frac{2e^x}{e^x + e^{-x}}\ dx = \int \frac{2e^{2x}}{e^{2x} + 1}\ dx.$
Let $u = e^{2x}$, then $du = 2e^{2x}\ dx$. By substitution, we have
$\int \frac{1}{u+1}\ du = \ln \mid u + 1 \mid + c = \ln\left(e^{2x} + 1\right) + c.$

Inverse Hyperbolic Functions
Properties the Inverse Hyperbolic Functions

- The function $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective, so it has an inverse function

$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$

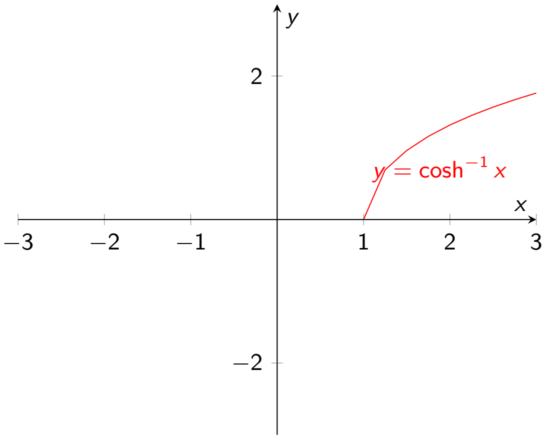
$$\sinh\ y = x \Leftrightarrow y = \sinh^{-1}\ x$$



■ The function cosh is injective on $[0, \infty)$, so $\cosh : [0, \infty) \to [1, \infty)$ is bijective on $[0, \infty)$. It has an inverse function

$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$

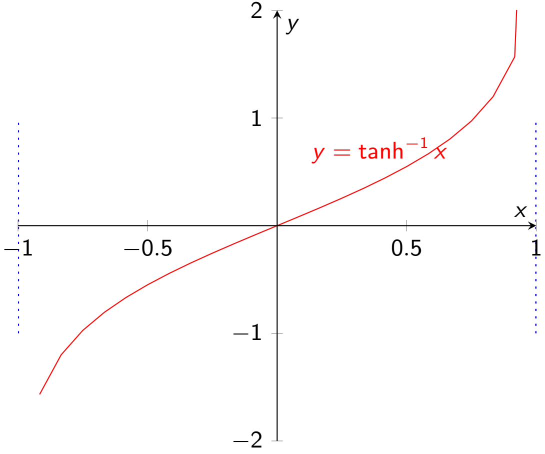
$$\cosh \; y = x \Leftrightarrow y = \cosh^{-1} \; x$$



- The function $\tanh : \mathbb{R} \to (-1, 1)$ is bijective, so it has an inverse function

$$\tanh^{-1} : (-1, 1) \to \mathbb{R}$$

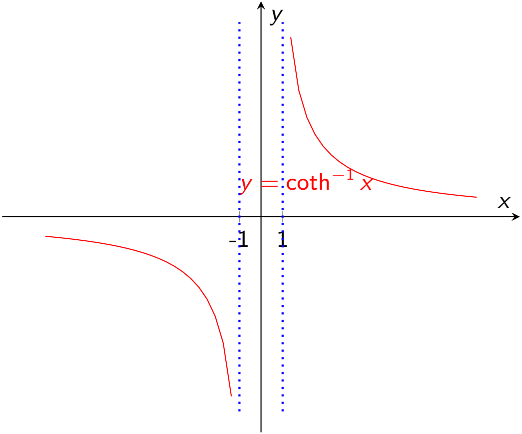
$$\tanh \ y = x \Leftrightarrow y = \tanh^{-1} \ x$$



The function coth : $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1, 1]$ is bijective, so it has an inverse function

$$\coth^{-1} : \mathbb{R} \setminus [-1, 1] \to \mathbb{R} \setminus \{0\}$$

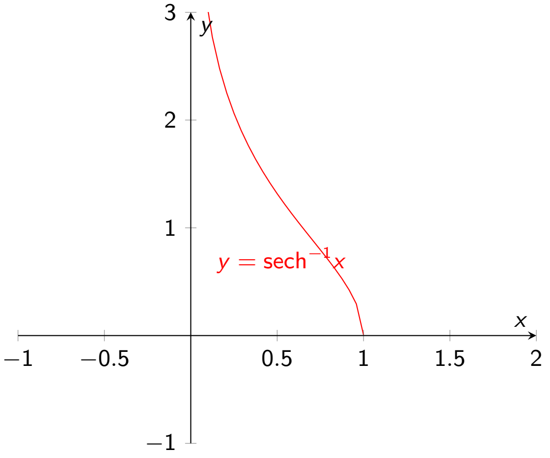
$$\coth\ y = x \Leftrightarrow y = \coth^{-1}\ x$$



The function sech is bijective on $[0, \infty)$, so sech : $[0, \infty) \to (0, 1]$ has an inverse function

$$\text{sech}^{-1} : (0, 1] \to [0, \infty)$$

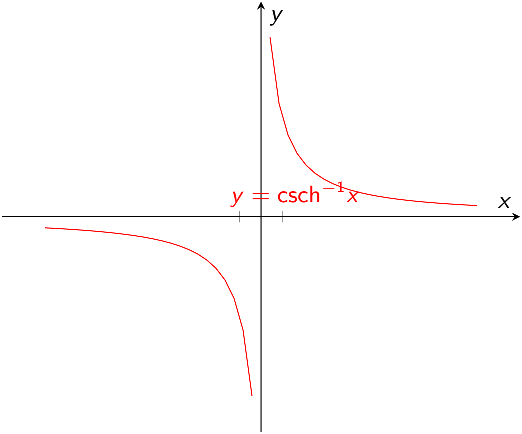
$$\text{sech } y = x \Leftrightarrow y = \text{sech}^{-1}\ x$$



The function csch : $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is bijective. The inverse function is

$$\text{csch}^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$$

$$\text{csch}\, y = x \Leftrightarrow y = \text{csch}^{-1}\, x$$



Theorem

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}),\ \forall x \in \mathbb{R}$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\ \forall x \in [1, \infty)$
3. $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),\ \forall x \in (-1, 1)$
4. $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right),\ \forall x \in \mathbb{R} \setminus [-1, 1]$
5. $sech^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ \forall x \in (0, 1]$
6. $csch^{-1} x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right),\ \forall x \in \mathbb{R} \setminus \{0\}$

Differentiating and Integrating the Inverse Hyperbolic Functions

Theorem

If $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx} \sinh^{-1} \ u = \frac{1}{\sqrt{u^2+1}} u'$
2. $\frac{d}{dx} \cosh^{-1} \ u = \frac{1}{\sqrt{u^2-1}} u', \ \forall u \in (1, \infty)$
3. $\frac{d}{dx} \tanh^{-1} \ u = \frac{1}{1-u^2} u', \ \forall u \in (-1, 1)$
4. $\frac{d}{dx} \coth^{-1} \ u = \frac{1}{1-u^2} u', \ \forall u \in \mathbb{R} \setminus [-1, 1]$
5. $\frac{d}{dx} sech^{-1} \ u = \frac{-1}{u\sqrt{1-u^2}} u', \ \forall u \in (0, 1)$
6. $\frac{d}{dx} csch^{-1} \ u = \frac{-1}{|u|\sqrt{u^2+1}} u', \ \forall u \in \mathbb{R} \setminus \{0\}$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
3. $y = \cosh^{-1} (4x^2)$
4. $y = \ln(\sinh^{-1} x)$
5. $y = \operatorname{csch}^{-1} 4x$
6. $y = x \tanh^{-1} \frac{1}{x}$
7. $y = (\tanh^{-1} x)^2$
8. $y = e^x \operatorname{sech}^{-1} x$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1}\ \sqrt{x}$
2. $y = \tanh^{-1}\ e^x$
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6. $y = x\ \tanh^{-1}\ \frac{1}{x}$
7. $y = (\tanh^{-1}\ x)^2$
8. $y = e^x\ \operatorname{sech}^{-1}\ x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}}\ \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.

Example

Find the derivative of the functions.

1. $y = \sinh^{-1}\ \sqrt{x}$
2. $y = \tanh^{-1}\ e^x$
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6. $y = x\ \tanh^{-1}\ \frac{1}{x}$
7. $y = (\tanh^{-1}\ x)^2$
8. $y = e^x\ \operatorname{sech}^{-1}\ x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}}\ \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
3. $y = \cosh^{-1} (4x^2)$
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7. $y = (\tanh^{-1} x)^2$
8. $y = e^x \operatorname{sech}^{-1} x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}$.

Example

Find the derivative of the functions.

1. $y = \sinh^{-1}\sqrt{x}$
2. $y = \tanh^{-1} e^x$
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8. $y = e^x \operatorname{sech}^{-1} x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}.$
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}.$
4. $y' = \frac{1}{\sinh^{-1} x} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}\,\sinh^{-1} x}.$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
3. $y = \cosh^{-1} (4x^2)$
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Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}$.
4. $y' = \frac{1}{\sinh^{-1} x} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1} \sinh^{-1} x}$.
5. $y' = \frac{-4}{|4x|\sqrt{16x^2+1}} = \frac{-1}{|x|\sqrt{16x^2+1}}$.

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
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Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}$.
4. $y' = \frac{1}{\sinh^{-1} x} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1} \sinh^{-1} x}$.
5. $y' = \frac{-4}{|4x|\sqrt{16x^2+1}} = \frac{-1}{|x|\sqrt{16x^2+1}}$.
6. $y' = \tanh^{-1}(\frac{1}{x}) + x \left(\frac{1}{1-(\frac{1}{x})^2}\right)\left(\frac{-1}{x^2}\right) = \tanh^{-1}(\frac{1}{x}) - \frac{x}{x^2-1}$.

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
3. $y = \cosh^{-1} (4x^2)$
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6. $y = x \tanh^{-1} \frac{1}{x}$
7. $y = (\tanh^{-1} x)^2$
8. $y = e^x \operatorname{sech}^{-1} x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}.$
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}.$
4. $y' = \frac{1}{\sinh^{-1} x} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1} \sinh^{-1} x}.$
5. $y' = \frac{-4}{|4x|\sqrt{16x^2+1}} = \frac{-1}{|x|\sqrt{16x^2+1}}.$
6. $y' = \tanh^{-1}(\frac{1}{x}) + x \left(\frac{1}{1-(\frac{1}{x})^2}\right)(\frac{-1}{x^2}) = \tanh^{-1}(\frac{1}{x}) - \frac{x}{x^2-1}.$
7. $y' = 2(\tanh^{-1} x) \frac{1}{1-x^2} = \frac{2\tanh^{-1} x}{1-x^2}$

Example

Find the derivative of the functions.

1. $y = \sinh^{-1} \sqrt{x}$
2. $y = \tanh^{-1} e^x$
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7. $y = (\tanh^{-1} x)^2$
8. $y = e^x \operatorname{sech}^{-1} x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}.$
2. $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}.$
3. $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}.$
4. $y' = \frac{1}{\sinh^{-1} x} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1} \sinh^{-1} x}.$
5. $y' = \frac{-4}{|4x|\sqrt{16x^2+1}} = \frac{-1}{|x|\sqrt{16x^2+1}}.$
6. $y' = \tanh^{-1}(\frac{1}{x}) + x\,(\frac{1}{1-(\frac{1}{x})^2})(\frac{-1}{x^2}) = \tanh^{-1}(\frac{1}{x}) - \frac{x}{x^2-1}.$
7. $y' = 2(\tanh^{-1} x) \frac{1}{1-x^2} = \frac{2\tanh^{-1} x}{1-x^2}$
8. $y' = e^x \operatorname{sech}^{-1} x - \frac{e^x}{x\sqrt{1-x^2}}$

- $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}\ x + c$
- $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}\ x + c,\ x > 1$
- $\displaystyle\int \frac{1}{1-x^2}\,dx = \tanh^{-1}\ x + c,\ \mid x \mid < 1$
- $\displaystyle\int \frac{1}{1-x^2}\,dx = \coth^{-1}\ x + c,\ \mid x \mid > 1$
- $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}\ \mid x \mid + c,\ \mid x \mid < 1$
- $\displaystyle\int \frac{1}{x\sqrt{x^2+1}}\,dx = -\operatorname{csch}^{-1}\ \mid x \mid + c,\ \mid x \mid > 1$

Theorem

1. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \sinh^{-1} \frac{x}{a} + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1} \frac{x}{a} + c,\ x > a$
3. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\tanh^{-1} \frac{x}{a} + c,\ |\, x \,| < a$
4. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\coth^{-1} \frac{x}{a} + c,\ |\, x \,| > a$
5. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = -\frac{1}{a} sech^{-1} \frac{|x|}{a} + c,\ |\, x \,| < a$
6. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = -\frac{1}{a} csch^{-1} \frac{|x|}{a} + c,\ |\, x \,| > a$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\, dx$
2. $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\, dx$
3. $\displaystyle\int \frac{1}{\sqrt{e^{2x}+9}}\, dx$
4. $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\, dx$
5. $\displaystyle\int_0^1 \frac{1}{16-x^2}\, dx$
6. $\displaystyle\int_5^7 \frac{1}{16-x^2}\, dx$

Example

Evaluate the integral.

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5. $\int_0^1 \frac{1}{16-x^2}\, dx$
6. $\int_5^7 \frac{1}{16-x^2}\, dx$

Solution:

1) $\int \frac{1}{\sqrt{x^2-4}}\, dx = \cosh^{-1} \frac{x}{2} + c.$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx$
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4. $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{16-x^2}\,dx$
6. $\displaystyle\int_5^7 \frac{1}{16-x^2}\,dx$

Solution:

1) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx = \cosh^{-1}\frac{x}{2} + c.$

2) $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{(2x)^2+9}}\,dx.$

Let $u = 2x$, then $du = 2dx$. By substitution, we have

$$\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}\,du = \frac{1}{2}\sinh^{-1}\frac{u}{3} + c = \frac{1}{2}\sinh^{-1}\frac{2x}{3} + c.$$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{e^{2x}+9}}\,dx$
4. $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{16-x^2}\,dx$
6. $\displaystyle\int_5^7 \frac{1}{16-x^2}\,dx$

Solution:

1) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx = \cosh^{-1}\ \frac{x}{2} + c.$

2) $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{(2x)^2+9}}\,dx.$

Let $u = 2x$, then $du = 2dx$. By substitution, we have

$$\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}\,du = \frac{1}{2}\sinh^{-1}\ \frac{u}{3} + c = \frac{1}{2}\sinh^{-1}\ \frac{2x}{3} + c.$$

3) $\displaystyle\int \frac{1}{\sqrt{e^{2x}+9}}\,dx = \int \frac{1}{\sqrt{(e^x)^2+9}}\,dx.$

Let $u = e^x$, then $du = e^x dx$. By substituting that into the integral, we have

$$\int \frac{1}{u\sqrt{u^2+9}}\,du = -\frac{1}{3}\mathrm{csch}^{-1}\ \frac{|\ u\ |}{3} + c = -\frac{1}{3}\mathrm{csch}^{-1}\ \frac{e^x}{3} + c.$$

4) $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\,dx = \int \frac{1}{x\sqrt{1-(x^3)^2}}\,dx.$
Let $u = x^3$, then $du = 3x^2 dx$. By substitution, we obtain

$$\frac{1}{3}\int \frac{1}{u\sqrt{1-u^2}}\,du = -\frac{1}{3}\text{sech}^{-1}\ |\,u\,| + c = -\frac{1}{3}\text{sech}^{-1}\ |\,x^3\,| + c.$$

4) $\int \frac{1}{x\sqrt{1-x^6}}\, dx = \int \frac{1}{x\sqrt{1-(x^3)^2}}\, dx.$
Let $u = x^3$, then $du = 3x^2 dx$. By substitution, we obtain

$$\frac{1}{3}\int \frac{1}{u\sqrt{1-u^2}}\, du = -\frac{1}{3}\text{sech}^{-1} \mid u \mid + c = -\frac{1}{3}\text{sech}^{-1} \mid x^3 \mid + c.$$

5) Since the interval of the integral is subinterval of $(-4, 4)$, then the value of the integral is $\tanh^{-1}$. Hence,

$$\int_0^1 \frac{1}{16-x^2}\, dx = \frac{1}{4}\Big[\tanh^{-1}\frac{x}{4}\Big]_0^1 = \frac{1}{4}\Big[\tanh^{-1}(\frac{1}{4}) - \tanh^{-1}(0)\Big] = \frac{1}{4}\Big[\frac{1}{2}\ln(\frac{5}{3}) - \frac{1}{2}\ln(1)\Big] = \frac{1}{8}\ln(\frac{5}{3})$$

4) $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\ dx = \int \frac{1}{x\sqrt{1-(x^3)^2}}\ dx.$
Let $u = x^3$, then $du = 3x^2 dx$. By substitution, we obtain

$$\frac{1}{3}\int \frac{1}{u\sqrt{1-u^2}}\ du = -\frac{1}{3}\text{sech}^{-1}\ \mid u \mid +c = -\frac{1}{3}\text{sech}^{-1}\ \mid x^3 \mid +c.$$

5) Since the interval of the integral is subinterval of $(-4, 4)$, then the value of the integral is $\tanh^{-1}$. Hence,

$$\int_0^1 \frac{1}{16-x^2}\ dx = \frac{1}{4}\Big[\tanh^{-1}\ \frac{x}{4}\Big]_0^1 = \frac{1}{4}\Big[\tanh^{-1}(\frac{1}{4}) - \tanh^{-1}(0)\Big] = \frac{1}{4}\Big[\frac{1}{2}\ln(\frac{5}{3}) - \frac{1}{2}\ln(1)\Big] = \frac{1}{8}\ln(\frac{5}{3})$$

6) The interval of the integral is not subinterval of $(-4, 4)$, so the value of the integral is $\coth^{-1}$. This implies

$$\int_5^7 \frac{1}{16-x^2}\ dx = \frac{1}{4}\Big[\coth^{-1}\ \frac{x}{4}\Big]_5^7 = \frac{1}{4}\Big[\coth^{-1}\ \frac{7}{4} - \coth^{-1}\ \frac{5}{4}\Big] = \frac{1}{8}\Big[\ln(11) - 3\ln(3)\Big].$$