Integral Calculus

Department of Mathematics

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3. Definite integrals.
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    - Simpson's rule.

Summation Notation

Definition

Let $\{a_1, a_2, ..., a_n\}$ *be a set of numbers. The symbol* $\sum_{k=1}^{n} a_k$ *represents their sum:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

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Example

Evaluate the sum.

1. $\sum_{i=1}^{3} i^3$
2. $\sum_{j=1}^{4} (j^2 + 1)$
3. $\sum_{k=1}^{3} (k+1)k^2$

Solution:

1. $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36.$
2. $\sum_{j=1}^{4} (j^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34.$
3. $\sum_{k=1}^{3} (k + 1)k^2 = (1 + 1)(1)^2 + (2 + 1)(2)^2 + (3 + 1)(3)^2 = 2 + 12 + 36 = 50.$

Solution:

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Theorem

Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ be sets of real numbers. If n is any positive integer, then

1. $\sum_{k=1}^{n} c = \underbrace{c + c + ... + c}_{n\text{-times}} = nc$ *for any* $c \in \mathbb{R}$.
2. $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$
3. $\sum_{k=1}^{n} c\ a_k = c \sum_{k=1}^{n} a_k$ *for any* $c \in \mathbb{R}$.

Example

Evaluate the sum.

1. $\sum_{k=1}^{10} 15$
2. $\sum_{k=1}^{4} (k^2 + 2k)$
3. $\sum_{k=1}^{3} 3(k + 1)$

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Solution:

1. $\sum_{k=1}^{10} 15 = (10)(15) = 150.$
2. $\sum_{k=1}^{4} (k^2 + 2k) = \sum_{k=1}^{4} k^2 + 2\sum_{k=1}^{4} k = (1^2 + 2^2 + 3^2 + 4^2) + 2(1 + 2 + 3 + 4) = 30 + 20 = 50.$
3. $\sum_{k=1}^{3} 3(k + 1) = 3\sum_{k=1}^{3} (k + 1) = 3(2 + 3 + 4) = 27.$

Theorem

1. $\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example

Evaluate the sum.

1. $\sum_{k=1}^{100} k$
2. $\sum_{k=1}^{10} k^2$
3. $\sum_{k=1}^{10} k^3$

Theorem

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Example

Evaluate the sum.

1. $\sum_{k=1}^{100} k$
2. $\sum_{k=1}^{10} k^2$
3. $\sum_{k=1}^{10} k^3$

Solution:

1. $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050.$
2. $\sum_{k=1}^{10} k^2 = \frac{10(11)(21)}{6} = 385.$
3. $\sum_{k=1}^{10} k^3 = \left[\frac{10(11)}{2}\right]^2 = 3025.$

Example

Express the sum in terms of n.

1. $\sum_{k=1}^{n}(k+1)$
2. $\sum_{k=1}^{n}(k^2-k-1)$

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1. $\sum_{k=1}^{n}(k+1)$
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Solution:

1. $\sum_{k=1}^{n}(k+1) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$.
2. $\sum_{k=1}^{n}(k^2-k-1) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = -\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - n = \frac{n(n^2-4)}{3}$.

Riemann Sum and Area

Definition

A set $P = \{x_0, x_1, x_2, ..., x_n\}$ *is called a partition of a closed interval* $[a, b]$ *if for any positive integer* n*,*

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b.$$

A partition of the interval $[a, b]$.

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A partition of the interval $[a, b]$.

Notes:

- The division of the interval $[a, b]$ by the partition P generates n subintervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$.
- The length of each subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
- The union of subintervals gives the whole interval $[a, b]$.

Definition

The norm of the partition of P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n$ *i.e.,*

$$\| P \| = max\{\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n\}.$$

Example

If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0, 4]$, find the norm of the partition P.

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If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0, 4]$, find the norm of the partition P.

Solution:

We need to find the subintervals and their lengths.

Subinterval $[x_{k-1}, x_k]$	Length Δx_k
$[0, 1.2]$	$1.2 - 0 = 1.2$
$[1.2, 2.3]$	$2.3 - 1.2 = 1.1$
$[2.3, 3.6]$	$3.6 - 2.3 = 1.3$
$[3.6, 4]$	$4 - 3.6 = 0.4$

The norm of P is the largest length among

$$\{\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4\}.$$

Hence, $\| P \| = \Delta x_3 = 1.3$

Remark

1. *The partition P of the interval $[a, b]$ is regular if* $\Delta x_0 = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \Delta x$.
2. *For any positive integer n, if the partition P is regular then*

$$\Delta x = \frac{b-a}{n} \quad \textit{and} \quad x_k = x_0 + k\ \Delta x.$$

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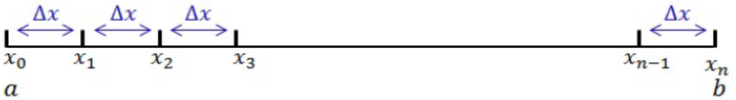
Let P be a regular partition of the interval $[a, b]$. Since $x_0 = a$ and $x_n = b$, then

$$\begin{aligned} x_1 &= x_0 + \Delta x\ , \\ x_2 &= x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x\ , \\ x_3 &= x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x. \end{aligned}$$

By continuing doing so, we have $x_k = x_0 + k\,\Delta x$.



A regular partition of the interval $[a, b]$.

Example

Define a regular partition P that divides the interval $[1, 4]$ into 4 subintervals.

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Therefore,

$x_0 = 1$

$x_1 = 1 + \frac{3}{4} = \frac{7}{4}$

$x_2 = 1 + 2(\frac{3}{4}) = \frac{5}{2}$

$x_3 = 1 + 3(\frac{3}{4}) = \frac{13}{4}$

$x_4 = 1 + 4(\frac{3}{4}) = 4$

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The regular partition is $P = \{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4\}$.

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The regular partition is $P = \{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4\}$.

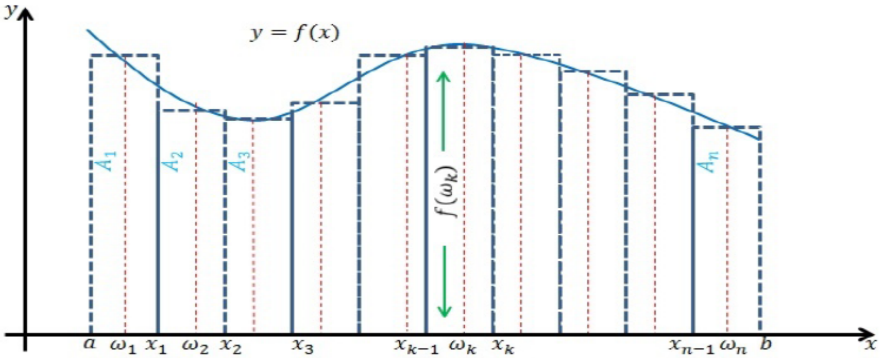
Definition

Let f be a function defined on a closed interval $[a,b]$ and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$. Let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark on the partition P where $\omega_k \in [x_{k-1}, x_k]$, $k = 1,2,3,...,n$. Then, a Riemann sum of f for P is

$$R_p = \sum_{k=1}^{n} f(\omega_k)\Delta x_k.$$

If f is a defined and positive function on a closed interval $[a, b]$ and P is a partition of that interval where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark on the partition P, then the Riemann sum estimates the area of the region under f from $x = a$ to $x = b$.

$$A = \lim_{\|P\| \to 0} R_p = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\omega_k)\Delta x_k$$



Example

Find a Riemann sum R_p of the function $f(x) = 2x - 1$ for the partition $P = \{-2, 0, 1, 4, 6\}$ of the interval $[-2, 6]$ by choosing the mark,

1. the left-hand endpoint,
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2. the right-hand endpoint,
3. the midpoint.

Solution:

1) Choose the left-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	-2	-5	-10
$[0, 1]$	$1 - 0 = 1$	0	-1	-1
$[1, 4]$	$4 - 1 = 3$	1	1	3
$[4, 6]$	$6 - 4 = 2$	4	7	14
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				6

2) Choose the right-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2, 0]$	$0 - (-2) = 2$	0	-1	-2
$[0, 1]$	$1 - 0 = 1$	1	1	1
$[1, 4]$	$4 - 1 = 3$	4	7	21
$[4, 6]$	$6 - 4 = 2$	6	11	22
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				42

[1] The midpoint of the subinterval $[x_{k-1}, x_k]$ is $\omega_k = \frac{x_{k-1}+x_k}{2}$.

2) Choose the right-hand endpoint of each subinterval.

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3) Choose the midpoint of each subinterval.[1]

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2, 0]$	$0-(-2)=2$	-1	-3	-6
$[0, 1]$	$1-0=1$	0.5	0	0
$[1, 4]$	$4-1=3$	2.5	4	12
$[4, 6]$	$6-4=2$	5	9	18
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				24

[1] The midpoint of the subinterval $[x_{k-1}, x_k]$ is $\omega_k = \frac{x_{k-1}+x_k}{2}$.

Example

Let A be the area under the graph of $f(x) = x + 1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right-hand endpoint of each subinterval.

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Solution:

For a regular partition P, we have

1. $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$, and
2. $x_k = x_0 + k\ \Delta x$ where $x_0 = 1$.

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2. $x_k = x_0 + k\ \Delta x$ where $x_0 = 1$.

Since the mark ω is the right endpoint of each subinterval, then $\omega_k = x_k = 1 + \frac{2k}{n}$. Therefore,

$$f(\omega_k) = (1 + \frac{2k}{n}) + 1 = \frac{2k}{n} + 2 = \frac{2}{n}(n + k).$$

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From the definition,

$$R_p = \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \frac{4}{n^2}\sum_{k=1}^{n}(n+k)$$

$$= \frac{4}{n^2}\left[n^2 + \frac{n(n+1)}{2}\right]$$

$$= 4 + \frac{2(n+1)}{n}.$$

(1) $\sum_{k=1}^{n}(n+k) = \sum_{k=1}^{n} n + \sum_{k=1}^{n} k$

(2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Hence, $\lim_{n\to\infty} R_p = 4 + 2 = 6$.

Definition

Let f be a defined function on a closed interval $[a, b]$ and let P be a partition of $[a, b]$. The definite integral of f on $[a, b]$ is

$$\int_a^b f(x)\ dx = \lim_{\|P\|\to 0} \sum_k f(\omega_k)\Delta x_k$$

if the limit exists. The numbers a and b are called the limits of the integration.

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Example

Evaluate the integral $\int_2^4 (x+2)\ dx$.

Solution: Let $P = \{x_0, x_1, ..., x_n\}$ be a regular partition of the interval $[2, 4]$, then $\Delta x = \frac{4-2}{n} = \frac{2}{n}$ and $x_k = x_0 + \Delta x$.

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Let f be a defined function on a closed interval $[a, b]$ *and let P be a partition of* $[a, b]$*. The definite integral of f on* $[a, b]$ *is*

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The Riemann sum of f for P is

$$R_p = \sum_k f(\omega_k)\Delta x_k = \frac{4}{n^2} \sum_k (2n + k) = \frac{4}{n^2}(2n^2 + \frac{n(n+1)}{2}) = 8 + \frac{2(n+1)}{n}.$$

Definition

Let f be a defined function on a closed interval $[a, b]$ and let P be a partition of $[a, b]$. The definite integral of f on $[a, b]$ is

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$$R_p = \sum_k f(\omega_k)\Delta x_k = \frac{4}{n^2}\sum_k (2n+k) = \frac{4}{n^2}(2n^2 + \frac{n(n+1)}{2}) = 8 + \frac{2(n+1)}{n}.$$

From the definition, $\displaystyle\int_2^4 (x+2)\ dx = \lim_{n\to\infty} R_p = 8 + \lim_{n\to\infty} \frac{2n(n+1)}{n^2} = 8 + 2 = 10.$

Properties of Definite Integrals

Theorem

1) $\int_a^b c \; dx = c(b-a)$,

2) $\int_a^a f(x) \; dx = 0$ *if* $f(a)$ *exists.*

3) Linearity of Definite Integrals:

- *If* f *and* g *are integrable on* $[a,b]$, *then* $f+g$ *and* $f-g$ *are integrable on* $[a,b]$ *and*

$$\int_a^b (f(x) \pm g(x)) \; dx = \int_a^b f(x) \pm \int_a^b g(x) \; dx.$$

- *If* f *is integrable on* $[a,b]$ *and* $k \in \mathbb{R}$, *then* $k \; f$ *is integrable on* $[a,b]$ *and*

$$\int_a^b k \; f(x) \; dx = k \; \int_a^b f(x) \; dx.$$

Theorem

4) Comparison of Definite Integrals:

- *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

- *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\ dx \geq 0.$$

5) Additive Interval of Definite Integrals:
If f is integrable on the intervals $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6) Reversed Interval of Definite Integrals:
If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

Example

Evaluate the integral.

1. $\displaystyle\int_0^2 3\ dx$
2. $\displaystyle\int_2^2 (x^2+4)\ dx$

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Solution:

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Example

If $\displaystyle\int_a^b f(x)\ dx = 4$ and $\displaystyle\int_a^b g(x)\ dx = 2$, then find $\displaystyle\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx.$

Example

Evaluate the integral.

1. $\displaystyle\int_0^2 3\ dx$
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Example

If $\displaystyle\int_a^b f(x)\ dx = 4$ and $\displaystyle\int_a^b g(x)\ dx = 2$, then find $\displaystyle\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx.$

Solution:

$$\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\ dx = 3\int_a^b f(x)\ dx - \frac{1}{2}\int_a^b g(x)\ dx = 3(4) - \frac{1}{2}(2) = 11.$$

Example

Prove that $\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx$ without evaluating the integrals.

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Solution: Let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 + 1$. We can find that $f(x) - g(x) = x^3 + 1 > 0$ for all $x \in [0, 2]$. This implies that $f(x) > g(x)$.

Example

Prove that $\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx$ without evaluating the integrals.

Solution: Let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 + 1$. We can find that $f(x) - g(x) = x^3 + 1 > 0$ for all $x \in [0, 2]$. This implies that $f(x) > g(x)$. From the theorem, we have

$$\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx.$$

The Fundamental Theorem of Calculus

Theorem

Suppose that f *is continuous on the closed interval* $[a, b]$*.*

1. *If* $F(x) = \int_a^x f(t)\, dt$ *for every* $x \in [a, b]$*, then* $F(x)$ *is an antiderivative of* f *on* $[a, b]$*.*
2. *If* $F(x)$ *is any antiderivative of* f *on* $[a, b]$*, then* $\int_a^b f(x)\, dx = F(b) - F(a)$*.*

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2. *If* $F(x)$ *is any antiderivative of* f *on* $[a, b]$*, then* $\int_a^b f(x)\,dx = F(b) - F(a)$.

Corollary

If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Notes:

■ From the previous corollary, a definite integral $\int_a^b f(x)\,dx$ is evaluated by two steps:

Step 1: Find an antiderivative F of the integrand,

Step 2: Evaluate the antiderivative F at upper and lower limits by substituting $x = b$ and $x = a$ (evaluate at lower limit) into F, then subtracting the latter from the former i.e., calculate $F(b) - F(a)$.

Notes:

- From the previous corollary, a definite integral $\int_a^b f(x)\ dx$ is evaluated by two steps:

Step 1: Find an antiderivative F of the integrand,
Step 2: Evaluate the antiderivative F at upper and lower limits by substituting $x = b$ and $x = a$ (evaluate at lower limit) into F, then subtracting the latter from the former i.e., calculate $F(b) - F(a)$.

- When using substitution to evaluate the definite integral $\int_a^b f(x)\ dx$, we have two options:

Option 1: Change the limits of integration to the new variable. For example, $\int_0^1 2x\sqrt{x^2+1}\ dx$. Let $u = x^2 + 1$, this implies $du = 2x\ dx$. Change the limits $u(0) = 1$ and $u(1) = 2$. By substitution, we have $\int_1^2 u^{1/2}\ du$. Then, evaluate the integral without returning to the original variable.
Option 2: Leave the limits in terms of the original variable. Evaluate the integral, then return to the original variable. After that, substitute $x = b$ and $x = a$ into the antiderivative as in step 2 above.

Example

Evaluate the integral.

1. $\int_{-1}^{2} (2x+1)\ dx$
2. $\int_{0}^{3} (x^2+1)\ dx$
3. $\int_{1}^{2} \frac{1}{\sqrt{x^3}}\ dx$
4. $\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\ dx$
5. $\int_{\frac{\pi}{4}}^{\pi} (\sec^2\ x-4)\ dx$
6. $\int_{0}^{\frac{\pi}{3}} (\sec\ x\ \tan\ x+x)\ dx$

Example

Evaluate the integral.

1. $\displaystyle\int_{-1}^{2} (2x+1)\ dx$
2. $\displaystyle\int_{0}^{3} (x^2+1)\ dx$
3. $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{x^3}}\ dx$
4. $\displaystyle\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\ dx$
5. $\displaystyle\int_{\frac{\pi}{4}}^{\pi} (\sec^2\ x-4)\ dx$
6. $\displaystyle\int_{0}^{\frac{\pi}{3}} (\sec\ x\ \tan\ x+x)\ dx$

Solution:

1) $\displaystyle\int_{-1}^{2} (2x+1)\ dx = \Big[x^2+x\Big]_{-1}^{2} = (4+2) - ((-1)^2+(-1)) = 6-0 = 6.$

2) $\displaystyle\int_{0}^{3} (x^2+1)\ dx = \Big[\tfrac{x^3}{3}+x\Big]_{0}^{3} = (\tfrac{27}{3}+3) - 0 = 12.$

3) $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{x^3}}\ dx = \Big[\tfrac{-2}{\sqrt{x}}\Big]_{1}^{2} = \tfrac{-2}{\sqrt{2}} - (-2) = \tfrac{-2+2\sqrt{2}}{\sqrt{2}} = -\sqrt{2}+2.$

4) $\displaystyle\int_{0}^{\frac{\pi}{2}} (\sin\ x+1)\ dx = \Big[-\cos\ x+x\Big]_{0}^{\frac{\pi}{2}} = (-\cos\ \tfrac{\pi}{2}+\tfrac{\pi}{2}) - (-\cos\ 0+0) = \tfrac{\pi}{2}+1.$

5) $\int_{\frac{\pi}{4}}^{\pi} (\sec^2 \ x - 4) \ dx = \left[\tan \ x - 4x\right]_{\frac{\pi}{4}}^{\pi} = (\tan \ \pi - 4\pi) - (\tan \ \frac{\pi}{4} - 4\frac{\pi}{4}) =$
$-4\pi - (1 - \pi) = -3\pi - 1.$

6) $\int_{0}^{\frac{\pi}{3}} (\sec \ x \ \tan \ x + x) \ dx = \left[\sec \ x + \frac{x^2}{2}\right]_{0}^{\frac{\pi}{3}} = (\sec \ \frac{\pi}{3} + \frac{(\frac{\pi}{3})^2}{2}) - (\sec \ 0 + \frac{0}{2}) =$
$2 + \frac{\pi^2}{18} - 1 = 1 + \frac{\pi^2}{18}.$

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Example

If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, find $\int_{-1}^{2} f(x)\, dx.$

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Example

If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, find $\int_{-1}^{2} f(x)\ dx.$

Solution:

The definition of the function f changes at 0. Since $[-1, 2] = [-1, 0] \cup [0, 2]$, then from the theorem,

$$\begin{aligned} \int_{-1}^{2} f(x)\ dx &= \int_{-1}^{0} f(x)\ dx + \int_{0}^{2} f(x)\ dx \\ &= \int_{-1}^{0} x^2\ dx + \int_{0}^{2} x^3\ dx \\ &= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{2} \\ &= \frac{1}{3} + \frac{16}{4} = \frac{13}{3}. \end{aligned}$$

Example

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Solution:

$$|\, x - 1 \,| = \begin{cases} -(x-1) & : x < 1 \\ x - 1 & : x \geq 1 \end{cases}$$

Example

Evaluate the integral $\int_0^2 | x - 1 | \ dx$.

Solution:

$$| x - 1 | = \begin{cases} -(x-1) & : x < 1 \\ x - 1 & : x \geq 1 \end{cases}$$

Since $[0,2] = [0,1] \cup [1,2]$, then from the theorem,

$$\int_0^2 | x - 1 | \ dx = \int_0^1 (-x+1) \ dx + \int_1^2 (x-1) \ dx$$

$$= \left[\frac{-x^2}{2} + x \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^2$$

$$= (\frac{1}{2} - 0) + (0 + \frac{1}{2}) = 1.$$

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Mean Value Theorem for Integrals

Theorem

If f is continuous on a closed interval $[a,b]$, then there is at least a number $z \in (a,b)$ such that

$$\int_a^b f(x) \; dx = f(z)(b-a).$$

Example

Find a number z that satisfies the conclusion of the Mean Value Theorem for the function f on the given interval.

1. $f(x) = 1 + x^2, \quad [0, 2]$
2. $f(x) = \sqrt[3]{x}, \quad [0, 1]$

Example

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Solution:

(1) From the theorem,

$$\int_0^2 (1 + x^2)\ dx = (2 - 0) f(z)$$

$$\left[x + \frac{x^3}{3}\right]_0^2 = 2(1 + z^2)$$

$$\frac{14}{3} = 2(1 + z^2)$$

$$\frac{7}{3} = 1 + z^2$$

This implies $z^2 = \frac{4}{3}$, then $z = \pm \frac{2}{\sqrt{3}}$. However, $-\frac{2}{\sqrt{3}} \notin (0, 2)$, so $z = \frac{2}{\sqrt{3}} \in (0, 2)$.

(2) From the theorem,

$$\int_0^1 \sqrt[3]{x}\ dx = (1-0)f(z)$$
$$\frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1 = \sqrt[3]{z}$$

This implies $z = \frac{27}{64} \in (0,1)$.

(2) From the theorem,

$$\int_0^1 \sqrt[3]{x}\, dx = (1-0)f(z)$$
$$\frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1 = \sqrt[3]{z}$$

This implies $z = \frac{27}{64} \in (0,1)$.

Definition

If f is continuous on the interval $[a,b]$, then the average value f_{av} of f on $[a,b]$ is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

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$$\int_0^1 \sqrt[3]{x}\, dx = (1-0) f(z)$$

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$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example

Find the average value of the function f on the given interval.

1. $f(x) = x^3 + x - 1, \quad [0, 2]$
2. $f(x) = \sqrt{x}, \quad [1, 3]$

Solution:

1. $f_{av} = \frac{1}{2-0}\int_0^2 (x^3 + x - 1)\ dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^2 = \frac{1}{2}\Big[(4 + 2 - 2) - (0)\Big] = 2.$

Solution:

1. $f_{av} = \frac{1}{2-0}\int_0^2 (x^3 + x - 1)\ dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^2 = \frac{1}{2}\Big[(4+2-2) - (0)\Big] = 2.$
2. $f_{av} = \frac{1}{3-1}\int_1^3 \sqrt{x}\ dx = \frac{1}{2}\frac{2}{3}\left[\ x^{\frac{3}{2}}\ \right]_1^3 = \frac{3\sqrt{3}-1}{3}.$

Solution:

1. $f_{av} = \frac{1}{2-0}\int_0^2 (x^3 + x - 1)\ dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^2 = \frac{1}{2}\left[(4+2-2) - (0)\right] = 2.$
2. $f_{av} = \frac{1}{3-1}\int_1^3 \sqrt{x}\ dx = \frac{1}{2}\frac{2}{3}\left[\ x^{\frac{3}{2}}\ \right]_1^3 = \frac{3\sqrt{3}-1}{3}.$

From the Fundamental Theorem, if f is continuous on $[a, b]$ and $F(x) = \displaystyle\int_c^x f(t)\ dt$ where $c \in [a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\ dt = \frac{d}{dx}\Big[F(x) - F(a)\Big] = f(x) \quad \forall x \in [a, b].$$

This result can be generalized as follows:

Theorem

Let f be continuous on $[a, b]$. *If g and h are in the domain of f and differentiable, then*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x) \quad \forall x \in [a, b].$$

Theorem

Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))h'(x) - f(g(x))g'(x) \ \ \forall x \in [a, b].$$

Corollary

Let f be continuous on $[a, b]$. If g and h are in the domain of f and differentiable, then

1. $\frac{d}{dx}\int_{a}^{h(x)} f(t)\ dt = f(h(x))h'(x) \ \ \forall x \in [a, b]$,
2. $\frac{d}{dx}\int_{g(x)}^{a} f(t)\ dt = -f(g(x))g'(x) \ \ \forall x \in [a, b].$

Example

Find the derivative.

1. $\frac{d}{dx}\int_1^x \sqrt{\cos\ t}\ dt$
2. $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\ dt$
3. $\frac{d}{dx}\left(x\int_x^{x^2} (t^3-1)\ dt\right)$
4. $\frac{d}{dx}\int_{x+1}^3 \sqrt{t+1}\ dt$
5. $\frac{d}{dx}\int_1^{\sin\ x} \frac{1}{1-t^2}\ dt$
6. $\frac{d}{dx}\int_{-x}^x \cos\ (t^2+1)\ dt$
7. $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^2+1}\ dt$
8. $\frac{d}{dx}\int_{\cos\ x}^{\sin\ x} \sqrt{1+t^4}\ dt$

Example

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2. $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\ dt$
3. $\frac{d}{dx}\left(x\int_x^{x^2} (t^3-1)\ dt\right)$
4. $\frac{d}{dx}\int_{x+1}^{3} \sqrt{t+1}\ dt$
5. $\frac{d}{dx}\int_1^{\sin\ x} \frac{1}{1-t^2}\ dt$
6. $\frac{d}{dx}\int_{-x}^{x} \cos\ (t^2+1)\ dt$
7. $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^2+1}\ dt$
8. $\frac{d}{dx}\int_{\cos\ x}^{\sin\ x} \sqrt{1+t^4}\ dt$

Solution:

1) $\frac{d}{dx}\int_1^x \sqrt{\cos\ t}\ dt = \sqrt{\cos\ x}\ (1) = \sqrt{\cos\ x}$.

2) $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\ dt = \frac{1}{(x^2)^3+1}(2x) = \frac{2x}{x^6+1}$.

3) $\frac{d}{dx}\left(x\int_x^{x^2} (t^3-1)\ dt\right) = \int_x^{x^2} (t^3-1)\ dt + x\big(2x(x^6-1)-(x^3-1)\big)$

4) $\frac{d}{dx}\int_{x+1}^{3} \sqrt{t+1}\, dt = 0 - \sqrt{(x+1)+1} = -\sqrt{x+2}.$

5) $\frac{d}{dx}\int_{1}^{\sin\ x} \frac{1}{1-t^2}\, dt = \frac{1}{1-\sin^2\ x}\cos\ x = \frac{\cos\ x}{\cos^2\ x} = \sec\ x.$

6) $\frac{d}{dx}\int_{-x}^{x} \cos\ (t^2+1)\, dt = \cos\ (x^2+1) + \cos\ (x^2+1) = 2\cos\ (x^2+1).$

7) $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^2+1}\, dt = \frac{2x}{x^4+1} + \frac{1}{x^2+1}.$

8) $\frac{d}{dx}\int_{\cos\ x}^{\sin\ x} \sqrt{1+t^4}\, dt = \sqrt{1+\sin^4\ x}\ \cos\ x + \sqrt{1+\cos^4\ x}\ \sin\ x.$

Example

If $F(x) = (x^2 - 2)\int_2^x \left(t + 3F'(t)\right)\, dt$, find $F'(2)$.

Example

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Solution:

$$F'(x) = 2x\int_2^x \big(t + 3F'(t)\big)\, dt + (x^2 - 2)\big(x + 3F'(x)\big)$$

Letting $x = 2$ gives

$$F'(2) = 4\int_2^2 \big(t + 3F'(t)\big)\, dt + (4 - 2)\big(2 + 3F'(2)\big)$$
$$\Rightarrow F'(2) = 2\big(2 + 3F'(2)\big).$$

Hence, $-5F'(2) = 4 \Rightarrow F'(2) = -\frac{4}{5}$.

Numerical Integration

Assume P is a regular partition of $[a, b]$. We divide the interval $[a, b]$ by the partition P into n subintervals : $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$. Then, we find the length of the subintervals: $\Delta x_k = \frac{b-a}{n}$. Using Riemann sum, we have

$$\int_a^b f(x)\ dx \approx \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \frac{b-a}{n}\sum_{k=1}^{n} f(\omega_k)\ ,$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is a mark on the partition P.

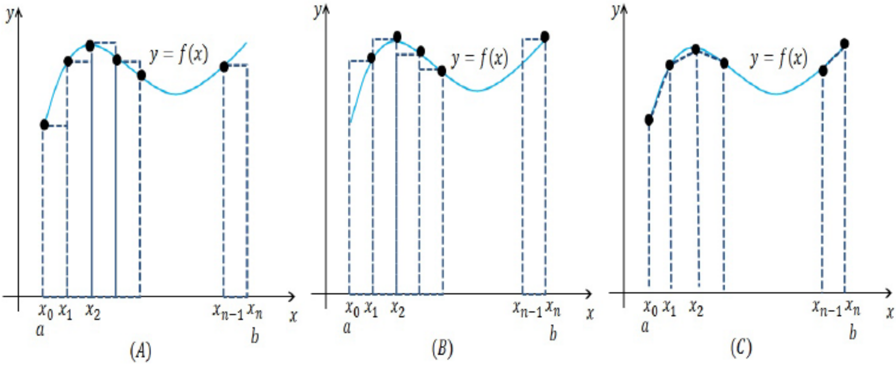

Figure: Approximation of a definite integral by using the trapezoidal rule.

As shown in the figure, we take the mark as follows:

1. The left-hand endpoint. We choose $\omega_k = x_{k-1}$ in each subinterval. Then,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(x_{k-1}).$$

2. The right-hand endpoint. We choose $\omega_k = x_k$ in each subinterval. Then,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(x_k).$$

3. The average of the previous two approximations is more accurate,

$$\frac{b-a}{2n}\left[\sum_{k=1}^{n} f(x_{k-1}) + \sum_{k=1}^{n} f(x_k)\right].$$

Trapezoidal Rule

Let f be continuous on $[a, b]$. If $P = \{x_0, x_1, ..., x_n\}$ is a regular partition of $[a, b]$, then

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\Big].$$

Error Estimation

Theorem

Suppose that f'' is continuous on $[a, b]$ and M is the maximum value for f'' over $[a, b]$. If E_T is the error in calculating $\int_a^b f(x)\, dx$ under the trapezoidal rule, then

$$\mid E_T \mid \leq \frac{M(b-a)^3}{12\ n^2}.$$

Example

By using the trapezoidal rule with $n = 4$, approximate the integral $\int_1^2 \frac{1}{x}\, dx$. Then, estimate the error.

Solution:

1) We approximate the integral $\int_1^2 \frac{1}{x}\, dx$ by the trapezoidal rule.

a) Find a regular partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.
We divide the interval $[1, 2]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{2-1}{4} = \frac{1}{4}$ as follows:

$x_0 = 1$
$x_1 = 1 + \frac{1}{4} = 1\frac{1}{4}$
$x_2 = 1 + 2(\frac{1}{4}) = 1\frac{1}{2}$
$x_3 = 1 + 3(\frac{1}{4}) = 1\frac{3}{4}$
$x_4 = 1 + 4(\frac{1}{4}) = 2$

The partition is $P = \{1, 1.25, 1.5, 1.75, 2\}$.

b) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1	1	1
1	1.25	0.8	2	1.6
2	1.5	0.6667	2	1.3334
3	1.75	0.5714	2	1.1428
4	2	0.5	1	0.5
Sum $= \sum_{k=1}^{4} m_k f(x_k)$				5.5762

Hence,

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{1}{8}\left[5.5762\right] = 0.697.$$

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Hence,

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{1}{8}\left[5.5762\right] = 0.697.$$

2) We estimate the error by using the theorem:

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} \Rightarrow f'''(x) = -\frac{6}{x^4}.$$

Since $f''(x)$ is a decreasing function on the interval $[1, 2]$, then $f''(x)$ is maximized at $x = 1$. Hence, $M =\mid f''(1) \mid= 2$ and $\mid E_T \mid\leq \frac{2(2-1)^3}{12(4)^2} = \frac{1}{96} = 0.0104$.

Simpson's Rule

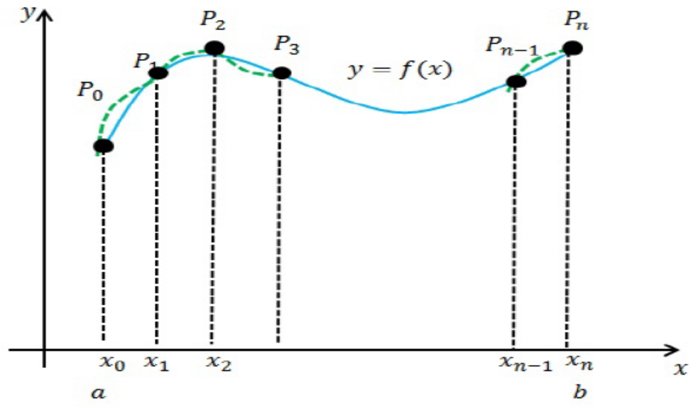


Figure: Approximation of a definite integral by using Simpson's rule.

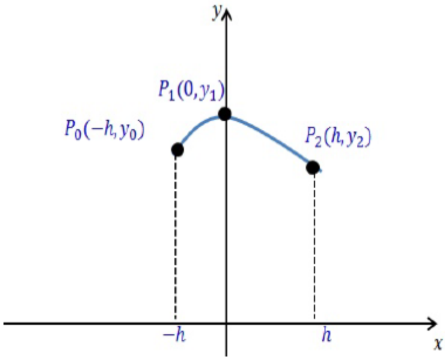
First, let P be a regular partition of the interval $[a, b]$ to generate n subintervals such that $\mid P \mid = \frac{(b-a)}{n}$ and n is an even number.

Take three points lying on the parabola as shown in the next figure. Assume for simplicity that $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. Since the equation of a parabola is

$$y = ax^2 + bx + c$$

, then from the figure, the area under the graph bounded by $[-h, h]$ is

$$\int_{-h}^{h} (ax^2 + bx + c)\ dx = \frac{h}{3}(2ah^2+6c).$$



figu

Thus, since the points P_0, P_1 and P_2 lie on the parabola, then

$$y_0 = ah^2 - bh + c$$
$$y_1 = c$$
$$y_2 = ah^2 + bh + c.$$

Some computations lead to $2ah^2 + 6c = y_0 + 4y_1 + y_2$. Therefore,

$$\int_{-h}^{h} (ax^2 + bx + c)\ dx = \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big).$$

Generally, for any three points P_{k-1}, P_k and P_{k+1}, we have

$$\frac{h}{3}(y_{k-1} + 4y_k + y_{k+1}) = \frac{h}{3}\big(f(x_{k-1}) + 4f(x_k) + f(x_{k+1})\big).$$

By summing the areas of all parabolas, we have

$$\begin{aligned}\int_a^b f(x)\ dx &= \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big)\\ &+ \frac{h}{3}\big(f(x_2) + 4f(x_3) + f(x_4)\big)\\ &\dots\\ &+ \frac{h}{3}\big(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)\\ &= \frac{b-a}{3n}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\end{aligned}$$

Simpson's Rule

Let f be continuous on $[a, b]$. If $P = \{x_0, x_1, ..., x_n\}$ is a regular partition of $[a, b]$ where n is even, then

$$\int_a^b f(x)\ dx \approx \frac{(b-a)}{3n}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big].$$

Error Estimation

The estimation of the error under Simpson's method is given by the following theorem.

Theorem

Suppose $f^{(4)}$ is continuous on $[a, b]$ and M is the maximum value for $f^{(4)}$ on $[a, b]$. If E_S is the error in calculating $\int_a^b f(x)\ dx$ under Simpson's rule, then

$$\mid E_S \mid \leq \frac{M(b-a)^5}{180\ n^4}.$$

Example

By using Simpson's rule with $n = 4$, approximate the integral $\int_1^3 \sqrt{x^2+1}\,dx$. Then, estimate the error.

Solution:

1) We approximate the integral $\int_1^3 \sqrt{x^2+1}\,dx$ under Simpson's rule.

a) Find the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.
We divide the interval $[1, 3]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$ as follows:

$x_0 = 1$
$x_1 = 1 + \frac{1}{2} = 1\frac{1}{2}$
$x_2 = 1 + 2(\frac{1}{2}) = 2$
$x_3 = 1 + 3(\frac{1}{2}) = 2\frac{1}{2}$
$x_4 = 1 + 4(\frac{1}{2}) = 3$

The partition is $P = \{1, 1.5, 2, 2.5, 3\}$.
b) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1.4142	1	2
1	1.5	1.8028	4	7.2112
2	2	2.2361	2	4.4722
3	2.5	2.6926	4	10.7704
4	3	3.1623	1	10
$\text{Sum} = \sum_{k=1}^{4} m_k f(x_k)$				27.0302

Hence, $\int_1^3 \sqrt{x^2+1}\, dx \approx \frac{2}{12}\left[27.0302\right] = 4.5050.$

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
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4	3	3.1623	1	10
Sum $= \sum_{k=1}^{4} m_k f(x_k)$				27.0302

Hence, $\int_1^3 \sqrt{x^2+1}\, dx \approx \frac{2}{12}\left[27.0302\right] = 4.5050.$

2) We estimate the error by using the theorem.

Since $f^{(5)}(x) = -(15x(4x^2-3))/\sqrt{(x^2+1)^9}$, then $f^{(4)}(x)$ is a decreasing function on the interval $[1,3]$. Therefore, $f^{(4)}(x)$ is maximized at $x = 1$. Then, $M = \mid f^{(4)}(1) \mid = 0.7955$ and

$$\mid E_s \mid < \frac{(0.7955)(3-1)^5}{180(4)^4} = 5.5243 \times 10^{-4}.$$