Lecture 6

Generating Functions and Sums of random Variables

Consider a portfolio of insurance risks covered by insurance policies issued by an insurance company. The total claims paid by the insurance company on all policies are the sum of the payments made by the insurer, $S_k = X_1 + X_2 + \dots + X_k$.

Defn For a random variable X, the moment generating fn (mgf) is $M_X(z) = E(e^{zX})$ $\forall z \in \mathbb{R}$, and the prob. generating fn (pgf) is

$$P_X(z) = E(z^X)$$

Note that: $M_X(z) = P_X(e^z)$ and $P_X(z) = M_X(\ln z)$

Theorem Let $S_k = X_1 + X_2 + \dots + X_k$, where the random variables in the sum are independent. Then, $M_{S_k}(z) = \prod_{j=1}^{k} M_{X_j}(z)$

and $P_{S_k}(z) = \prod_{j=1}^{k} P_{X_j}(z)$, provided that all the component mgfs and pgfs exist.

EX 1 Show that the sum of independent gamma random variables, each with the same value of θ, has a gamma distribution

Ans:

For $X \sim \text{gamma}(\theta, \alpha)$, $f(x) = \dfrac{(x/\theta)^{\alpha} e^{-x/\theta}}{x\,\Gamma(\alpha)}$

$$M_X(z) = E(e^{zX}) = \int_0^\infty \frac{e^{zx}\, x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\,\theta^{\alpha}}\,dx$$

$$M_X(z) = \frac{\int_0^\infty x^{\alpha-1} e^{-x(-z+1/\theta)}\,dx}{\Gamma(\alpha)\,\theta^{\alpha}}$$

$$M_X(z) = \frac{\int_0^\infty y^{\alpha-1} (-z+1/\theta)^{-\alpha+1}\, e^{-y}\,dy}{\Gamma(\alpha)\,\theta^{\alpha}\,(-z+1/\theta)}, \quad \text{where } y = x(-z+1/\theta), \quad dx = \frac{dy}{(-z+1/\theta)}$$

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$$M_X(z) = \frac{\int_0^{\infty} y^{\alpha-1} (-z + 1/\theta)^{-\alpha} e^{-y}\, dy}{\Gamma(\alpha)\, \theta^{\alpha}}$$

$$M_X(z) = \left[\frac{1}{(-z+1/\theta)\theta}\right]^{\alpha} \frac{\int_0^{\infty} y^{\alpha-1} e^{-y}\, dy}{\Gamma(\alpha)}$$

, when $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$

$$M_X(z) = \left(\frac{1}{1-\theta z}\right)^{\alpha} \cdot \cancel{\frac{\Gamma(\alpha)}{\Gamma(\alpha)}}$$

$$\therefore M_X(z) = \left(\frac{1}{1-\theta z}\right)^{\alpha}, \quad z < 1/\theta$$

Now let X_j have a gamma distn with parameters α_j and θ. Then, the moment generating fn of the sum is

$$M_{S_k}(z) = \prod_{j=1}^{k} \left(\frac{1}{1-\theta z}\right)^{\alpha_j}$$

$$M_{S_k}(z) = \left(\frac{1}{1-\theta z}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_k}$$

which is the moment generating fn of a gamma distribution with parameters $\alpha_1+\alpha_2+\cdots+\alpha_k$ and θ.

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EX

Obtain the mgf and pgf for the Poisson distribution

Ans: the pgf is

$$P_X(z) = \sum_{x=0}^{\infty} z^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$P_X(z) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(z\lambda)^x}{x!} = e^{-\lambda} e^{z\lambda}$$

$$\therefore P_X(z) = e^{\lambda(z-1)}$$

The mgf is $M_X(z) = P_X(e^z) = \exp[\lambda(e^z - 1)]$

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pb 3.21 p-33

A portfolio contains 16 independent risks, each with a gamma distribution with parameters $\alpha = 1$ and $\theta = 250$. Give an expression using the incomplete gamma function for the probability that the sum of the losses exceeds 6000. Then approximate this probability using the central limit theorem.

Ans:

The sum has a gamma distn with parameters $\alpha = 16$ and $\theta = 250$

$$pr(S_{16} > 6,000) = 1 - F(x)$$
$$= 1 - \Gamma(\alpha; x/\theta)$$
$$= 1 - \Gamma\left(16, \frac{6000}{250}\right)$$
$$= 1 - \Gamma(16, 24)$$

Note: the incomplete gamma function is defined in p. 459 Textbook.

* You can find the result by using MATLAB Program as follows:

$pr(S_{16} > 6,000) = 1 - \text{gammainc}(24, 16) = 0.0344$

gammainc الدالة

or

$pr(S_{16} > 6,000) = 1 - \text{gamcdf}(6000, 16, 250) = 0.0344$

gamcdf الدالة

* By using central limit theorem

$\because$ The sum $S_k \sim \text{gamma}\left(\theta, \sum_{j=1}^{k} \alpha_j\right)$

$S_{16} \sim \text{gamma}(250, 16)$

From the central limit theorem, the sum has an approximate normal distribution with mean $\alpha\theta = 16(250) = 4,000$ and variance $\alpha\theta^2 = 16(250)^2 = 1,000,000$ (standard deviation of 1,000).

$\Longrightarrow$

$$pr(S_{16} > 6000) = 1 - \Phi\left(\frac{6000 - 4000}{1000}\right)$$
$$= 1 - \Phi(2) = 0.0228$$

, You can obtain the same result by using MATLAB as follows:

$pr(S_{16} > 6000) = 1 - \text{normcdf}(6000, 4000, 1000)$

$x \quad \mu \quad \sigma$

normcdf الدالة