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Lecture (20)
Mathematical Statistics
Ch.11 p. (229)

Maximum likelihood Estimation

Defn : The likelihood function is

$$L(\theta) = \prod_{j=1}^{n} pr(X_j \in A_j \mid \theta)$$

where A_j results from observing the random variable X_j. If A_j is a single point and the distribution is continuous, then $pr(X_j \in A_j \mid \theta)$ is interpreted as the probability density fn evaluated at that point.

Ex (1) p. 231
Suppose that the data in Data Set B, introduced in ch. 10 and reproduced here as Table 11.1 were such that the exact value of all observations above 250 was unknown. All that is known is that the value was greater than 250. Determine the maximum likelihood estimate of θ for an exponential distribution.

Ans:

Table 11.1 Data Set B

27	82	115	126	155	161	243	~~294~~	~~340~~	~~384~~
~~457~~	~~680~~	~~855~~	~~877~~	~~974~~	~~1,193~~	~~1,340~~	~~1,884~~	~~2,558~~	~~15,743~~

For the first seven values, the set A_j contains the single point equal to the observation x_j.

$\Rightarrow f(27)f(82)\ldots f(243)$

$= \theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\ldots\theta^{-1}e^{-243/\theta}$

$= \theta^{-7}e^{-909/\theta}$

$X \sim \exp(\theta)$

$f(x) = \dfrac{e^{-x/\theta}}{\theta}$

$S(x) = e^{-x/\theta}$

For each of the final 13 terms, the set A_j is the interval from 250 to infinity and, therefore,

$pr(X_j \in A_j) = pr(X_j > 250) = e^{-250/\theta}$

$\therefore L(\theta) = \theta^{-7}e^{-909/\theta}\left(e^{-250/\theta}\right)^{13} = \theta^{-7}e^{-4159/\theta}$

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$\therefore \quad \ell(\theta) = -7\ln\theta - 4159\theta^{-1}$

which is known as loglikelihood $f^{\underline{n}}$

$\ell(\theta) = \ln L(\theta)$

$\Rightarrow \ell'(\theta) = -7\theta^{-1} + 4{,}159\,\theta^{-2} = 0$

$\Rightarrow \hat{\theta} = \dfrac{4{,}159}{7} = 594.14$

which is the likelihood estimate of the parameter θ.

Note that: clearly $\ell''(\hat{\theta})$ is $-$ve quantity, i.e $\hat{\theta}$ is a maximum estimate for the parameter θ.

- Individual Data

Consider the special case where the value of each observation is recorded. It is easy to write the likelihood $f^{\underline{n}}$:

$$L(\theta) = \prod_{j=1}^{n} f_{X_j}(x_j|\theta) \qquad \text{(This for Complete data)}$$

and loglikelihood $f^{\underline{n}}$:

$$\ell(\theta) = \sum_{j=1}^{n} \ln f_{X_j}(x_j|\theta)$$

Ex. ② p. 232

Using Data set B, determine the maximum likelihood estimates for an exponential distribution, for a gamma dist$^{\underline{n}}$ where α is known to equal 2, and for a gamma dist$^{\underline{n}}$ where both parameters are unknown.

Ans:

* ① For exp. dist$^{\underline{n}}$:

$$L(\theta) = \prod_{j=1}^{n} \frac{e^{-x_j/\theta}}{\theta}$$

$$\therefore \ell(\theta) = \sum_{j=1}^{n} (-\ln\theta - x_j\theta^{-1})$$

$$\therefore \ell(\theta) = -n\ln\theta - n\bar{x}\theta^{-1}, \qquad \bar{x} = \frac{\sum_{j=1}^{n} x_j}{n}$$

let $\ell'(\theta)=0 \Rightarrow \ell'(\theta) = -n\theta^{-1} + n\bar{x}\theta^{-2} = 0$

$\Rightarrow \cancel{n}\theta = \cancel{n}\bar{x} \Rightarrow \therefore \hat{\theta} = \bar{x}$

From Data Set B, $\hat{\theta} = 1424.4$ and $\ell(\hat{\theta}) = -165.23$

* ② For the gamma dist$\underline{n}$ with $\alpha = 2$:

$$f(x|\theta) = \frac{x^{2-1}\, e^{-x/\theta}}{\Gamma(2)\,\theta^2}$$

$$f(x|\theta) = x\,\theta^{-2}\, e^{-x/\theta}$$

$X \sim \text{gamma}(\alpha, \theta)$

$f(x) = \dfrac{(x/\theta)^{\alpha}\, e^{-x/\theta}}{x\,\Gamma(\alpha)}$

$$\ln f(x_j|\theta) = \ln x_j - 2\ln\theta - x_j\theta^{-1}$$

$$\therefore\ l(\theta) = \sum_{j=1}^{n} \ln x_j - 2n\ln\theta - \theta^{-1}\sum_{j=1}^{n} x_j$$

$$\therefore\ l(\theta) = \sum_{j=1}^{n} \ln x_j - 2n\ln\theta - n\bar{x}\,\theta^{-1} \quad , \quad \sum_{j=1}^{n} x_j = n\bar{x}$$

$$\Rightarrow l'(\theta) = -2n\theta^{-1} + n\bar{x}\,\theta^{-2} = 0$$

$$\Rightarrow 2\theta^{-1} = \bar{x}\,\theta^{-2}$$

$$\therefore\ \hat{\theta} = \frac{\bar{x}}{2} \qquad \left(\text{i.e. } \hat{\theta} = \tfrac{1}{2}\bar{x}\right)$$

For Data Set B, $\hat{\theta} = 1{,}424.4/2 = 712.2$ and $l(\hat{\theta}) = -179.98$

* ③ For the gamma dist$\underline{n}$ with unknown parameters:

$$f(x|\alpha,\theta) = \frac{x^{\alpha-1}\, e^{-x/\theta}}{\Gamma(\alpha)\,\theta^{\alpha}}$$

$$\Rightarrow \ln f(x|\alpha,\theta) = (\alpha-1)\ln x - x\theta^{-1} - \ln\Gamma(\alpha) - \alpha\ln\theta$$

$$\therefore\ l(\alpha,\theta) = \sum_{j=1}^{n} \ln f_j(x_j|\alpha,\theta)$$

$$\Rightarrow l(\alpha,\theta) = (\alpha-1)\sum_{j=1}^{n}\ln x_j - \frac{n\bar{x}}{\theta} - n\ln\Gamma(\alpha) - n\alpha\ln\theta \qquad ①$$

To get $\hat{\alpha}, \hat{\theta}$, set $l'_{\theta}(\alpha,\theta) = 0$ and $l'_{\alpha}(\alpha,\theta) = 0$ and solving the resulting Eq$\underline{ns}$

$$l'_{\theta}(\alpha,\theta) = \frac{n\bar{x}}{\theta^2} - \frac{n\alpha}{\theta} = 0 \Rightarrow \frac{\bar{x}}{\theta} = \alpha \qquad ②$$

$$\& \ l'_{\alpha}(\alpha,\theta) = \sum_{j=1}^{n}\ln x_j - \frac{n}{\Gamma(\alpha)}\,\text{diff}[\Gamma(\alpha)] - n\ln\theta = 0 \qquad ③$$

The resulting eq$\underline{ns}$ ② and ③ cannot be solved analytically. Using numerical methods (by using MATLAB or Excel), the estimates are $\hat{\alpha} = 0.55616$ and $\hat{\theta} = 2561.1$ and $l(\hat{\alpha},\hat{\theta}) = -162.29$ ∥