# Lecture (17)

## Selected distributions and their relationships

See p. (72) Textbook

5.3.1 Introduction

5.3.2 Two parametric families

See transformed beta family and the transformed/inverse transformed gamma family in Figures 5.2 and 5.3 p. (72, 73)

5.3.3 Limiting distributions

Another way to relate distributions is to see what happens as parameters go to their limiting values of zero or infinity.

We have two important limits, to do this.

(1)
$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1$$
, which is known as Stirling's formula.

and

(2)
$$\lim_{a\to\infty}\left(1+\frac{x}{a}\right)^{a+b} = e^{x}$$
, It's a standard result found in most calculus texts.

**Example 5.10 p. (73)**

Show that the transformed gamma distribution is a limiting case of the transformed beta distribution as $\theta\to\infty$, $\alpha\to\infty$, and $\theta/\alpha^{1/\gamma}\to\xi$, a constant.

Ans: The transformed beta pdf is

$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\, x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\,\theta^{\gamma\tau}(1+x^{\gamma}\theta^{-\gamma})^{\alpha+\tau}} \quad (1)$$

Let $\theta = \xi\alpha^{1/\gamma}$ where $\theta/\alpha^{1/\gamma} = \xi = \text{Const.}$

when $\theta\to\infty$, $\alpha\to\infty$ and $\theta/\alpha^{1/\gamma}\to\xi$, a constant

$\Rightarrow f(x) \longrightarrow$ ???

By using Stirling's formula,

$$\lim_{\alpha\to\infty}\frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \quad (2)$$

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$$\Rightarrow \lim_{\alpha+\tau\to\infty} \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)} = 1 \quad (3)$$

(2), (3) in (1)

$$\Rightarrow f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}\cancel{(2\pi)^{1/2}}\,\gamma\, x^{\gamma\tau-1}}{\cancel{e^{-\alpha}}\,\alpha^{\alpha-1/2}\,\cancel{(2\pi)^{1/2}}\,\Gamma(\tau)(\xi\alpha^{1/\gamma})^{\gamma\tau}(1+x^{\gamma}\xi^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

where $\theta = \xi\alpha^{1/\gamma} \Rightarrow \theta^{-\gamma} = \xi^{-\gamma}\alpha^{-1}$

$$\therefore f(x) = \frac{e^{-\tau}\left(\frac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-1/2}\gamma\, x^{\gamma\tau-1}}{\Gamma(\tau)\,\xi^{\gamma\tau}\left[1+\frac{(x/\xi)^{\gamma}}{\alpha}\right]^{\alpha+\tau}}$$

$$\because \lim_{\alpha\to\infty}\left(\frac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-1/2} = \lim_{\alpha\to\infty}\left(1+\frac{\tau}{\alpha}\right)^{\alpha+\tau-1/2} = e^{\tau}$$

Note that
$$\lim_{\alpha\to\infty}\left(1+\frac{\tau}{\alpha}\right)^{\tau} = 1$$

and $$\lim_{\alpha\to\infty}\left[1+\frac{(x/\xi)^{\gamma}}{\alpha}\right]^{\alpha+\tau} = e^{(x/\xi)^{\gamma}}$$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{\cancel{e^{-\tau}}\,\cancel{e^{\tau}}\,\gamma\, x^{\gamma\tau-1}}{\Gamma(\tau)\,\xi^{\gamma\tau}\,e^{(x/\xi)^{\gamma}}}$$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{\gamma\, x^{\gamma\tau-1}\, e^{-(x/\xi)^{\gamma}}}{\Gamma(\tau)\,\xi^{\gamma\tau}}$$

which is the pdf of the transformed gamma dist$^n$ with parameters $\tau$, $\xi$ and $\gamma$.

For transformed gamma — $\alpha, \theta, \tau$

$$f(x) = \frac{\tau u^{\alpha} e^{-u}}{x\Gamma(\alpha)}, \quad u = (x/\theta)^{\tau}$$

See p. 467

$\alpha \to \tau$, $\theta \to \xi$, $\tau \to \gamma$