Lecture 16

Some Solved pbs For 5.2.7 Exercises p. 68 Textbook

- pb 5.8 p. 69
  Let N have a poisson distribution with mean Λ. Let Λ have a gamma distribution with mean 1 and variance 2. Determine the unconditional probability that $N=1$.

Ans:

$$\Lambda \sim gamma(\alpha, \theta) \quad , N|\Lambda \sim poisson(\lambda)$$

$$E(\Lambda) = \alpha\theta = 1 \quad , Var(\Lambda) = \alpha\theta^2 = 2$$

$$\Rightarrow \alpha\theta \cdot \theta = 2$$

$$\therefore 1\theta = 2 \Rightarrow \theta = 2, \alpha = 1/2$$

$$\because f_N(n) = \int f_{N|\Lambda}(n|\lambda) f(\lambda) d\lambda$$

$$\therefore f_N(1) = \int_0^\infty \frac{e^{-\lambda}\lambda^1}{1!} \cdot \frac{\lambda^{-0.5} e^{-0.5\lambda}}{\Gamma(0.5) \cdot 2^{0.5}}$$

$\Lambda \sim gamma(\alpha, \theta)$

$$\Rightarrow f(\lambda) = \frac{(\lambda/\theta)^\alpha e^{-\lambda/\theta}}{\lambda \Gamma(\alpha)}$$

$$f_N(1) = \frac{1}{\sqrt{2}\,\Gamma(0.5)} \int_0^\infty \lambda^{0.5} e^{-1.5\lambda} d\lambda$$

let $y = 1.5\lambda \Rightarrow d\lambda = \frac{dy}{1.5}, \lambda = \frac{y}{1.5}$

$$\therefore f_N(1) = \frac{1}{1.5\,\Gamma(0.5)\sqrt{2}} \int_0^\infty \frac{y^{0.5}}{\sqrt{1.5}} e^{-y} dy$$

$$\sqrt{1.5} = \sqrt{\frac{3}{2}}$$

$$f_N(1) = \frac{\Gamma(1.5)}{1.5\,\Gamma(0.5)\sqrt{3}}$$

$$f_N(1) = \frac{0.5\,\cancel{\Gamma(0.5)}}{1.5\,\cancel{\Gamma(0.5)}\sqrt{3}}$$

$$\therefore f_N(1) = 0.1925$$

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pb 5.12 p. 69

Determine the probability density function and the hazard rate of the frailty distribution.

Ans:

For frailty model

$$S_X(x) = M_\Lambda[-A(x)] \quad (1)$$

moment generating fn

Revise Lecture (15)

where $A(x) = \int_0^x a(t)\,dt$

$\therefore f_X(x) = -S'(x)$

$\therefore f_X(x) = -M'_\Lambda[-A(x)][-a(x)]$

$\therefore f_X(x) = a(x)\,M'_\Lambda[-A(x)] \quad (2)$

$\therefore h_X(x) = \dfrac{f_X(x)}{S_X(x)}$

(1), (2) $\Rightarrow$

$$\therefore h_X(x) = \frac{a(x)\,M'_\Lambda[-A(x)]}{M_\Lambda[-A(x)]} \quad (3)$$

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Pb 5.13 p. 69

Suppose that $X|\Lambda$ has a Weibull survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$, $x \geq 0$, and Λ has an exponential distribution. Demonstrate that the unconditional distribution of X is loglogistic.

Ans:

$$X|\Lambda \sim \text{Weibull}(\lambda, \gamma)$$

$$\Rightarrow S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}} = e^{-\lambda A(x)}, \quad x \geq 0$$

$$\Rightarrow \boxed{A(x) = x^{\gamma}} \quad ①$$

$\because \Lambda \sim$ exponential distn $- \theta$

$$\therefore M_{\Lambda}(z) = E(e^{z\Lambda}) = \frac{1}{1-\theta z}, \quad z < \frac{1}{\theta}$$

$$\therefore M_{\Lambda}(z) = (1-\theta z)^{-1} \quad ②$$

See Apendix A p. 470

$$\& \because S_X(x) = E[e^{-\Lambda A(x)}]$$

$$= M_{\Lambda}[-A(x)] \quad ③$$

Revise lecture 15

$$②,③ \Rightarrow S_X(x) = [1+\theta A(x)]^{-1}$$

$① \Rightarrow S_X(x) = (1+\theta x^{\gamma})^{-1}$ which is the survival function for loglogistic distn, where θ replaced by $\theta^{-1/\gamma}$, see Appendix A p. 466