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# Lecture (13)

## Ch 5

③ Exponentiation

**Theorem**

Let $X$ be a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$ and with $f_X(x) > 0$ for all real $x$. Let $Y = \exp(X)$. Then for $y > 0$

$$F_Y(y) = F_X(\ln y), \quad f_Y(y) = \frac{1}{y} f_X(\ln y)$$

**proof**

$$F_Y(y) = pr(Y \leq y)$$
$$= pr(e^X \leq y)$$
$$F_Y(y) = pr(X \leq \ln y) = F_X(\ln y)$$

Also, $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y)$

$$f_Y(y) = \frac{1}{y} f_X(\ln y) \qquad \#$$

EX 5.3 p.62 Textbook

Let $X$ have a normal dist^n with mean $\mu$ and variance $\sigma^2$. Determine the cdf and pdf of $Y = e^X$

Ans: $X \sim N(\mu, \sigma^2)$

$$\therefore F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$\therefore F_Y(y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right) \text{ and } f_Y(y) = \frac{1}{y\sigma} \phi\left(\frac{\ln y - \mu}{\sigma}\right)$$

$$\Rightarrow f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right]$$, which is the lognormal dist^n

④ Mixing

The concept of mixing can be extended from mixing a finite number of random variables to mixing an uncountable number. In the following theorem, the pdf $f(\lambda)$ plays the role of the discrete probabilities $a_j$ in the K-point mixture.

Theorem let $X$ have pdf $f_{X|\Lambda}(x|\lambda)$ and cdf $F_{X|\Lambda}(x|\lambda)$, where $\lambda$ is a parameter of $X$.
While $X$ may have other parameters, they are not relevant.
Let $\lambda$ be a realization of the random variable $\Lambda$ with pdf $f_{\Lambda}(\lambda)$.
Then, the unconditional pdf of $X$ is

$$f_X(x) = \int f_{X|\Lambda}(x|\lambda)\, f_{\Lambda}(\lambda)\, d\lambda, \quad \forall \lambda, \text{ s.t } f_{\Lambda}(\lambda)>0$$

The resulting distribution is a mixture distribution.
The distribution function is

$$F_X(x) = \int_{-\infty}^{x} \int f_{X|\Lambda}(y|\lambda)\, f_{\Lambda}(\lambda)\, d\lambda\, dy$$

$$= \int \int_{-\infty}^{x} f_{X|\Lambda}(y|\lambda)\, f_{\Lambda}(\lambda)\, dy\, d\lambda$$

$$F_X(x) = \int F_{X|\Lambda}(x|\lambda)\, f_{\Lambda}(\lambda)\, d\lambda$$

Moments of the mixture distribution can be found from

$$E(X^k) = E\left[E(X^k|\Lambda)\right]$$

and

$$Var(X) = E\left[Var(X|\Lambda)\right] + Var\left[E(X|\Lambda)\right].$$

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See, proof p. 63 Textbook.

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Ex (5.4) p. 63 Textbook

Let $X|\Lambda$ have an exponential distribution with parameter $1/\Lambda$. Let $\Lambda$ have a gamma distribution. Determine the unconditional distribution of $X$.

Ans: $X|\Lambda \sim exp(\lambda)$, $\Lambda \sim gamma(\alpha, \theta)$

$$f_X(x) = \int f_{X|\Lambda}(x|\lambda)\, f_\Lambda(\lambda)\, d\lambda$$

for $X \sim gamma(\alpha, \theta)$

$$f(x) = \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\,\Gamma(\alpha)}$$

$$f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} \int \lambda e^{-\lambda x} \lambda^{\alpha-1} e^{-\theta\lambda}\, d\lambda$$

(note that the parameter $\theta$ here in gamma distn has been replaced by its reciprocal)

$$\therefore f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)}\, d\lambda$$

$$f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)\,(x+\theta)^{\alpha+1}} \int_0^\infty u^\alpha e^{-u}\, du$$

where $u = \lambda(x+\theta) \Rightarrow \lambda = \frac{u}{x+\theta}$, $\lambda^\alpha = \frac{u^\alpha}{(x+\theta)^\alpha}$, $d\lambda = \frac{du}{x+\theta}$

$$\int_0^\infty u^\alpha e^{-u}\, du = \Gamma(\alpha+1)$$

gamma function, $\alpha > 0$

$$\therefore f_X(x) = \frac{\theta^\alpha}{(x+\theta)^{\alpha+1}} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \frac{\theta^\alpha\, \alpha \cancel{\Gamma(\alpha)}}{(x+\theta)^{\alpha+1}\, \cancel{\Gamma(\alpha)}}$$

$$\therefore f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}},$$

which is a pareto distribution. #

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