Ch 6 تابع

Continuous time Markov chains

Lecture (30)

عملية يول

• Yule process (linear pure birth process model)

Sec 6.1.3
+ pb 6.1.5
p. 282 & p. 284
see

Pb If $X(t)$ represents a size of a population

where $X(0) = 1$

Using the differential Eqns

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) \quad (1)$$

$$\frac{dP_n(t)}{dt} = \lambda_{n-1} P_{n-1}(t) - \lambda_n P_n(t) \quad (2)$$

$$, n = 1, 2, \dots$$

prove that $X(t) \sim geom(P)$, $P = e^{-\lambda t}$

where $\lambda_0 = 0$ and $\lambda_n = n\lambda$

and then find the mean and Variance of this process

Ans: $X(0) = 1 \Rightarrow P_n(0) = \begin{cases} 1 , & n = 1 \\ 0 , & \text{otherwise} \end{cases}$

أي أن العملية بدأت من الحالة 1 بنسبة 100% (أي $P_1(0) = 1$)

$\because \lambda_0 = 0 \quad (1) \Rightarrow \frac{dP_0(t)}{dt} = 0$

$\therefore P_0(t) = \text{Constant}$

$\because P_0(t) = 0$ at $t = 0$ i.e ($P_0(0) = 0$)

$\therefore \boxed{P_0(t) = 0} \quad (3)$

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$$② \Rightarrow \frac{dP_n(t)}{dt} + \lambda_n P_n(t) = \lambda_{n-1} P_{n-1}(t) \quad , n = 1, 2, \dots$$

$$\because \lambda_n = n\lambda \Rightarrow \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \frac{dP_n(t)}{dt} + n\lambda P_n(t) = (n-1)\lambda\, P_{n-1}(t) \,, n = 1, 2, \dots$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t} \frac{dP_n(t)}{dt} + n\lambda\, e^{n\lambda t} P_n(t) = (n-1)\lambda\, e^{n\lambda t} P_{n-1}(t)$$

$P_n(t)$??

$$\therefore \frac{d}{dt}\left[P_n(t)\, e^{n\lambda t}\right] = (n-1)\lambda\, P_{n-1}(t)\, e^{n\lambda t}$$

نفصل المتغيرات

تذكر

$\boxed{\frac{dy}{dx} = 5 \Rightarrow dy = 5dx}$

فصل المتغيرات

$$\therefore d\left[P_n(t)\, e^{n\lambda t}\right] = (n-1)\lambda\, P_{n-1}(t)\, e^{n\lambda t}\, dt$$

By Integration from 0 to t

$$\int_0^t d\left[P_n(x)\, e^{n\lambda x}\right] = (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx$$

$$\therefore \left[P_n(x)\, e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx \quad , n = 1, 2, \dots$$

$$\therefore P_n(t)\, e^{n\lambda t} - P_n(0) = (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx \quad , n = 1, 2, \dots$$

$$\therefore P_n(t)\, e^{n\lambda t} = P_n(0) + (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx$$

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Multiply both sides by $e^{-n\lambda t}$

$$\therefore P_n(t) = e^{-n\lambda t}\left[P_n(0) + (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx\right] \quad (4)$$

علاقة تكرارية

recurrence relation

at $n=1$ $\quad (4) \Rightarrow P_1(t) = e^{-\lambda t}\left[P_1(0) + 0\right]$

$$\therefore \boxed{P_1(t) = e^{-\lambda t}} \quad (5) \quad \text{where } P_1(0) = 1$$

at $n=2$ $\quad (4) \Rightarrow$

$$P_2(t) = e^{-2\lambda t}\left[P_2(0) + \lambda \int_0^t P_1(x)\, e^{2\lambda x}\, dx\right]$$

$$(5) \Rightarrow P_1(x) = e^{-\lambda x}$$

$$\therefore P_2(t) = e^{-2\lambda t} \cdot \lambda \int_0^t e^{-\lambda x}\, e^{2\lambda x}\, dx$$

$$P_2(t) = \lambda e^{-2\lambda t} \int_0^t e^{\lambda x}\, dx$$

$$P_2(t) = \cancel{\lambda} e^{-2\lambda t}\left[\frac{e^{\lambda x}}{\cancel{\lambda}}\right]_0^t$$

$$P_2(t) = e^{-2\lambda t}\left[e^{\lambda t} - 1\right]$$

$$P_2(t) = e^{-2\lambda t} \cdot e^{\lambda t}\left[1 - e^{-\lambda t}\right]$$

$$\therefore P_2(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{1} \quad (6)$$

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From 5, 6, We can deduce that

$$P_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$$

$$P_n(t) = p(1-p)^{n-1}, \quad p = e^{-\lambda t}$$

$\Rightarrow$ geometric dist$^{\underline{n}}$

$\therefore X(t) \sim \text{geom}(p), \; p = e^{-\lambda t}$

$$\text{Mean}[X(t)] = 1/p = e^{\lambda t}$$

$$\text{Variance}[X(t)] = \frac{1-p}{p^2}$$

$$= \frac{1 - e^{-\lambda t}}{e^{-2\lambda t}}$$

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