

PHYS 404

Lecture 2: Associated Legendre Functions

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Associated Legendre Functions

- As we have seen the associated Legendre d.e. is given by Eq. (1.9):

$$\frac{d}{dx} \left[(1-x^2) \frac{du}{dx} \right] + \left[n(n+1) - \frac{m^2}{1-x^2} \right] u = 0 \quad (2.1)$$

- We can prove that the solution of (2.1) is written as:

$$u(x) = (1-x^2)^{m/2} \frac{d^m y(x)}{dx^m}, \quad (2.2)$$

- Where $y(x)$ is a solution of Legendre d.e. (1.11)

Associated Legendre Functions

- The d.e. has two linearly independent solutions: the associated Legendre functions of the first and the second kind respectively:

$$P_n^m(x) = (1-x^2)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_n(x), \quad (2.3)$$

$$Q_n^m(x) = (1-x^2)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} Q_n(x) \quad (2.4)$$

Obviously the associated Legendre functions are not polynomials if m is odd.

The generating function

The *generating* function of the associated Legendre polynomials is given by:

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2xt+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^m(x) t^s \quad (2.5)$$

$|t| < 1$

This form lacks of any direct physical application and it is seldom used

The Recurrence Relations

Associated Legendre functions obey the following recurrence relations:

- $$P_n^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}} P_n^m(x) + [n(n+1) - m(m-1)] P_n^{m-1}(x) = 0$$
- $$(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + [n-m+1]P_{n+1}^m(x) = 0$$
- $$(2n+1)(1-x^2)^{1/2} P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)$$

$$= (n+m)(n+m-1)P_{n-1}^{m-1}(x) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(x)$$
- $$(1-x^2)^{1/2} \left[P_n^m(x) \right]' = \frac{1}{2} P_n^{m+1}(x) - \frac{1}{2} (n+m)(n-m+1) P_n^{m-1}(x)$$

Associated Legendre Functions

Special Properties

- The Parity property:(with respect to $x = 0$, $\theta = \pi/2$)

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x) \quad (2.7a)$$

$$P_n^m(\cos(\pi - \theta)) = (-1)^{n+m} P_n^m(\cos(\theta)) \quad (2.7b)$$

- When Rodriguez formula (1.22) could be shown that it takes the form:

$$P_n^m(x) = \frac{1}{2^n n!} (1 - x^2)^{m/2} \left(\frac{d}{dx} \right)^{n+m} (x^2 - 1)^n \quad (2.8)$$

Associated Legendre Functions

Orthogonality

Legendre's equation is self-adjoint. Which satisfies Sturm-Liouville theory where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Associated Legendre functions are a set of orthogonal functions on $(-1,1)$.

$$\int_{-1}^1 P_p^m(x) P_q^m(x) dx = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!} \delta_{p,q} \quad (2.9)$$

$$\int_0^\pi P_p^m(\cos\theta) P_q^m(\cos\theta) \sin\theta d\theta = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!} \delta_{p,q} \quad (2.10)$$

$$\int_{-1}^1 P_n^m(x) P_n^k(x) (1-x^2)^{-1} dx = \frac{(n+m)!}{m(n-m)!} \delta_{m,k} \quad (2.11)$$

Associated Legendre Functions *Series*

- According to Sturm-Liouville theory that Legendre functions form a complete set.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^m(x) \quad (2.12)$$

- The coefficients a_n are obtained by multiplying the series by $P_m(x)$ and integrating in the interval $[-1,1]$

$$a_n = \frac{(2n+1)}{2(2n)!} \int_{-1}^1 f(x) P_n^m(x) dx \quad (2.13)$$

Useful notes

- The associated Legendre functions are the same for m and $-m$ provided that m^2 appears in d. e. (2.1).
- Given that a polynomial can be differentiated as many times as its degree then the inequality must hold: $|m| \leq n \Rightarrow -n \leq m \leq n$
- This means that for given n there are $2n+1$ different values of m .

$$m = -n, -n+1, \dots, n-1, n$$

Useful notes

- We must be careful when work with m having values from 0 to $-l$. In this case:

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$