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Lecture (10)

Random Sums

p.59 Textbook

Theorem: the moments of a random sum

If $E(\xi_k) = \mu$, $Var(\xi_k) = \sigma^2$

, $E(N) = \nu$, $Var(N) = \tau^2$

μ — متوسط ξ
σ^2 — تباين ξ

$X = \xi_1 + \xi_2 + \cdots + \xi_k + \cdots \xi_N$

then $E(X) = \mu\nu$

and $Var(X) = \nu\sigma^2 + \mu^2\tau^2$

N متوسط , ξ_k تباين , ξ_k متوسط , N تباين

proof The random sum is

$$X = \begin{cases} 0 & , N = 0 \\ \xi_1 + \xi_2 + \cdots + \xi_N & , N > 0 \end{cases}$$

i) To prove that $\boxed{E(X) = \mu\nu}$

$$E(X) = \sum_{n=0}^{\infty} E[X \mid N = n]\, P_N(n)$$

$$E[g(X)] = \sum_{n=0}^{\infty} E[g(X) \mid N = n]\, P_N(n)$$

$$\therefore E(X) = \sum_{n=1}^{\infty} E[\xi_1 + \xi_2 + \cdots + \xi_N \mid N = n]\, P_N(n)$$

Def$^{\underline{n}}$ of X

$$\Rightarrow E(X) = \sum_{n=1}^{\infty} E\left[\xi_1 + \xi_2 + \dots + \xi_n\right] P_N(n)$$

where N is independent of $\xi_1, \xi_2, \dots$

Remember
For Independent
A, B
$pr(A|B) = pr(A)$
$\Rightarrow E(A|B) = E(A)$

$\because E(\xi_k) = \mu, \; k = 1, 2, \dots, n$

$$\therefore E(X) = \sum_{n=1}^{\infty} n \mu P_N(n)$$

$$= \mu \sum_{n=1}^{\infty} n P_N(n)$$

$$\therefore E(X) = \mu E(N) = \mu \upsilon$$

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ii) To prove that $\boxed{Var(X) = \upsilon \sigma^2 + \mu^2 \tau^2}$

$$Var(X) = E\left[(X - \underbrace{\mu\upsilon}_{X})^2\right]$$

$$= E\left[X - N\mu + N\mu - \upsilon\mu\right]^2$$

$$\therefore Var(X) = E\left[(X - N\mu)^2\right] + E\left[\mu^2 (N - \upsilon)^2\right] + 2E\left[\mu (X - N\mu)(N - \upsilon)\right]$$

$$\because E\left[(X - N\mu)^2\right] = \sum_{n=0}^{\infty} E\left[(X - N\mu)^2 \mid N = n\right] P_N(n) \quad (1)$$

$$= \sum_{n=1}^{\infty} E\left[(\xi_1 + \xi_2 + \dots + \xi_n - n\mu)^2 \mid N = n\right] P_N(n)$$

$$= \sum_{n=1}^{\infty} E(\xi_1 + \xi_2 + \dots + \xi_n - n\mu)^2 \cdot P_N(n)$$

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$\because Var(\xi_k) = \sigma^2, \; k = 1, 2, \dots, n$

$Var(\xi_k) = E(\xi_k - \mu)^2 = \sigma^2$

$$\therefore E[(X - N\mu)^2] = \sum_{n=1}^{\infty} n \sigma^2 P_N(n)$$

$$= \sigma^2 \sum_{n=1}^{\infty} n P_N(n)$$

$$\therefore E[(X - N\mu)^2] = \boxed{\upsilon \sigma^2} \quad \text{where} \quad \sum_{n=1}^{\infty} n P_N(n) = \upsilon$$

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$$\& \; E[\mu^2 (N - \upsilon)^2] = \mu^2 E[(N - \upsilon)^2]$$

$$= \mu^2 Var(N) = \mu^2 \tau^2$$

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Also, $E[\mu (X - N\mu)(N - \upsilon)]$

$$= \mu \sum_{n=1}^{\infty} E[(X - n\mu)(n - \upsilon) \mid N = n] \, P_N(n)$$

$$= \mu \sum (n - \upsilon) E[(X - n\mu) \mid N = n] \, P_N(n)$$

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$= 0$

where $E[(X - n\mu) \mid N = n] = E(X - n\mu)$

independent

$$= E(\xi_1 + \xi_2 + \dots + \xi_n - n\mu)$$

$$= n\mu - n\mu = 0$$

Substitute (2), (3) and (4) in (1), we get

$$\boxed{Var(X) = \nu\sigma^2 + \mu^2\tau^2}$$

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Pb 2.3.1 Text book

$N \sim poisson(\lambda)$, $\xi_k \sim Bernoulli(p)$

$\Rightarrow E(N) = \nu = \lambda$, $Var(N) = \tau^2 = \lambda$

$\Rightarrow E(\xi_k) = \mu = p$, $Var(\xi_k) = \sigma^2 = p(1-p)$

For $Z = \xi_1 + \xi_2 + \cdots + \xi_N$, $N > 0$, $\boxed{E(Z) = \mu\nu = \lambda p}$

$$\boxed{Var(Z) = \nu\sigma^2 + \mu^2\tau^2}$$

$$= \lambda p(1-p) + p^2\lambda$$

$$\therefore Var(Z) = \lambda p - \cancel{\lambda p^2} + \cancel{p^2\lambda}$$

$\therefore \boxed{E(Z) = \lambda p}$, $\boxed{Var(Z) = \lambda p}$

$\therefore Z \sim poisson(\lambda p)$ #