GENERAL MATHEMATICS 2

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1. Polar Coordinates
2. The Relationship between Cartesian and Polar Coordinates
3. Polar Curves
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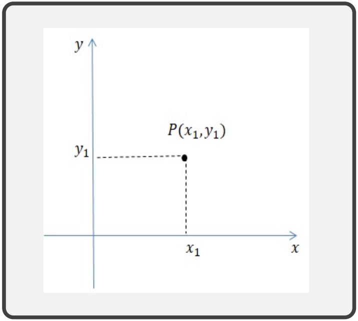
Polar Coordinates

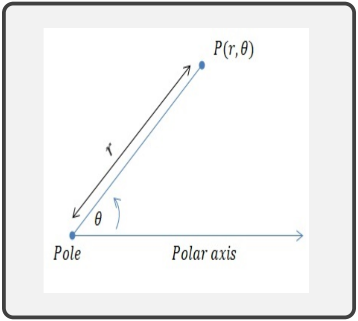
Definition

The polar coordinate system is a two-dimensional system consisted of a pole and polar axis (half line). Each point P in a polar plane is determined by a distance r from a fixed point O called the pole (or origin) and an angle θ from a fixed direction.

Cartesian Coordinates (Rectangular Coordinates)



Polar Coordinate



Polar Coordinates

Notes:
(1) From the definition, the point P in the polar coordinate system is represented by the ordered pair (r, θ) where r, θ are called polar coordinates.

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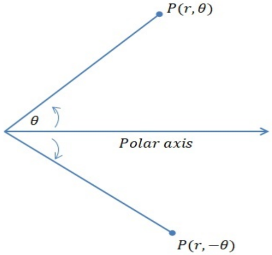

(2) The angle θ is positive if it is measured counterclockwise from the axis, but if it is measured clockwise the angle is negative.

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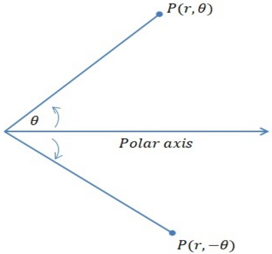
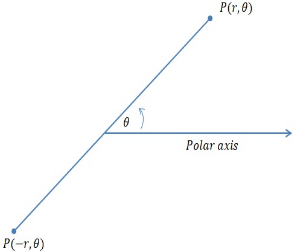
(1) From the definition, the point P in the polar coordinate system is represented by the ordered pair (r, θ) where r, θ are called polar coordinates.

(2) The angle θ is positive if it is measured counterclockwise from the axis, but if it is measured clockwise the angle is negative.

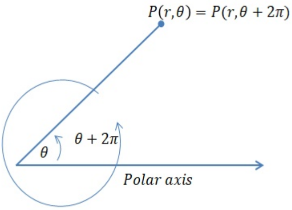


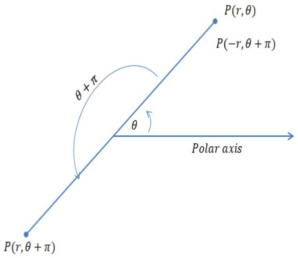
(3) In the polar coordinates, if $r > 0$, the point $P(r, \theta)$ will be in the same quadrant as θ; if $r < 0$, it will be in the quadrant on the opposite side of the pole with the half line. That is, the points $P(r, \theta)$ and $P(-r, \theta)$ lie in the same line through the pole O, but on opposite sides of O.



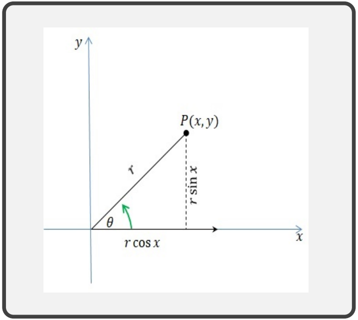
(4) In the Cartesian coordinate system, every point has only one representation while in a polar coordinate system each point has many representations. The following formula gives all representations of a point $P(r, \theta)$ in the polar coordinate system

$$P(r, \theta + 2n\pi) = P(r, \theta) = P(-r, \theta + (2n+1)\pi), \quad n \in \mathbb{Z}\ . \qquad (1)$$





The Relationship between Cartesian and Polar Coordinates



As shown in the figure:

1. Let (x, y) be a Cartesian coordinate and (r, θ) be a polar coordinate of the same point P.
2. Let the pole be at the origin of the Cartesian coordinates system.
3. Let the polar axis lies on the positive x-axis and the line $\theta = \frac{\pi}{2}$ lies on the positive y-axis.

From the right triangle, we have

$$\cos\ \theta = \frac{x}{r} \Rightarrow x = r\cos\ \theta \ \text{ and}$$

$$\sin\ \theta = \frac{y}{r} \Rightarrow y = r\sin\ \theta.$$

Hence,

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2 \qquad\qquad \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

This implies, $x^2 + y^2 = r^2$ and $\tan\ \theta = \frac{y}{x}$ for $x \neq 0$.

Polar Equations & Graphs in Polar Coordinates

Polar Equations

A polar equation is an equation in r and θ, $r = f(\theta)$.

A solution of the polar equation is an ordered pair (r_0, θ_0) satisfies the equation i.e., $r_0 = f(\theta_0)$. For example, $r = 2\cos\ \theta$ is a polar equation and $(1, \frac{\pi}{3})$, and $(\sqrt{2}, \frac{\pi}{4})$ are solutions of that equation.

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Graphs in Polar Coordinates

Lines in polar coordinates

1. The polar equation of a straight line $ax + by = c$ is $r = \dfrac{c}{a\cos\ \theta + b\sin\ \theta}$.

   Since $x = r\cos\ \theta$ and $y = r\sin\ \theta$, then

$$ax + by = c \Rightarrow r(a\cos\ \theta + b\sin\ \theta) = c \Rightarrow r = \frac{c}{(a\cos\ \theta + b\sin\ \theta)}$$

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2. The polar equation of a vertical line $x = k$ is $r = k\sec\ \theta$.

   Let $x = k$, then $r\cos\ \theta = k$. This implies $r = \dfrac{k}{\cos\ \theta} = k\sec\ \theta$.

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   Let $x = k$, then $r\cos\ \theta = k$. This implies $r = \dfrac{k}{\cos\ \theta} = k\sec\ \theta$.
3. The polar equation of a horizontal line $y = k$ is $r = k\csc\ \theta$.
   Let $y = k$, then $r\sin\ \theta = k$. This implies $r = \dfrac{k}{\sin\ \theta} = r\csc\ \theta$.

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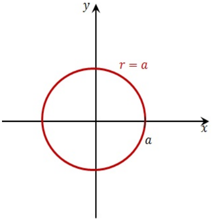
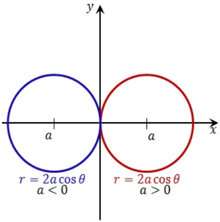
4. The polar equation of a line that passes the origin point and makes an angle θ_0 with the positive x-axis is $\theta = \theta_0$.

Circles in polar coordinates

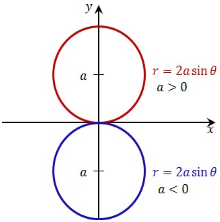
(1) The circle equation with center at the pole O and radius $|a|$ is $r = a$.



(2) The circle equation with center at $(a, 0)$ and radius $|a|$ is $r = 2a\cos\ \theta$.



(3) The circle equation with center at $(0, a)$ and radius $|a|$ is $r = 2a\sin\ \theta$.



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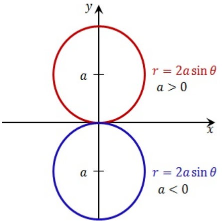
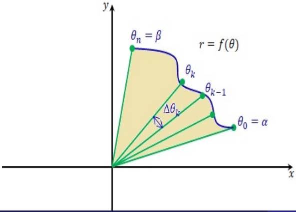
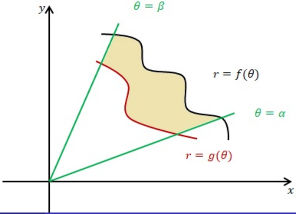


Area in Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2\ d\theta$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[(f(\theta))^2 - (g(\theta))^2\right]\ d\theta$$





Area in Polar Coordinates

Example

Find the area of the region bounded by the graph of the polar equation $r = 3$.

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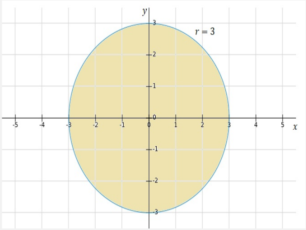
Solution:

From the figure, the area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2\, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\Big[\ \theta\ \Big]_0^{2\pi} = 9\pi.$$

Note that we can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2\, d\theta\Big) = 2\int_0^{\frac{\pi}{2}} 9\, d\theta = 18\Big[\ \theta\ \Big]_0^{\frac{\pi}{2}} = 9\pi.$$



Example

Find the area of the region that is between the curves $r = 2$ *and* $r = 3$ *in the first quadrant.*

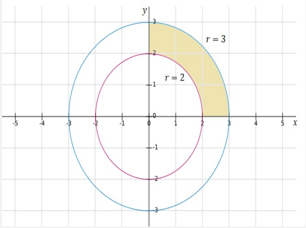
Area in Polar Coordinates

Example

Find the area of the region that is between the curves $r = 2$ and $r = 3$ in the first quadrant.

Solution: The region bounded by the two curves $r_1 = 2$ and $r_2 = 3$ is displayed in the figure.

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (r_2^2 - r_1^2)\, d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (9-4)\, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 5\, d\theta \\
&= \frac{5}{2}\Big[\theta\Big]_0^{\frac{\pi}{2}} \\
&= \frac{5}{2}\left[\frac{\pi}{2} - 0\right] \\
&= \frac{5\pi}{4}\ .
\end{aligned}
$$



Example

Find the area of the region that is between the curves $r = 2$ *and* $r = 3$ *where* $\frac{\pi}{2} \leq \theta \leq \pi$.

Example

Find the area of the region that is between the curves $r = 2$ and $r = 3$ where $\frac{\pi}{2} \leq \theta \leq \pi$.

Solution: The region bounded by the two curves $r_1 = 2$ and $r_2 = 3$ is displayed in the figure.

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (r_2^2 - r_1^2)\, d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (9 - 4)\, d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} 5\, d\theta \\ &= \frac{5}{2}\Big[\theta\Big]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{5}{2}\Big[\pi - \frac{\pi}{2}\Big] \\ &= \frac{5\pi}{4}\,. \end{aligned}$$