# GENERAL MATHEMATICS 2

Dr. M. Alghamdi

Department of Mathematics

November 3, 2022

## Main Contents

1. Chain Rule for Partial Derivatives
2. Implicit Differentiation

# Chain Rule for Partial Derivatives

## Definition

*If $g$ is a differentiable function at $x$ and $f$ is differentiable at $g(x)$, then the composite function*

$$F(x) = (f \circ g)(x) = f(g(x))$$

*is differentiable at $x$ as follows:*

$$\frac{dF}{dx} = \frac{df}{dg(x)} \frac{dg(x)}{dx} .$$

## Example

*If $y = \cos\ x^2$, calculate $\frac{dy}{dx}$.*

Solution:
Let $f(x) = \cos\ x$ and $g(x) = x^2$, then $(f \circ g)(x) = f(g(x)) = \cos\ x^2$.
It follows that

$$\frac{df}{dg(x)} = -\sin\ (g(x)) \text{ and } \frac{dg(x)}{dx} = 2x .$$

By applying the chain rule, we have

$$\frac{dy}{dx} = \frac{df}{dg(x)} \frac{dg(x)}{dx}$$
$$= -\sin\ (g(x))\ (2x) = -2x\ \sin\ x^2 .$$

# Chain Rule for Partial Derivatives

In the following, we expanded the chain rule for composite functions of two or three functions. Thus, we need to use the chain rule more than once.

1. If $w = f(x, y)$, $x = g(t)$, and $y = h(t)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} .$$

2. If $w = f(x, y)$, $x = g(t, s)$, and $y = h(t, s)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} .$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} .$$

3. If $w = f(x, y, z)$, $x = g(t, s)$, $y = h(t, s)$, and $z = k(t, s)$ such that $f$, $g$, $h$ and $k$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} .$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} .$$

# Chain Rule for Partial Derivatives

In the following, we expanded the chain rule for composite functions of two or three functions. Thus, we need to use the chain rule more than once.

1. If $w = f(x, y)$, $x = g(t)$, and $y = h(t)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}\ .$$

2. If $w = f(x, y)$, $x = g(t, s)$, and $y = h(t, s)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}\ .$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s}\ .$$

3. If $w = f(x, y, z)$, $x = g(t, s)$, $y = h(t, s)$, and $z = k(t, s)$ such that $f$, $g$, $h$ and $k$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\,\frac{\partial z}{\partial t}\ .$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\,\frac{\partial z}{\partial s}\ .$$

# Chain Rule for Partial Derivatives

In the following, we expanded the chain rule for composite functions of two or three functions. Thus, we need to use the chain rule more than once.

1. If $w = f(x, y)$, $x = g(t)$, and $y = h(t)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

2. If $w = f(x, y)$, $x = g(t, s)$, and $y = h(t, s)$ such that $f$, $g$ and $h$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

3. If $w = f(x, y, z)$, $x = g(t, s)$, $y = h(t, s)$, and $z = k(t, s)$ such that $f$, $g$, $h$ and $k$ are differentiable, then

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= y\, s^2 + (x+2y)\,(1)\\ &= (s+t)\, s^2 + s^2 t + 2s + 2t\\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= y\,(2st) + (x+2y)(1)\\ &= (s+t)(2st) + s^2 t + 2s + 2t\\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\, s^2 + (x + 2y)\,(1)$$
$$= (s + t)\, s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\,(2st) + (x + 2y)(1)$$
$$= (s + t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\,s^2 + (x + 2y)\,(1)$$
$$= (s+t)\,s^2 + s^2t + 2s + 2t$$
$$= s^3 + 2s^2t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\,(2st) + (x + 2y)(1)$$
$$= (s+t)(2st) + s^2t + 2s + 2t$$
$$= 3s^2t + 2st^2 + 2s + 2t\ .$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\, s^2 + (x + 2y)\,(1)$$
$$= (s+t)\, s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\,(2st) + (x+2y)(1)$$
$$= (s+t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x,y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\, s^2 + (x + 2y)\,(1)$$
$$= (s+t)\, s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\,(2st) + (x + 2y)(1)$$
$$= (s+t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y\, s^2 + (x + 2y)\,(1) \\ &= (s+t)\, s^2 + s^2 t + 2s + 2t \\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\,(2st) + (x + 2y)(1) \\ &= (s+t)(2st) + s^2 t + 2s + 2t \\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y\, s^2 + (x + 2y)\,(1) \\ &= (s+t)\, s^2 + s^2 t + 2s + 2t \\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\,(2st) + (x + 2y)(1) \\ &= (s+t)(2st) + s^2 t + 2s + 2t \\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x,y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\, s^2 + (x + 2y)\,(1)$$
$$= (s+t)\, s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\,(2st) + (x + 2y)(1)$$
$$= (s+t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y\, s^2 + (x + 2y)\,(1) \\ &= (s+t)\, s^2 + s^2 t + 2s + 2t \\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\,(2st) + (x + 2y)(1) \\ &= (s+t)(2st) + s^2 t + 2s + 2t \\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x,y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\ s^2 + (x + 2y)\ (1)$$
$$= (s+t)\ s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\ (2st) + (x + 2y)(1)$$
$$= (s+t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x, y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y\, s^2 + (x + 2y)\,(1) \\ &= (s + t)\, s^2 + s^2 t + 2s + 2t \\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\,(2st) + (x + 2y)(1) \\ &= (s + t)(2st) + s^2 t + 2s + 2t \\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x, y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\ s^2 + (x + 2y)\ (1)$$
$$= (s + t)\ s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\ (2st) + (x + 2y)(1)$$
$$= (s + t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\
&= y\ s^2 + (x + 2y)\ (1) \\
&= (s + t)\ s^2 + s^2 t + 2s + 2t \\
&= s^3 + 2s^2 t + 2s + 2t\ .
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\
&= y\ (2st) + (x + 2y)(1) \\
&= (s + t)(2st) + s^2 t + 2s + 2t \\
&= 3s^2 t + 2st^2 + 2s + 2t\ .
\end{aligned}$$

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= y\ s^2 + (x+2y)\ (1)\\ &= (s+t)\ s^2 + s^2 t + 2s + 2t\\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= y\ (2st) + (x+2y)(1)\\ &= (s+t)(2st) + s^2 t + 2s + 2t\\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y) = xy + y^2$, $x = s^2 t$, *and* $y = s + t$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= y\ s^2 + (x+2y)\ (1)$$
$$= (s+t)\ s^2 + s^2 t + 2s + 2t$$
$$= s^3 + 2s^2 t + 2s + 2t\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= y\ (2st) + (x+2y)(1)$$
$$= (s+t)(2st) + s^2 t + 2s + 2t$$
$$= 3s^2 t + 2st^2 + 2s + 2t\ .$$

## Example

*If $f(x, y) = xy + y^2$, $x = s^2 t$, and $y = s + t$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= y\ s^2 + (x + 2y)\ (1)\\ &= (s + t)\ s^2 + s^2 t + 2s + 2t\\ &= s^3 + 2s^2 t + 2s + 2t\ .\end{aligned}$$

(2)

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= y\ (2st) + (x + 2y)(1)\\ &= (s + t)(2st) + s^2 t + 2s + 2t\\ &= 3s^2 t + 2st^2 + 2s + 2t\ .\end{aligned}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ \left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\ \cos(ts\ (s+t)) + \left(\left(\frac{s}{t^2}\right)\ ts - \left(\frac{s}{t}\right)\ s\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\ \cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\; s + x\;\cos(xy)\;(1) - x\;\sin(xz)\;\left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\cos(ts\,(s+t)) + \left(\left(\frac{s}{t^2}\right) ts - \left(\frac{s}{t}\right) s\right)\;\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\;.$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\;t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\;s\;t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\;\frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\; t + x\;\cos(xy)\;(1) - x\;\sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\;.$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If* $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ \left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + \left(\left(\frac{s}{t^2}\right) ts - \left(\frac{s}{t}\right) s\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\, s + x\ \cos(xy)\,(1) - x\ \sin(xz)\left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\cos(ts\,(s+t)) + \left(\left(\frac{s}{t^2}\right) ts - \left(\frac{s}{t}\right) s\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\,(1) - x\ \sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\ \cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\ \cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ \left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + \left(\left(\frac{s}{t^2}\right) ts - \left(\frac{s}{t}\right) s\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

  $$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

  $$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$*,* $x = ts$*,* $y = s + t$*, and* $z = \frac{s}{t}$*, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$*,* $x = ts$*,* $y = s + t$*, and* $z = \frac{s}{t}$*, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\ \cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\ \cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x, y, z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:

(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ \left(\frac{-s}{t^2}\right)$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + \left(\left(\frac{s}{t^2}\right) ts - \left(\frac{s}{t}\right) s\right)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)\left(\frac{1}{t}\right)$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - \left(\left(\frac{s}{t}\right)t + \left(\frac{s}{t}\right)t\right)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

# Chain Rule for Partial Derivatives

## Example

*If $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, and $z = \frac{s}{t}$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$*,* $x = ts$*,* $y = s + t$*, and* $z = \frac{s}{t}$*, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

## Example

*If $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, and $z = \frac{s}{t}$, calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

# Chain Rule for Partial Derivatives

## Example

*If* $f(x,y,z) = x + \sin(xy) + \cos(xz)$, $x = ts$, $y = s + t$, *and* $z = \frac{s}{t}$, *calculate* **(1)** $\frac{\partial f}{\partial t}$ **(2)** $\frac{\partial f}{\partial s}$.

Solution:
(1)

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ s + x\ \cos(xy)\ (1) - x\ \sin(xz)\ (\frac{-s}{t^2})$$

$$= s + ((s+t)s + ts)\cos(ts\ (s+t)) + ((\frac{s}{t^2})\ ts - (\frac{s}{t})\ s)\ \sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t))\ .$$

(2)

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1 + y\cos(xy) - z\sin(xz))\ t + x\ \cos(xy)\ (1) - x\ \sin(xz)(\frac{1}{t})$$

$$= t + ((s+t)s + ts)\cos(ts(s+t)) - ((\frac{s}{t})t + (\frac{s}{t})t)\sin(s^2)$$

$$= s + (s^2 + 2ts)\cos(ts(s+t)) - 2s\sin(s^2)\ .$$

- For $\frac{\partial z}{\partial t}$, let $z = \frac{s}{t} = s\ t^{-1}$

$$\frac{\partial z}{\partial t} = (-1)\ s\ t^{-2} = \frac{-s}{t^2}$$

- For $\frac{\partial z}{\partial s}$, let $z = \frac{s}{t} = s\ \frac{1}{t}$

$$\frac{\partial z}{\partial s} = \frac{1}{t}$$

## Definition

1. *Suppose that the equation $F(x, y) = 0$ defines $y$ implicitly as a function of $x$, $y = f(x)$ such that $f$ is differentiable. Then,*
$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$
2. *Suppose that the equation $F(x, y, z) = 0$ defines $z$ implicitly as a function of $x$ and $y$, $z = f(x, y)$ such that $f$ is differentiable. Then,*
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

## Example

*Let $y^2 - xy + 3x^2 = 0$, find $\frac{dy}{dx}$.*

Solution:
Let $F(x, y) = y^2 - xy + 3x^2 = 0$, then
$$F_x = -y + 6x \text{ and } F_y = 2y - x$$
Hence,
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y + 6x}{2y - x} = \frac{y - 6x}{2y - x}.$$

## Definition

1. *Suppose that the equation $F(x, y) = 0$ defines $y$ implicitly as a function of $x$, $y = f(x)$ such that $f$ is differentiable. Then,*

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

2. *Suppose that the equation $F(x, y, z) = 0$ defines $z$ implicitly as a function of $x$ and $y$, $z = f(x, y)$ such that $f$ is differentiable. Then,*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

## Example

*Let $y^2 - xy + 3x^2 = 0$, find $\frac{dy}{dx}$.*

Solution:
Let $F(x, y) = y^2 - xy + 3x^2 = 0$, then

$$F_x = -y + 6x \quad \text{and} \quad F_y = 2y - x$$

Hence,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y + 6x}{2y - x} = \frac{y - 6x}{2y - x}.$$

# Implicit Differentiation

## Definition

1. *Suppose that the equation $F(x, y) = 0$ defines $y$ implicitly as a function of $x$, $y = f(x)$ such that $f$ is differentiable. Then,*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} .$$

2. *Suppose that the equation $F(x, y, z) = 0$ defines $z$ implicitly as a function of $x$ and $y$, $z = f(x, y)$ such that $f$ is differentiable. Then,*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} .$$

## Example

*Let $y^2 - xy + 3x^2 = 0$, find $\frac{dy}{dx}$.*

Solution:
Let $F(x, y) = y^2 - xy + 3x^2 = 0$, then

$$F_x = -y + 6x \text{ and } F_y = 2y - x$$

Hence,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y + 6x}{2y - x} = \frac{y - 6x}{2y - x} .$$

# Implicit Differentiation

## Definition

1. *Suppose that the equation* $F(x, y) = 0$ *defines* $y$ *implicitly as a function of* $x$, $y = f(x)$ *such that* $f$ *is differentiable. Then,*
$$\frac{dy}{dx} = -\frac{F_x}{F_y} .$$

2. *Suppose that the equation* $F(x, y, z) = 0$ *defines* $z$ *implicitly as a function of* $x$ *and* $y$, $z = f(x, y)$ *such that* $f$ *is differentiable. Then,*
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} .$$

## Example

*Let* $y^2 - xy + 3x^2 = 0$, *find* $\frac{dy}{dx}$.

Solution:
Let $F(x, y) = y^2 - xy + 3x^2 = 0$, then
$$F_x = -y + 6x \text{ and } F_y = 2y - x$$

Hence,
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y + 6x}{2y - x} = \frac{y - 6x}{2y - x} .$$

## Example

*Let* $F(x, y, z) = x^2 y + z^2 + \sin(xyz) = 0$, *calculate* **(1)** $\frac{\partial z}{\partial x}$ **(2)** $\frac{\partial z}{\partial y}$.

Solution: First, we find $F_x$, $F_y$ and $F_z$.

$$F_x = 2xy + yz\cos(xyz) \ ,$$

$$F_y = x^2 + xz\cos(xyz)$$

and

$$F_z = 2z + xy\cos(xyz) \ .$$

Hence,

1. $$\frac{\partial z}{dx} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}.$$
2. $$\frac{\partial z}{dy} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}.$$

## Example

*Let* $F(x, y, z) = x^2 y + z^2 + \sin(xyz) = 0$, *calculate* **(1)** $\frac{\partial z}{\partial x}$ **(2)** $\frac{\partial z}{\partial y}$.

Solution: First, we find $F_x$, $F_y$ and $F_z$.

$$F_x = 2xy + yz\cos(xyz) \ ,$$

$$F_y = x^2 + xz\cos(xyz)$$

and

$$F_z = 2z + xy\cos(xyz) \ .$$

Hence,

1. $$\frac{\partial z}{dx} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}.$$
2. $$\frac{\partial z}{dy} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}.$$

# Implicit Differentiation

## Example

*Let* $F(x, y, z) = x^2y + z^2 + \sin(xyz) = 0$, *calculate* **(1)** $\frac{\partial z}{\partial x}$ **(2)** $\frac{\partial z}{\partial y}$.

Solution: First, we find $F_x$, $F_y$ and $F_z$.

$$F_x = 2xy + yz\cos(xyz) \ ,$$

$$F_y = x^2 + xz\cos(xyz)$$

and

$$F_z = 2z + xy\cos(xyz) \ .$$

Hence,

1. $$\frac{\partial z}{dx} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}.$$
2. $$\frac{\partial z}{dy} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}.$$

## Example

*Let* $F(x, y, z) = x^2 y + z^2 + \sin(xyz) = 0$, *calculate* **(1)** $\frac{\partial z}{\partial x}$ **(2)** $\frac{\partial z}{\partial y}$.

Solution: First, we find $F_x$, $F_y$ and $F_z$.

$$F_x = 2xy + yz\cos(xyz) \ ,$$

$$F_y = x^2 + xz\cos(xyz)$$

and

$$F_z = 2z + xy\cos(xyz) \ .$$

Hence,

1. $$\frac{\partial z}{dx} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}.$$
2. $$\frac{\partial z}{dy} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}.$$