GENERAL MATHEMATICS 2

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Main Contents

1. Method of Cylindrical Shells

Volumes of Revolution Solids (Cylindrical Shells Method)

In this section, we study a new method to evaluate the volume of revolution solid called cylindrical shells method. In the washer method, we assume that the rectangle from each subinterval is vertical to the revolution axis while in the cylindrical shells method, the rectangle will be parallel to the revolution axis.

The figure shows a cylindrical shell. Let

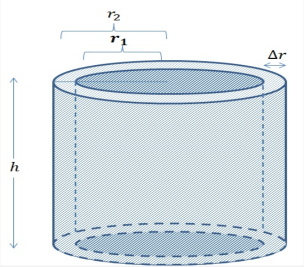
r_1 be the inner radius of the shell,

r_2 be the outer radius of the shell,

h be high of the shell,

$\Delta r = r_2 - r_1$ be the thickness of the shell,

$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.



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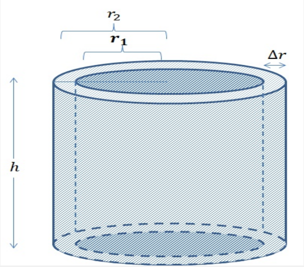

The volume of the cylindrical shell:
$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right)h(r_2 - r_1) \\ &= 2\pi r h \Delta r. \end{aligned}$$

Now, let R be a region R on the interval $[a, b]$ and S be a solid generated by revolving the region about y-axis. Let P be a partition of the interval $[a, b]$ and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a mark on P where ω_k is the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about the y-axis generates a cylindrical shell where
the high $= f(\omega_k)$,
the average radius $= \omega_k$,
the thickness $= \Delta x_k$.

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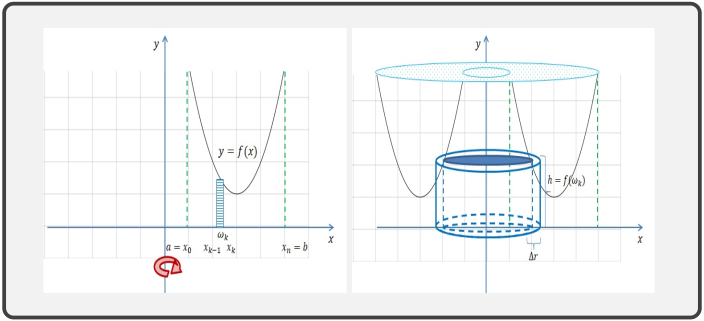

Volumes of Revolution Solids (Cylindrical Shells Method)

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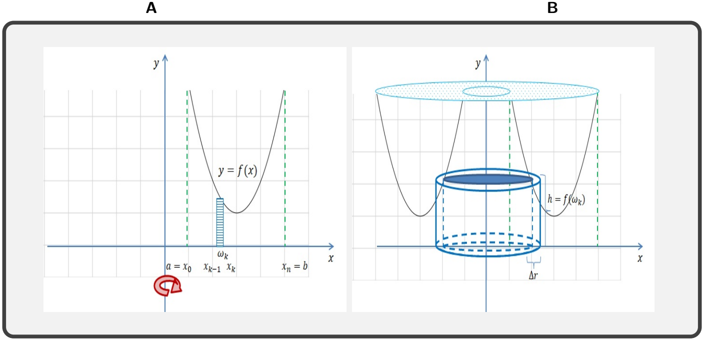


Hence, the volume of the cylindrical shell is $V_k = 2\pi\omega_k f(\omega_k)\Delta x_k$. To evaluate the volume of the whole solid, we sum the volumes of all cylindrical shells. This implies

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k.$$

Volumes of Revolution Solids (Cylindrical Shells Method)

From the Riemann sum

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k = \int_a^b x f(x)\ dx\ .$$

This implies

$$V = 2\pi \int_a^b x\ f(x)\ dx.$$

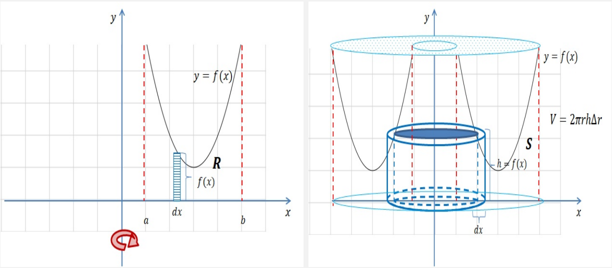
Similarly, we can find that if the revolution of the region is about x-axis, the volume of the revolution solid is

$$V = 2\pi \int_c^d y\ f(y)\ dy.$$

Volumes of Revolution Solids (Cylindrical Shells Method)

Theorem:

(1) If R is a region bounded by the graph of $y = f(x)$ on the interval $[a, b]$, the volume of the revolution solid generated by revolving R about y-axis is



1. The two points (area boundaries) on the $x-$axis.

2. Rotation about the $y-$axis.

3. The rectangle is parallel to the axis of rotation ($y-$axis).

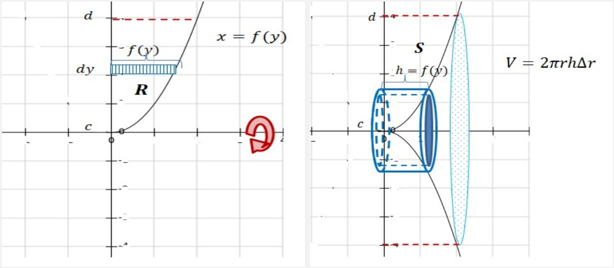
$$V = 2\pi \int_a^b x f(x)\, dx.$$

١. حدود المنطقة على محور س

٢. الدوران حول محور ص

٣. المستطيل يوازي محور الدوران (محور ص)

(2) If R is a region bounded by the graph of $x = f(y)$ on the interval $[c, d]$, the volume of the revolution solid generated by revolving R about x-axis is



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Note: The importance of the cylindrical shells method appears when solving equations for one variable in terms of another (i.e., solving x in terms of y). For example, let S be a solid generated by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about y-axis. By the washer method, we have to solve the cubic equation for x in terms of y, but this is not simple.

Example

Sketch the region R bounded by the graph of $y = 2x - x^2$ and x-axis. Then, by the cylindrical shells method, find the volume of the solid generated by revolving R about y-axis.

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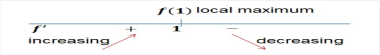
■ First derivative test.

$$y' = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$$



Volumes of Revolution Solids (Cylindrical Shells Method)

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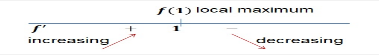

- Second derivative test.

$$y'' = 0 \Rightarrow y'' = -2 < 0$$

The graph of f is concave downward.

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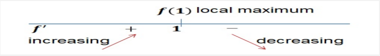

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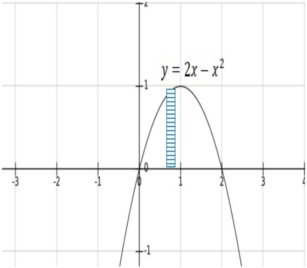

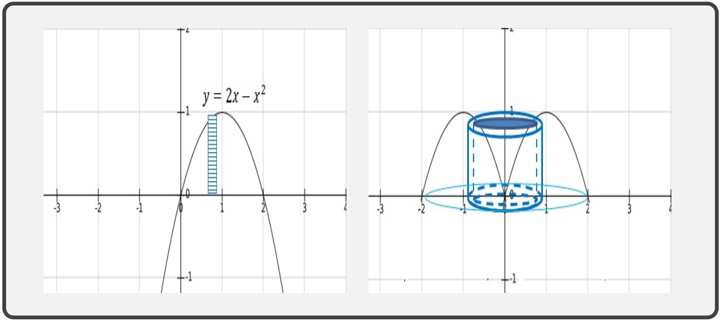

Since the revolution is about the y-axis, the rectangle is vertical and by revolving it, we obtain a cylindrical shell where
the high: $y = 2x - x^2$,
the average radius: x,
the thickness: dx.

The volume of the cylindrical shell is $dV = 2\pi x(2x - x^2)\,dx = 2\pi(2x^2 - x^3)\,dx$.

Thus, the volume of the solid over the interval $[0, 2]$ is

$$V = 2\pi \int_0^2 x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{8\pi}{3}$$

Volumes of Revolution Solids (Cylindrical Shells Method)

Example

Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $y = 4$, and y-axis. Then, find the volume of the solid generated by revolving R about x-axis.

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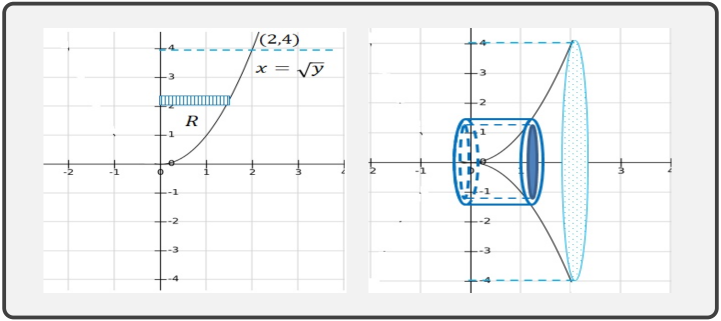

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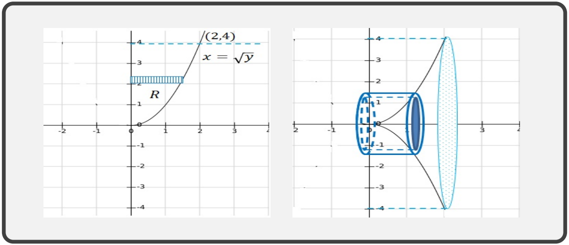


Since the revolution is about the x-axis, the rectangle is horizontal and by revolving it, we have a cylindrical shell where
the high: $x = \sqrt{y}$,
the average radius: y,
the thickness: dy.

The volume of the cylindrical shell is $dV = 2\pi\ y\ \sqrt{y}\ dy$.

Thus, the volume of the solid over the interval $[0, 4]$ is

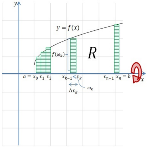
$$V = 2\pi \int_0^4 y\sqrt{y}\ dy = 2\pi \int_0^4 y^{\frac{3}{2}}\ dy$$
$$= \frac{4\pi}{5}\left[\ y^{\frac{5}{2}}\ \right]_0^4 = \frac{4\pi}{5}\Big[32 - 0\Big] = \frac{128\pi}{5}.$$

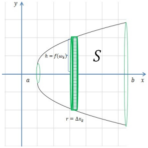
The difference between disk method and method of cylindrical shells

Disk Method

- **Region R:** bounded by :
  ◇ $y = f(x)$
  ◇ x-axis
  ◇ $[a, b]$ on x-axis
- **Revolution:** about x-axis
- **Rectangle:** vertical on the axis of revolution (x-axis)

$$V = \pi \int_a^b (f(x))^2 \, dx$$





Cylindrical Shells Method

- **Region R:** bounded by :
  ◇ $y = f(x)$
  ◇ x-axis
  ◇ $[a, b]$ on x-axis
- **Revolution:** about y-axis
- **Rectangle:** parallel to the axis of revolution (y-axis)

$$V = 2\pi \int_a^b x \, f(x) \, dx$$