GENERAL MATHEMATICS 2

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Main Contents

1. Volumes of Revolution Solids (Washer Method)

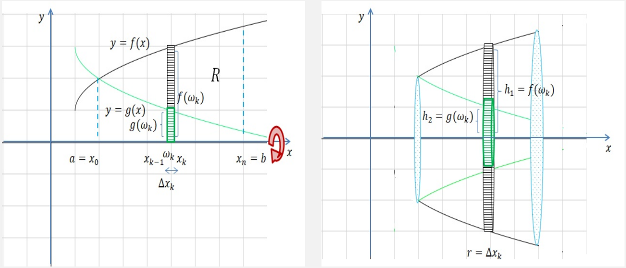
Volumes of Revolution Solids (Washer Method)

Let R be a region bounded by the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$ as shown in the figure. The volume of the solid S generated by revolving the region R about x-axis can be found by calculating the difference between the volumes of the two solids generated by revolving the regions under f and g about the x-axis as follows:

The outer radius: $y_1 = f(x)$
The inner radius: $y_2 = g(x)$
The thickness: dx
The volume of a washer is $dV = \pi \left[(\text{the outer radius})^2 - (\text{the inner radius})^2 \right]$. thickness.

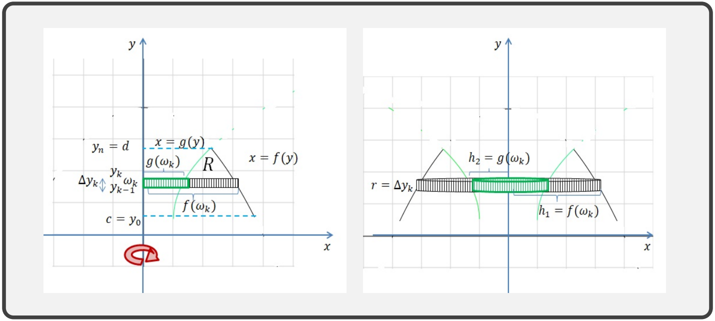
This implies $dV = \pi \left[(f(x))^2 - (g(x))^2 \right] dx$.
Hence, the volume of the solid over the interval $[a, b]$ is

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx.$$



Similarly, let R be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) \geq g(y)$ for all $y \in [c, d]$ as shown in Figure 4. The volume of the solid S generated by revolving R about the y-axis is

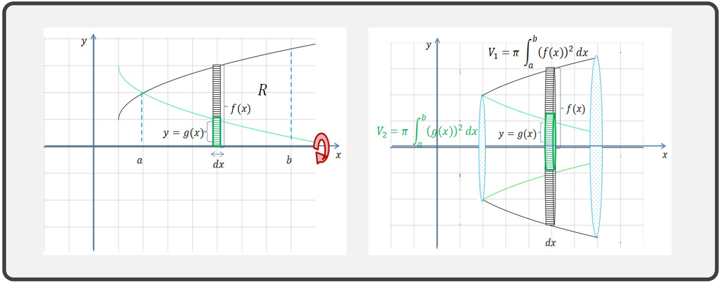
$$V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2 \right] dy.$$



Volumes of Revolution Solids (Washer Method)

Theorem:

(1) If R is a region bounded by the graphs of $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ such that $f \geq g$, the volume of the revolution solid generated by revolving R about x-axis is

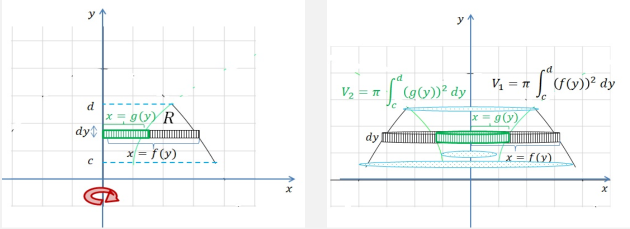


$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx.$$

1. The two points (area boundaries) on the $x-$axis.
2. Rotation about the $x-$axis.
3. The rectangles are perpendicular to the axis of rotation ($x-$axis).

Volumes of Revolution Solids (Washer Method)

(2) If R is a region bounded by the graphs of $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ such that $f \geq g$, the volume of the revolution solid generated by revolving R about y-axis is



$$V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2 \right) dy.$$

1. The two points (area boundaries) on the $y-$axis.

2. Rotation about the $y-$axis.

3. The rectangles are perpendicular to the axis of rotation ($y-$axis).

Volumes of Revolution Solids (Washer Method)

Example

Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about x-axis.

Volumes of Revolution Solids (Washer Method)

Example

Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about x-axis.

Solution: First, we check whether the graphs of the two functions are intersecting or not.

$$\begin{aligned} f(x) = g(x) \Rightarrow x^2 = 2x &\Rightarrow x^2 - 2x = 0 \\ &\Rightarrow x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Volumes of Revolution Solids (Washer Method)

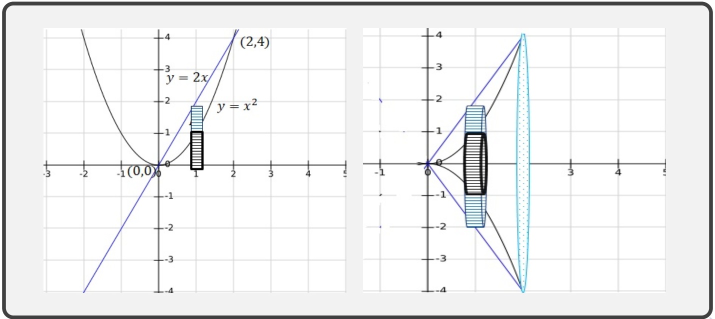
Example

Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about x-axis.

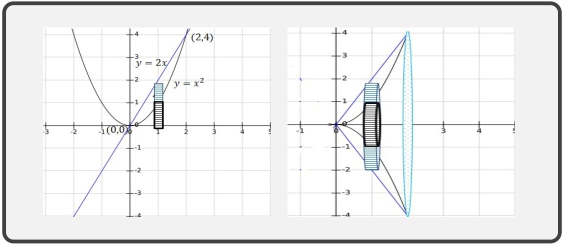
Solution: First, we check whether the graphs of the two functions are intersecting or not.

$$\begin{aligned} f(x) = g(x) \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x &= 0 \\ \Rightarrow x(x-2) &= 0 \\ \Rightarrow x = 0 \text{ or } x &= 2. \end{aligned}$$

By substitution, we have that the two curves intersect in two points $(0, 0)$ and $(2, 4)$.



Volumes of Revolution Solids (Washer Method)



The figure shows the region R and the solid generated by revolving the region about the x-axis. A vertical rectangle generates a washer where

- the outer radius: $y_1 = 2x$,
- the inner radius: $y_2 = x^2$ and
- the thickness: dx.

The volume of the washer is $dV = \pi \left[(2x)^2 - (x^2)^2 \right] \, dx$.

Hence, the volume of the solid over the interval $[0, 2]$ is

$$V = \pi \int_0^2 \left((2x)^2 - (x^2)^2 \right) \, dx = \pi \int_0^2 (4x^2 - x^4) \, dx$$

$$= \pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \pi \left[\frac{32}{3} - \frac{32}{5} \right] = \frac{64}{15}\pi.$$

Volumes of Revolution Solids (Washer Method)

Example

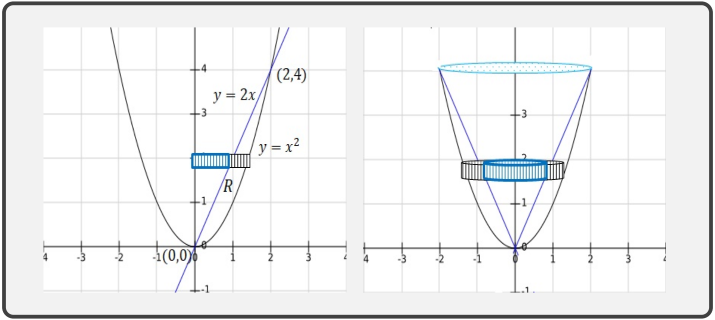
Consider the same region as in the previous example enclosed by the graphs of $y = x^2$ and $y = 2x$. Revolve the region about y-axis instead and find the volume of the generated solid.

Volumes of Revolution Solids (Washer Method)

Example

Consider the same region as in the previous example enclosed by the graphs of $y = x^2$ and $y = 2x$. Revolve the region about y-axis instead and find the volume of the generated solid.

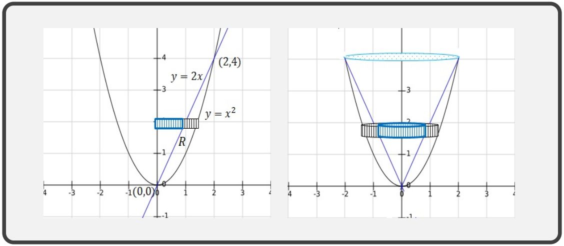
Solution: The figure shows the region R and the solid generated by revolving the region about the y-axis.



Since the revolution is about the y-axis, we need to rewrite the equations in term of y i.e., $x = f(y)$ and $x = g(y)$.

$$y = x^2 \Rightarrow x = \sqrt{y} = f(y) \text{ and } y = 2x \Rightarrow x = \frac{y}{2} = g(y) .$$

Volumes of Revolution Solids (Washer Method)



The two horizontal rectangles generate a washer where

- the outer radius: $x_1 = \sqrt{y}$,
- the inner radius: $x_2 = \frac{y}{2}$ and
- the thickness: dy.

The volume of the washer is $dV = \pi \left[(\sqrt{y})^2 - (\frac{y}{2})^2 \right] dy$.

Hence, the volume of the solid over the interval $[0, 4]$ is

$$V = \pi \int_0^4 \left((\sqrt{y})^2 - (\frac{y}{2})^2 \right) dy = \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \frac{8}{3} \pi .$$

Volumes of Revolution Solids (Washer Method)

Example

Consider a region R bounded by the graphs of the functions $y = \sqrt{x}$, $y = 6 - x$ *and* x*-axis. Revolve the region about* y*-axis and find the volume of the generated solid.*

Example

Consider a region R bounded by the graphs of the functions $y = \sqrt{x}$, $y = 6 - x$ and x-axis. Revolve the region about y-axis and find the volume of the generated solid.

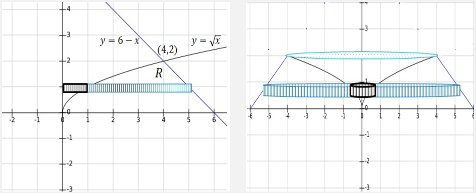
Solution: Since the revolution is about the y-axis, we need to rewrite the functions in terms of y i.e., $x = f(y)$ and $x = g(y)$.

$$y = \sqrt{x} \Rightarrow x = y^2 = f(y) \text{ and } y = 6 - x \Rightarrow x = 6 - y = g(y).$$

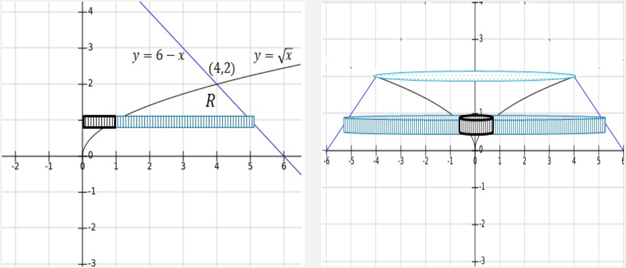
Now, we find the intersection points:
$f(y) = g(y) \Rightarrow y^2 = 6 - y \Rightarrow y^2 + y - 6 = 0 \Rightarrow (y + 3)(y - 2) = 0 \Rightarrow y = -3$ or $y = 2$.

Note: since $y = \sqrt{x}$, we ignore the value $y = -3$.

By substituting $y = 2$ into the two functions, we have $x = 4$. Thus, the two curves intersect in one point $(4, 2)$. The solid S generated by revolving the region R about y-axis is shown in the figure.



Volumes of Revolution Solids (Washer Method)



Also, the revolution is about the y-axis, so we have a horizontal rectangle that generates a washer where

- the outer radius: $x_1 = 6 - y$,
- the inner radius: $x_2 = y^2$ and
- the thickness: dy.

The volume of the washer is $dV = \pi\left[(6-y)^2 - (y^2)^2\right]\, dy$.

The volume of the solid over the interval $[0, 2]$ is

$$V = \pi \int_0^2 \left[(6-y)^2 - (y^2)^2\right] dy = \pi\left[-\frac{(6-y)^3}{3} - \frac{y^5}{5}\right]_0^2 = \pi\left[\left(-\frac{64}{3} - \frac{32}{5}\right) - \left(-\frac{216}{3} - 0\right)\right] = \frac{664}{15}\pi.$$

Volumes of Revolution Solids (Washer Method)

Example

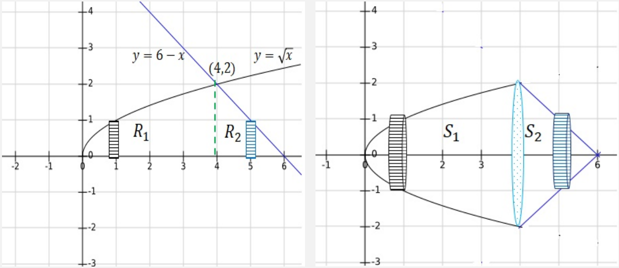
Consider the same region as in the previous example enclosed by the graphs of $y = \sqrt{x}$, $y = 6 - x$ *and* x*-axis. Revolve the region about* x*-axis instead and find the volume of the generated solid.*

Volumes of Revolution Solids (Washer Method)

Example

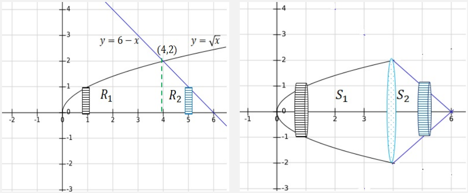
Consider the same region as in the previous example enclosed by the graphs of $y = \sqrt{x}$, $y = 6 - x$ *and* x*-axis. Revolve the region about* x*-axis instead and find the volume of the generated solid.*

Solution:



Note: the solid is made up of **two separate regions**: R_1 and R_2, and each requires its own integral. We use the disk method to evaluate the volume of the solid generated by revolving each region.

Volumes of Revolution Solids (Washer Method)



(1) Region R_1: revolution it about the x-axis generates a vertical disk with radius $y = \sqrt{x}$ and thickness dx:

$$V_1 = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{\pi}{2} \left[x^2 \right]_0^4 = 8\pi.$$

(2) Region R_2: revolution it about the x-axis generates a vertical disk with radius $y = 6 - x$ and thickness dx:

$$V_2 = \pi \int_4^6 (6-x)^2 \, dx = \pi \int_4^6 (6-x)^2 \, dx = -\frac{\pi}{3} \left[(6-x)^3 \right]_4^6 = \frac{8}{3}\pi.$$

The volume of the total solid:

$$\begin{aligned} V &= V_1 + V_2 \\ &= 8\pi + \frac{8}{3}\pi = \frac{32}{3}\pi. \end{aligned}$$