GENERAL MATHEMATICS 2

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Main Contents

Find the derivative of the given function.
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Section 1: Antiderivatives

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Section 1: Antiderivatives

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A function F is called an antiderivative function of a function f on an interval I if

$$F^{'}(x) = f(x) \text{ for every } x \in I.$$

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Example:

1. Consider the functions $F(x) = x^3 + 4x^2 - x$ and $f(x) = 3x^2 + 8x - 1$.

   Since $F^{'}(x) = 3x^2 + 8x - 1 = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.

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2. Consider the functions $G(x) = \tan\ x + x^2$ and $g(x) = \sec^2\ x + 2x$.

   Since $G^{'}(x) = \sec^2\ x + 2x = g(x)$, then the function $G(x)$ is an antiderivative of $g(x)$.

Definition

Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x)\, dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

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Section 2: Indefinite Integrals

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Properties of Indefinite Integrals

Theorem

Assume f and g have antiderivatives on an interval I, then

1. $\dfrac{d}{dx}\displaystyle\int f(x)\, dx = f(x).$
2. $\displaystyle\int \frac{d}{dx}(F(x))\, dx = F(x) + c.$
3. $\displaystyle\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx.$
4. $\displaystyle\int kf(x)\, dx = k\int f(x)\, dx$, *where k is a constant.*

Section 2: Indefinite Integrals

Integration as an Inverse Process of Differentiation

Rule 1: Power of x.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \;\Rightarrow\; \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1 \; .$$

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Special case: For $n = 0$, we have $\int 1 \, dx = x + c$.

From this, $\int 2 \, dx = 2x + c$ and $\int 3 \, dx = 3x + c$ etc.

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Note that **Rule 1** cannot be applied for $n = -1$.

For this value, the formula gives

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Example

Evaluate the integral.

1. $\int x \, dx$
2. $\int x^3 \, dx$

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Example

Evaluate the integral.

1. $\int x\, dx$
2. $\int x^3\, dx$

Solution:

1. $\int x\, dx = \frac{x^2}{2} + c$

Integration as an Inverse Process of Differentiation

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Example

Evaluate the integral.

1. $\int x\, dx$
2. $\int x^3\, dx$

Solution:

1. $\int x\, dx = \frac{x^2}{2} + c$
2. $\int x^3\, dx = \frac{x^4}{4} + c$

Example

Evaluate the integral.

1. $\int 4\,x\;dx$
2. $\int 7\,x^3\;dx$
3. $\int \frac{1}{x^2}\;dx$
4. $\int \frac{1}{\sqrt{x}}\;dx$

Example

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4. $\int \frac{1}{\sqrt{x}}\,dx$

Solution:

1. $\int 4\,x\,dx = 4\int x\,dx = 4\,\frac{x^2}{2} + c = 2\,x^2 + c$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Example

Evaluate the integral.

1. $\int 4\,x\;dx$
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Solution:

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2. $\int 7\,x^3\;dx = 7\,\frac{x^4}{4} + c$ $\qquad \int kf(x)\;dx = k\int f(x)\;dx$

Example

Evaluate the integral.

1. $\int 4\,x\;dx$
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Solution:

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2. $\int 7\,x^3\;dx = 7\,\frac{x^4}{4} + c$ — $\int kf(x)\;dx = k\int f(x)\;dx$
3. $\int \frac{1}{x^2}\;dx = \int x^{-2}\;dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$ — *Remember:* $x^{-n} = \frac{1}{x^n}$

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4. $\int \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{x^{\frac{1}{2}}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

   $-\frac{1}{2} + \frac{1}{1} = \frac{-1+2}{2} = \frac{1}{2}$

Remember:

(1) $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[m]{x^n} = x^{\frac{n}{m}}$

(2) $\frac{a}{b} \pm \frac{c}{d} = \frac{a\times d \pm c\times b}{b\times d}$

Example

Evaluate the integral.

1. $\int (x+1)\ dx$
2. $\int (4x^3 + 2x^2 + 1)\ dx$
3. $\int \left(x^2 - \frac{1}{x^3}\right)\ dx$

Example

Evaluate the integral.

1. $\int (x+1)\ dx$
2. $\int (4x^3 + 2x^2 + 1)\ dx$
3. $\int (x^2 - \frac{1}{x^3})\ dx$

Solution:

1. $\int (x+1)\ dx = \int x\ dx + \int 1\ dx = \frac{x^2}{2} + x + c$

$$\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$$

Example

Evaluate the integral.

1. $\int (x+1)\ dx$
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Solution:

1. $\int (x+1)\ dx = \int x\ dx + \int 1\ dx = \frac{x^2}{2} + x + c$ $\quad \int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$
2. $\int (4x^3 + 2x^2 + 1)\ dx = \int 4x^3\ dx + \int 2x^2\ dx + \int 1\ dx = \frac{4x^4}{4} + \frac{2}{3}x^3 + x + c = x^4 + \frac{2}{3}x^3 + x + c$.

Example

Evaluate the integral.

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Solution:

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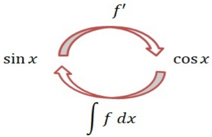
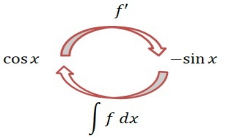
3. $\int \left(x^2-\frac{1}{x^3}\right)\ dx = \int x^2\ dx - \int x^{-3}\ dx = \frac{x^3}{3}+\frac{x^{-2}}{2}+c = \frac{x^3}{3}+\frac{1}{2x^2}+c\ .$

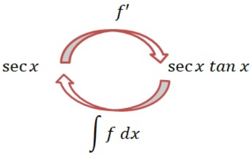
Rule 2: Trigonometric Functions.

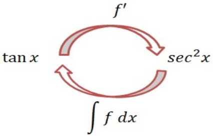
- $\frac{d}{dx}(\sin\ x) = \cos\ x \Rightarrow \int \cos\ dx = \sin\ x + c$
- $\frac{d}{dx}(\cos\ x) = -\sin\ x \Rightarrow \int -\sin\ x\ dx = \cos\ x + c$ OR $\int \sin\ x\ dx = -\cos\ x + c$
- $\frac{d}{dx}(\tan\ x) = \sec^2\ x \Rightarrow \int \sec^2\ x\ dx = \tan\ x + c$
- $\frac{d}{dx}(\cot\ x) = -\csc\ x \Rightarrow \int -\csc^2\ x\ dx = \cot\ x + c$ OR $\int \csc^2\ x\ dx = -\cot\ x + c$
- $\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x \Rightarrow \int \sec\ x\ \tan\ x\ dx = \sec\ x + c$
- $\frac{d}{dx}(\csc\ x) = -\csc\ x\ \cot\ x \Rightarrow \int -\csc\ x\ \cot\ x\ dx = \csc\ x + c$ OR $\int \csc\ x\ \cot\ x\ dx = -\csc\ x + c$

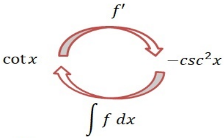




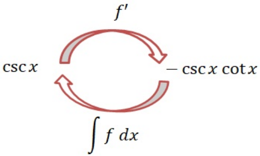
$$\int \sin x \;\; dx = -\cos x + c$$







$$\int csc^2\, x \;\; dx = -\cot x + c$$



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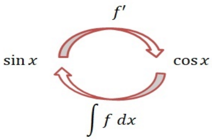

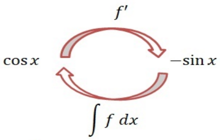

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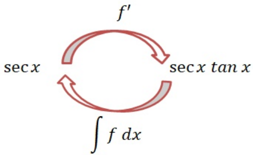

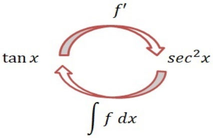

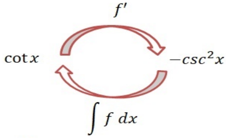

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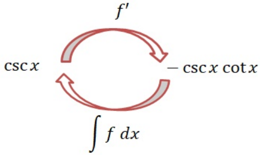

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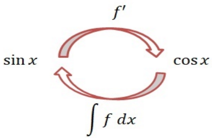
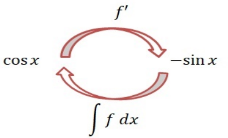
Example

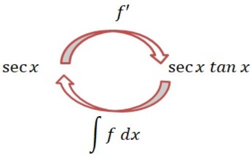
Evaluate the integral $\displaystyle\int (\cos x + \sec x \;\tan x)\; dx$

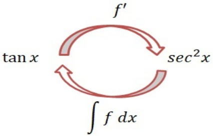
Section 2: Indefinite Integrals

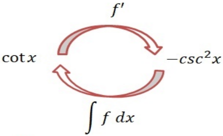




$$\int \sin x \; dx = -\cos x + c$$







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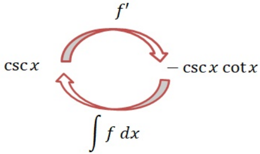

$$\int csc\, x \cot x \;\; dx = -\csc x + c$$

Example

Evaluate the integral $\int (\cos x + \sec x \;\tan x)\; dx$

Solution:

$$\int (\cos x + \sec x \;\tan x)\; dx = \int \cos x \; dx + \int \sec x \;\tan x \; dx = \sin x + \sec x + c$$

Example

Evaluate the integral.

1. $\displaystyle\int \left(\frac{1}{\cos^2 x} - \sin x \right) \, dx$
2. $\displaystyle\int \sec x \, (\sec x + \tan x) \, dx$

Example

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Solution:

1. $\int \left(\frac{1}{\cos^2 x} - \sin x \right) dx = \int \sec^2 x \, dx - \int \sin x \, dx = \tan x + \cos x + c$.

   $\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x}$

Example

Evaluate the integral.

1. $\int \left(\frac{1}{\cos^2 x} - \sin x \right) \, dx$
2. $\int \sec x \, (\sec x + \tan x) \, dx$

Solution:

1. $\int \left(\frac{1}{\cos^2 x} - \sin x \right) \, dx = \int \sec^2 x \, dx - \int \sin x \, dx = \tan x + \cos x + c$.

   $\sec x = \frac{1}{\cos x} \Rightarrow \sec^2 x = \frac{1}{\cos^2 x}$

2. $\int \sec x \, (\sec x + \tan x) \, dx = \int \sec^2 x \, dx + \int \sec x \, \tan x \, dx = \tan x + \sec x + c$.

Rule 3: Natural Logarithmic and Exponential Functions.

If $u = g(x)$ is a differentiable function, then

- $\frac{d}{dx}\left(\ln\ |u|\right) = \frac{u'}{u} \Longrightarrow \int \frac{u'}{u}\ dx = \ln|u| + c$
- $\frac{d}{dx}\left(e^{u}\right) = e^{u}.u' \Longrightarrow \int e^{u}\ u'\ dx = e^{u} + c$

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Example

Evaluate the integral.

1. $\int \frac{2}{2x-7}\ dx$
2. $\int \frac{x+3}{x^2+6x+5}\ dx$
3. $\int 3x^2\ e^{x^3}\ dx$
4. $\int \cos x\ e^{\sin x}\ dx$

Rule 3: Natural Logarithmic and Exponential Functions.

If $u = g(x)$ is a differentiable function, then

- $\frac{d}{dx}\left(\ln\ |u|\right) = \frac{u'}{u} \Longrightarrow \int \frac{u'}{u}\ dx = \ln|u| + c$
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Solution:

1. $\int \frac{2}{2x-7}\ dx = \ln|2x-7| + c$
2. $\int \frac{x+3}{x^2+6x+5}\ dx = \frac{1}{2}\int \frac{2(x+3)}{x^2+6x+5}\ dx$

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Solution:

1. $\displaystyle\int \frac{2}{2x-7}\ dx = \ln|2x-7| + c$
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Solution:

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3. $\int 3x^2\ e^{x^3}\ dx = e^{x^3} + c$
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Section 2: Indefinite Integrals

Rule 4: Inverse Trigonometric Functions.

- $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
- $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

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Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$
2. $\displaystyle\int \frac{1}{9 + x^2}\, dx$

Rule 4: Inverse Trigonometric Functions.

- $$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$
- $$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$
- $$\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$$

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Evaluate the integral.

1. $$\int \frac{1}{\sqrt{4 - x^2}}\, dx$$
2. $$\int \frac{1}{9 + x^2}\, dx$$

Solution:

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Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$
2. $\displaystyle\int \frac{1}{9 + x^2}\, dx$

Solution:

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx = \int \frac{1}{\sqrt{2^2 - x^2}}\, dx$

Rule 4: Inverse Trigonometric Functions.

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Example

Evaluate the integral.

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$
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Solution:

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx = \int \frac{1}{\sqrt{2^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{2}\right) + c$

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Example

Evaluate the integral.

1. $\int \frac{1}{\sqrt{4 - x^2}}\, dx$
2. $\int \frac{1}{9 + x^2}\, dx$

Solution:

1. $\int \frac{1}{\sqrt{4 - x^2}}\, dx = \int \frac{1}{\sqrt{2^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{2}\right) + c$
2. $\int \frac{1}{9 + x^2}\, dx$

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Example

Evaluate the integral.

1. $$\int \frac{1}{\sqrt{4 - x^2}}\, dx$$
2. $$\int \frac{1}{9 + x^2}\, dx$$

Solution:

1. $$\int \frac{1}{\sqrt{4 - x^2}}\, dx = \int \frac{1}{\sqrt{2^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{2}\right) + c$$
2. $$\int \frac{1}{9 + x^2}\, dx = \int \frac{1}{3^2 + x^2}\, dx$$

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Example

Evaluate the integral.

1. $\displaystyle\int \frac{5\,x}{4+x^2}\,dx$
2. $\displaystyle\int \frac{5}{4+x^2}\,dx$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{5\ x}{4+x^2}\ dx$
2. $\displaystyle\int \frac{5}{4+x^2}\ dx$

Solution:

1.

$$\int \frac{5\ x}{4+x^2}\ dx = 5\ \int \frac{x}{4+x^2}\ dx$$
$$= 5\ \frac{1}{2} \int \frac{2x}{4+x^2}\ dx$$
$$= \frac{5}{2}\ \ln(4+x^2) + c\ .$$

Example

Evaluate the integral.

1. $\displaystyle\int \frac{5\,x}{4+x^2}\,dx$
2. $\displaystyle\int \frac{5}{4+x^2}\,dx$

Solution:

1.

$$\begin{aligned}\int \frac{5\,x}{4+x^2}\,dx &= 5\int \frac{x}{4+x^2}\,dx\\ &= 5\,\frac{1}{2}\int \frac{2x}{4+x^2}\,dx\\ &= \frac{5}{2}\,\ln(4+x^2)+c\,.\end{aligned}$$

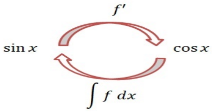
2.

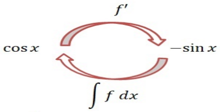
$$\begin{aligned}\int \frac{5}{4+x^2}\,dx &= 5\int \frac{1}{2^2+x^2}\,dx\\ &= 5\,\frac{1}{2}\,\tan^{-1}\left(\frac{x}{2}\right)+c\\ &= \frac{5}{2}\,\tan^{-1}\left(\frac{x}{2}\right)+c\,.\end{aligned}$$

Section 2: Indefinite Integrals

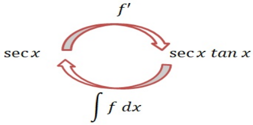
Rule 1: Power of x: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.

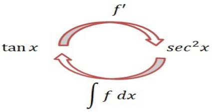
Rule 2: Trigonometric Functions.

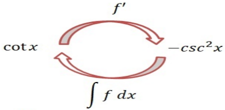




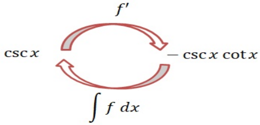
$$\int \sin x \; dx = -\cos x + c$$







$$\int csc^2 x \; dx = -\cot x + c$$



$$\int csc\, x \cot x \; dx = -\csc x + c$$

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- $\frac{d}{dx}(e^u) = e^u . u' \Longrightarrow \int e^u \; u' \; dx = e^u + c$

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- $\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
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5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx =$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
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5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$
3. $\int x(3x - 2)\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
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Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$
3. $\int x(3x - 2)\ dx = \int (3x^2 - 2x)\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
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Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$
3. $\int x(3x - 2)\ dx = \int (3x^2 - 2x)\ dx = 3\frac{x^3}{3} - 2\frac{x^2}{2} + c$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x \left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$
3. $\int x(3x - 2)\ dx = \int (3x^2 - 2x)\ dx = 3\frac{x^3}{3} - 2\frac{x^2}{2} + c = x^3 - x^2 + c$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
2. $\int (x^2 + 3)^2\ dx$
3. $\int x(3x - 2)\ dx$
4. $\int \cos x\left(2 + \sec x\right)\ dx$
5. $\int x(e^{x^2} + \frac{3}{x^2})\ dx$

Solution:

1. $\int (x^2 + 5x + 3)\ dx = \frac{x^3}{3} + 5\frac{x^2}{2} + 3x + c$
2. $\int (x^2 + 3)^2\ dx = \int (x^4 + 6x^2 + 9)\ dx = \frac{x^5}{5} + 6\frac{x^3}{3} + 9x + c = \frac{x^5}{5} + 2x^3 + 9x + c$
3. $\int x(3x - 2)\ dx = \int (3x^2 - 2x)\ dx = 3\frac{x^3}{3} - 2\frac{x^2}{2} + c = x^3 - x^2 + c$
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Example

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Example

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4. $\int \cos x \Big(2 + \sec x\Big)\ dx = \int \Big(2\cos x + \cos x \sec x\Big)\ dx = \int \Big(2\cos x + 1\Big)\ dx$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
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Example

Evaluate the integral.

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Solution:

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5. $$\int x(e^{x^2} + \frac{3}{x^2})\ dx = \int (xe^{x^2} + \frac{3}{x})\ dx$$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
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5. $$\int x(e^{x^2} + \frac{3}{x^2})\ dx = \int (xe^{x^2} + \frac{3}{x})\ dx = \int xe^{x^2}\ dx + \int \frac{3}{x}\ dx$$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
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5. $$\int x(e^{x^2} + \frac{3}{x^2})\ dx = \int (xe^{x^2} + \frac{3}{x})\ dx = \int xe^{x^2}\ dx + \int \frac{3}{x}\ dx = \frac{1}{2}\int 2xe^{x^2}\ dx + 3\int \frac{1}{x}\ dx$$

Example

Evaluate the integral.

1. $\int (x^2 + 5x + 3)\ dx$
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5. $$\int x(e^{x^2} + \frac{3}{x^2})\ dx = \int (xe^{x^2} + \frac{3}{x})\ dx = \int xe^{x^2}\ dx + \int \frac{3}{x}\ dx = \frac{1}{2}\int 2xe^{x^2}\ dx + 3\int \frac{1}{x}\ dx = \frac{1}{2}e^{x^2} + 3\ln|x| + c$$

Section 3: Definite Integrals

Indefinite Integrals

$$\int f(x)\ dx = F(x) + c$$

Definite Integrals

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Indefinite Integrals

$$\int f(x)\ dx = F(x) + c$$

Definite Integrals

$$\int_a^b f(x)\ dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Properties of Definite Integrals

Theorem

Assume f and g have antiderivatives on an interval $[a, b]$*, then*

1. $\displaystyle\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx.$
2. $\displaystyle\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$, *where k is a constant.*
3. $\displaystyle\int_a^a f(x)\ dx = 0$
4. $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
3. $\int_0^3 (x^2 + 5) \, dx$

Example

Evaluate the integral

1. $\int_1^2 x\ dx$
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Solution:

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Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
3. $\int_0^3 (x^2 + 5) \, dx$

Solution:

1. $\int_1^2 x \, dx = \left[\frac{x^2}{2}\right]_1^2$

Section 3: Definite Integrals

Example

Evaluate the integral

1. $\int_1^2 x\ dx$
2. $\int_0^1 \sqrt{x}\ dx$
3. $\int_0^3 (x^2 + 5)\ dx$

Solution:

1. $\int_1^2 x\ dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\ \left[x^2\right]_1^2$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
3. $\int_0^3 (x^2 + 5) \, dx$

Solution:

1. $\int_1^2 x \, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2} \left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right]$

Example

Evaluate the integral

1. $\int_1^2 x\ dx$
2. $\int_0^1 \sqrt{x}\ dx$
3. $\int_0^3 (x^2+5)\ dx$

Solution:

1. $\int_1^2 x\ dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\ \left[x^2\right]_1^2 = \frac{1}{2}\left[2^2-1^2\right] = \frac{1}{2}\ \left[3\right] = \frac{3}{2}$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
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Solution:

1. $\int_1^2 x \, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2} \left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2} \left[3\right] = \frac{3}{2}$
2. $\int_0^1 \sqrt{x} \, dx$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
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2. $\int_0^1 \sqrt{x} \, dx = \int_0^1 x^{\frac{1}{2}} \, dx$

Example

Evaluate the integral

1. $\int_1^2 x\, dx$
2. $\int_0^1 \sqrt{x}\, dx$
3. $\int_0^3 (x^2 + 5)\, dx$

Solution:

1. $\int_1^2 x\, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2}\left[3\right] = \frac{3}{2}$

2. $\int_0^1 \sqrt{x}\, dx = \int_0^1 x^{\frac{1}{2}}\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
3. $\int_0^3 (x^2 + 5) \, dx$

Solution:

1. $\int_1^2 x \, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2} \left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2} \left[3\right] = \frac{3}{2}$
2. $\int_0^1 \sqrt{x} \, dx = \int_0^1 x^{\frac{1}{2}} \, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}$ $\quad \frac{1}{\frac{a}{b}} = \frac{b}{a}$

   $\frac{1}{2} + 1 = \frac{1}{2} + \frac{1}{1} = \frac{1+2}{2} = \frac{3}{2}$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
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Example

Evaluate the integral

1. $\int_1^2 x \; dx$
2. $\int_0^1 \sqrt{x} \; dx$
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1. $\int_1^2 x \; dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2}\left[3\right] = \frac{3}{2}$
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   $\frac{1}{2} + 1 = \frac{1}{2} + \frac{1}{1} = \frac{1+2}{2} = \frac{3}{2}$
3. $\int_0^3 (x^2+5) \; dx = \left[\frac{x^3}{3} + 5x\right]_0^3$

Example

Evaluate the integral

1. $\int_1^2 x \, dx$
2. $\int_0^1 \sqrt{x} \, dx$
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1. $\int_1^2 x \, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2}\left[3\right] = \frac{3}{2}$

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   $\frac{1}{2} + 1 = \frac{1}{2} + \frac{1}{1} = \frac{1+2}{2} = \frac{3}{2}$

3. $\int_0^3 (x^2+5) \, dx = \left[\frac{x^3}{3} + 5x\right]_0^3 = \left[\frac{3^3}{3} + 5(3)\right] - \left[\frac{0^3}{3} + 5(0)\right]$

Example

Evaluate the integral

1. $\int_1^2 x\ dx$
2. $\int_0^1 \sqrt{x}\ dx$
3. $\int_0^3 (x^2+5)\ dx$

Solution:

1. $\int_1^2 x\ dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}\ \left[x^2\right]_1^2 = \frac{1}{2}\left[2^2 - 1^2\right] = \frac{1}{2}\ \left[3\right] = \frac{3}{2}$

2. $\int_0^1 \sqrt{x}\ dx = \int_0^1 x^{\frac{1}{2}}\ dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}$ $\quad \frac{1}{\frac{a}{b}} = \frac{b}{a}$

   $\frac{1}{2} + 1 = \frac{1}{2} + \frac{1}{1} = \frac{1+2}{2} = \frac{3}{2}$

3. $\int_0^3 (x^2+5)\ dx = \left[\frac{x^3}{3} + 5x\right]_0^3 = \left[\frac{3^3}{3} + 5(3)\right] - \left[\frac{0^3}{3} + 5(0)\right] = \left[9 + 15\right] - \left[0\right] = 24$

Example

Evaluate the integral

1. $\displaystyle\int_0^2 \frac{1}{x+1}\,dx$
2. $\displaystyle\int_0^1 x\,e^{x^2}\,dx$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \cos\,x\,dx$

Example

Evaluate the integral

1. $\int_0^2 \frac{1}{x+1}\, dx$
2. $\int_0^1 x\, e^{x^2}\, dx$
3. $\int_0^{\frac{\pi}{2}} \cos\, x\, dx$

Solution:

1. $\int_0^2 \frac{1}{x+1}\, dx$

Example

Evaluate the integral

1. $\displaystyle\int_0^2 \frac{1}{x+1}\, dx$
2. $\displaystyle\int_0^1 x\, e^{x^2}\, dx$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \cos\, x\, dx$

Solution:

1. $\displaystyle\int_0^2 \frac{1}{x+1}\, dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1|$

Example

Evaluate the integral

1. $\int_0^2 \frac{1}{x+1}\, dx$
2. $\int_0^1 x\, e^{x^2}\, dx$
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Solution:

1. $\int_0^2 \frac{1}{x+1}\, dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$

   Note: $\ln(1) = 0$

Example

Evaluate the integral

1. $\int_0^2 \frac{1}{x+1}\,dx$
2. $\int_0^1 x\,e^{x^2}\,dx$
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1. $\int_0^2 \frac{1}{x+1}\,dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$

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Example

Evaluate the integral

1. $\displaystyle\int_0^2 \frac{1}{x+1}\, dx$
2. $\displaystyle\int_0^1 x\, e^{x^2}\, dx$
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1. $\displaystyle\int_0^2 \frac{1}{x+1}\, dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$

   Note: $\ln(1) = 0$

2. $\displaystyle\int_0^1 x\, e^{x^2}\, dx = \frac{1}{2}\int_0^1 2\, x\, e^{x^2}\, dx = \frac{1}{2}\Big[e^{x^2}\Big]_0^1 = \frac{1}{2}\Big[e^1 - e^0\Big] = \frac{1}{2}(e-1) = \frac{e-1}{2}$ $\quad\displaystyle\int u'\, e^u\, dx = e^u + c$

   Note: $e^0 = 1$ and $e \approx 2.71828$

Example

Evaluate the integral

1. $\int_0^2 \frac{1}{x+1}\ dx$
2. $\int_0^1 x\ e^{x^2}\ dx$
3. $\int_0^{\frac{\pi}{2}} \cos\ x\ dx$

Solution:

1. $\int_0^2 \frac{1}{x+1}\ dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$

   Note: $\ln(1) = 0$

2. $\int_0^1 x\ e^{x^2}\ dx = \frac{1}{2}\int_0^1 2\ x\ e^{x^2}\ dx = \frac{1}{2}\Big[e^{x^2}\Big]_0^1 = \frac{1}{2}\Big[e^1 - e^0\Big] = \frac{1}{2}(e-1) = \frac{e-1}{2}$ $\quad \int u'\ e^u\ dx = e^u + c$

   Note: $e^0 = 1$ and $e \approx 2.71828$

3. $\int_0^{\frac{\pi}{2}} \cos\ x\ dx$

Example

Evaluate the integral

1. $\int_0^2 \frac{1}{x+1}\, dx$
2. $\int_0^1 x\, e^{x^2}\, dx$
3. $\int_0^{\frac{\pi}{2}} \cos\, x\, dx$

Solution:

1. $\int_0^2 \frac{1}{x+1}\, dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$

   Note: $\ln(1) = 0$

2. $\int_0^1 x\, e^{x^2}\, dx = \frac{1}{2} \int_0^1 2\, x\, e^{x^2}\, dx = \frac{1}{2}\Big[e^{x^2}\Big]_0^1 = \frac{1}{2}\Big[e^1 - e^0\Big] = \frac{1}{2}(e-1) = \frac{e-1}{2}$ $\quad \int u'\, e^u\, dx = e^u + c$

   Note: $e^0 = 1$ and $e \approx 2.71828$

3. $\int_0^{\frac{\pi}{2}} \cos\, x\, dx = \Big[\sin\, x\Big]_0^{\frac{\pi}{2}}$

Example

Evaluate the integral

1. $\displaystyle\int_0^2 \frac{1}{x+1}\ dx$
2. $\displaystyle\int_0^1 x\ e^{x^2}\ dx$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \cos\ x\ dx$

Solution:

1. $$\int_0^2 \frac{1}{x+1}\ dx = \Big[\ln|x+1|\Big]_0^2 = \ln|2+1| - \ln|0+1| = \ln(3) - \ln(1) = \ln(3)$$

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2. $$\int_0^1 x\ e^{x^2}\ dx = \frac{1}{2}\int_0^1 2\ x\ e^{x^2}\ dx = \frac{1}{2}\Big[e^{x^2}\Big]_0^1 = \frac{1}{2}\Big[e^1 - e^0\Big] = \frac{1}{2}(e-1) = \frac{e-1}{2} \qquad \int u'\ e^u\ dx = e^u + c$$

   Note: $e^0 = 1$ and $e \approx 2.71828$

3. $$\int_0^{\frac{\pi}{2}} \cos\ x\ dx = \Big[\sin\ x\Big]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin\ (0) = 1$$

   Note: $\sin(\frac{\pi}{2}) = 1$ and $\sin(0) = 0$

Degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Integration By Substitution

Remember Rule 1:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1 \qquad \qquad \textit{Example}: \int x^3 \, dx = \frac{x^4}{4} + c$$

Integration By Substitution

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Can we use **Rule 1** to evaluate $\int 2x \, (x^2 + 1)^3 \, dx$?

Integration By Substitution

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Can we use **Rule 1** to evaluate $\int 2x \, (x^2+1)^3 \, dx$?

Theorem

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on an interval J that contains the range of the function g. If $\int f(x) \, dx = F(x) + c$, then

$$\int f(g(x)) \, g'(x) \, dx = F(g(x)) + c, \quad \forall x \in I.$$

Integration By Substitution

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$$\int f(g(x)) \, g'(x) \, dx = F(g(x)) + c, \quad \forall x \in I.$$

Example

Evaluate the integral $\int 2x \, (x^2 + 1)^3 \, dx$.

Integration By Substitution

Remember Rule 1:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{for} \quad n \neq -1 \qquad\qquad Example: \int x^3 \, dx = \frac{x^4}{4} + c$$

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Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on an interval J that contains the range of the function g. If $\int f(x) \, dx = F(x) + c$, then

$$\int f(g(x)) \, g'(x) \, dx = F(g(x)) + c, \quad \forall x \in I.$$

Example

Evaluate the integral $\int 2x \, (x^2+1)^3 \, dx$.

Solution: Let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $(f \circ g)(x) = f(g(x)) = (x^2+1)^3$.

$$g(x) = x^2 + 1 \Rightarrow g'(x) = 2x$$

From the theorem, we have

$$\int \underbrace{2x}_{g'(x)} \underbrace{(x^2+1)^3}_{f(g(x))} \, dx = \frac{(x^2+1)^4}{4} + c \, .$$

Integration By Substitution

We can end with the same solution by using the five steps of the substitution method given below.

Steps of the integration by substitution:

Step 1: Choose a new variable u.
Step 2: Determine the value of du.
Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral in terms of u.
Step 4: Evaluate the new integral.
Step 5: Return the evaluation to the initial variable x.

Exercise: Evaluate the integral $\int u^3 \, du$

$$\int u^3 \, du = \frac{u^4}{4} + c$$

Example

Evaluate the integral $\int 2x \, (x^2 + 1)^3 \, dx$.

Solution: Let

$$u = x^2 + 1 \Rightarrow du = 2x \, dx \Rightarrow \frac{du}{2x} = dx$$

. By substituting that into the original integral, we have

$$\int 2x \, u^3 \, \frac{du}{2x} = \int u^3 \, du = \frac{u^4}{4} + c = \underbrace{\frac{(x^2+1)^4}{4}}_{\text{Returning the evaluation to } x} + c$$

Exercise: Evaluate the integral $\int \sqrt{x}\ dx$

Integration By Substitution

Exercise: Evaluate the integral $\int \sqrt{x}\, dx$

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{3}{2}}}{3/2} + c = \frac{2x^{\frac{3}{2}}}{3} + c$$

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5}\, dx$

Integration By Substitution

Exercise: Evaluate the integral $\int \sqrt{x}\, dx$

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\ dx = \frac{x^{\frac{3}{2}}}{3/2} + c = \frac{2x^{\frac{3}{2}}}{3} + c$$

Example

Evaluate the integral $\int x^2\sqrt{2x^3 - 5}\, dx$

Solution: $\int x^2\sqrt{2x^3 - 5}\, dx = \int x^2(2x^3 - 5)^{\frac{1}{2}}\ dx$

Integration By Substitution

Exercise: Evaluate the integral $\int \sqrt{x}\, dx$

$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \dfrac{x^{\frac{3}{2}}}{3/2} + c = \dfrac{2x^{\frac{3}{2}}}{3} + c$

Example

Evaluate the integral $\int x^2\sqrt{2x^3-5}\, dx$

Solution: $\int x^2\sqrt{2x^3-5}\, dx = \int x^2(2x^3-5)^{\frac{1}{2}}\, dx$

Let $f(x) = x^{\frac{1}{2}}$ and $g(x) = 2x^3 - 5$, then $(f \circ g)(x) = f\big(g(x)\big) = (2x^3-5)^{\frac{1}{2}}$.

$$g(x) = 2x^3 - 5 \Rightarrow g'(x) = 6x^2$$

From the theorem $\int f(g(x))g'(x)\, dx = F(g(x)) + c$, we have

$$\frac{1}{6}\int \underbrace{6x^2}_{g'(x)}\underbrace{(2x^3-5)^{\frac{1}{2}}}_{f(g(x))}\, dx = \frac{1}{6}\,\frac{(2x^3-5)^{\frac{3}{2}}}{3/2} + c = \frac{1}{6}\,\frac{2}{3}\,(2x^3-5)^{\frac{3}{2}} + c = \frac{(2x^3-5)^{\frac{3}{2}}}{9} + c\,.$$

Integration By Substitution

Example

Evaluate the integral $\int x^2 \sqrt{2x^3 - 5}\, dx$

Integration By Substitution

Example

Evaluate the integral $\int x^2\sqrt{2x^3-5}\,dx$

Solution:

$$\int x^2\sqrt{2x^3-5}\,dx = \int x^2(2x^3-5)^{\frac{1}{2}}\,dx$$

Let

$$u = 2x^3 - 5 \Rightarrow du = 6x^2\,dx \Rightarrow \frac{du}{6x^2} = dx$$

By substitution, we have

$$\int x^2\,u^{\frac{1}{2}}\,\frac{du}{6x^2} = \frac{1}{6}\int u^{\frac{1}{2}}\,du = \frac{1}{6}\frac{u^{\frac{3}{2}}}{3/2} + c = \frac{1}{6}\frac{2}{3}u^{\frac{3}{2}} + c = \frac{u^{\frac{3}{2}}}{9} + c = \underbrace{\frac{(2x^3-5)^{\frac{3}{2}}}{9}}_{\text{Returning the evaluation to } x} + c$$

Integration By Substitution

Exercise: Evaluate the integral $\int \sec^2 x \, dx$

$\int \sec^2 x \, dx = \tan x + c$

Exercise: Evaluate the integral $\int \sec^2 x \, dx$

$\int \sec^2 x \, dx = \tan x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$

Integration By Substitution

Exercise: Evaluate the integral $\int \sec^2 \ x \ dx$

$\int \sec^2 \ x \ dx = \tan \ x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \ \sqrt{x}}{\sqrt{x}} \ dx$

Solution: Let $f(x) = \sec^2 \ x$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = f(g(x)) = \sec^2 \ \sqrt{x}$.

$$g(x) = \sqrt{x} \Rightarrow g^{'}(x) = \frac{1}{2\sqrt{x}}$$

From the theorem $\int f(g(x))g^{'}(x) \ dx = F(g(x)) + c$, we have

$$2 \int \frac{\sec^2 \ \sqrt{x}}{2\sqrt{x}} \ dx = 2 \tan \ \sqrt{x} + c.$$

Integration By Substitution

Exercise: Evaluate the integral $\int \sec^2 x \, dx$

$\int \sec^2 x \, dx = \tan x + c$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx$

Solution: Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = f(g(x)) = \sec^2 \sqrt{x}$.

$$g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

From the theorem $\int f(g(x))g'(x) \, dx = F(g(x)) + c$, we have

$$2 \int \frac{\sec^2 \sqrt{x}}{2\sqrt{x}} \, dx = 2 \tan \sqrt{x} + c.$$

Example

Evaluate the integral $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx.$

Solution: Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow 2\sqrt{x} \, du = dx$. By substitution, we obtain

$$\int \frac{\sec^2 u}{\sqrt{x}} \, 2\sqrt{x} \, du = 2 \int \sec^2 u \, du = 2 \tan u + c = 2 \tan \sqrt{x} + c$$

Integration By Substitution

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\, dx$
2. $\int \cos\,(3x+4)\, dx$

Integration By Substitution

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

Integration By Substitution

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\,(3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x - 5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2}$$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos(3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\ du$$

Example

Evaluate the integral

1. $\displaystyle\int \sqrt{2x-5}\ dx$
2. $\displaystyle\int \cos\,(3x+4)\ dx$

Solution:

1. $\displaystyle\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\ \int u^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{u^{\frac{3}{2}}}{3/2} + c$$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2}\ \frac{u^{\frac{3}{2}}}{3} + c$$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\, dx$
2. $\int \cos(3x+4)\, dx$

Solution:

1. $\int \sqrt{2x-5}\, dx = \int (2x-5)^{\frac{1}{2}}\, dx$

   Let $u = 2x - 5 \Rightarrow du = 2\, dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}}\, du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2} \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2}\ \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

OR

$$\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\int 2\ (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\ \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos\ (3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2}\ \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

   OR

$$\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\int 2\ (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\ \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

2. $\int \cos\ (3x+4)\ dx$

Integration By Substitution

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\, dx$
2. $\int \cos(3x+4)\, dx$

Solution:

1. $\int \sqrt{2x-5}\, dx = \int (2x-5)^{\frac{1}{2}}\, dx$

   Let $u = 2x-5 \Rightarrow du = 2\, dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\, \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}}\, du = \frac{1}{2}\, \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2}\, \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

   OR

$$\int \sqrt{2x-5}\, dx = \int (2x-5)^{\frac{1}{2}}\, dx = \frac{1}{2} \int 2\,(2x-5)^{\frac{1}{2}}\, dx = \frac{1}{2}\, \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

2. $\int \cos(3x+4)\, dx = \frac{1}{3} \int 3\, \cos(3x+4)\, dx$

Example

Evaluate the integral

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos(3x+4)\ dx$

Solution:

1. $\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx$

   Let $u = 2x-5 \Rightarrow du = 2\ dx \Rightarrow dx = \frac{du}{2}$. By substitution, we have

$$\int u^{\frac{1}{2}}\ \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\ du = \frac{1}{2}\ \frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2}{2}\ \frac{u^{\frac{3}{2}}}{3} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

   OR

$$\int \sqrt{2x-5}\ dx = \int (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\int 2\ (2x-5)^{\frac{1}{2}}\ dx = \frac{1}{2}\ \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(2x-5)^{\frac{3}{2}}}{3} + c$$

2. $\int \cos(3x+4)\ dx = \frac{1}{3}\int 3\ \cos(3x+4)\ dx = \frac{1}{3}\ \sin(3x+4) + c$

Integration By Substitution

Example

Evaluate the integral

1. $\displaystyle\int 5x(x^2+3)^7\,dx$
2. $\displaystyle\int \sec^2(4x)\,dx$
3. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$
4. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$

Integration By Substitution

Example

Evaluate the integral

1. $\int 5x(x^2+3)^7 \, dx$
2. $\int \sec^2(4x) \, dx$
3. $\int \frac{(\ln x)^2}{x} \, dx$
4. $\int \frac{\cos x}{1+\sin^2 x} \, dx$

Solution:

1. $5\int x(x^2+3)^7 \, dx$

Integration By Substitution

Example

Evaluate the integral

1. $\displaystyle\int 5x(x^2+3)^7\ dx$
2. $\displaystyle\int \sec^2(4x)\ dx$
3. $\displaystyle\int \frac{(\ln x)^2}{x}\ dx$
4. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\ dx$

Solution:

1. $\displaystyle 5\int x(x^2+3)^7\ dx = \frac{5}{2}\int 2x(x^2+3)^7\ dx$

Example

Evaluate the integral

1. $\int 5x(x^2+3)^7 \, dx$
2. $\int \sec^2(4x) \, dx$
3. $\int \frac{(\ln x)^2}{x} \, dx$
4. $\int \frac{\cos x}{1+\sin^2 x} \, dx$

Solution:

1. $5\int x(x^2+3)^7 \, dx = \frac{5}{2}\int 2x(x^2+3)^7 \, dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$

Example

Evaluate the integral

1. $\int 5x(x^2+3)^7 \; dx$
2. $\int \sec^2(4x) \; dx$
3. $\int \frac{(\ln x)^2}{x} \; dx$
4. $\int \frac{\cos x}{1+\sin^2 x} \; dx$

Solution:

1. $5\int x(x^2+3)^7 \; dx = \frac{5}{2}\int 2x(x^2+3)^7 \; dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$
2. $\frac{1}{4}\int 4\sec^2(4x) \; dx$

Integration By Substitution

Example

Evaluate the integral

1. $\displaystyle\int 5x(x^2+3)^7 \, dx$
2. $\displaystyle\int \sec^2(4x) \, dx$
3. $\displaystyle\int \frac{(\ln x)^2}{x} \, dx$
4. $\displaystyle\int \frac{\cos x}{1+\sin^2 x} \, dx$

Solution:

1. $\displaystyle 5\int x(x^2+3)^7 \, dx = \frac{5}{2}\int 2x(x^2+3)^7 \, dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$
2. $\displaystyle \frac{1}{4}\int 4\sec^2(4x) \, dx = \frac{1}{4}\tan(4x) + c$

Integration By Substitution

Example

Evaluate the integral

1. $\int 5x(x^2+3)^7 \, dx$
2. $\int \sec^2(4x) \, dx$
3. $\int \frac{(\ln x)^2}{x} \, dx$
4. $\int \frac{\cos x}{1+\sin^2 x} \, dx$

Solution:

1. $5\int x(x^2+3)^7 \, dx = \frac{5}{2}\int 2x(x^2+3)^7 \, dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$
2. $\frac{1}{4}\int 4\sec^2(4x) \, dx = \frac{1}{4}\tan(4x) + c$
3. $\int (\ln x)^2 \frac{1}{x} \, dx$

Integration By Substitution

Example

Evaluate the integral

1. $\displaystyle\int 5x(x^2+3)^7 \, dx$
2. $\displaystyle\int \sec^2(4x) \, dx$
3. $\displaystyle\int \frac{(\ln x)^2}{x} \, dx$
4. $\displaystyle\int \frac{\cos x}{1+\sin^2 x} \, dx$

Solution:

1. $\displaystyle 5\int x(x^2+3)^7 \, dx = \frac{5}{2}\int 2x(x^2+3)^7 \, dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$
2. $\displaystyle \frac{1}{4}\int 4\sec^2(4x) \, dx = \frac{1}{4}\tan(4x) + c$
3. $\displaystyle \int (\ln x)^2 \, \frac{1}{x} \, dx = \frac{(\ln x)^3}{3} + c$

Integration By Substitution

Example

Evaluate the integral

1. $\int 5x(x^2+3)^7\ dx$
2. $\int \sec^2(4x)\ dx$
3. $\int \frac{(\ln x)^2}{x}\ dx$
4. $\int \frac{\cos x}{1+\sin^2 x}\ dx$

Solution:

1. $5\int x(x^2+3)^7\ dx = \frac{5}{2}\int 2x(x^2+3)^7\ dx = \frac{5}{2}\frac{(x^2+3)^8}{8} + c = 5\frac{(x^2+3)^8}{16} + c$
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3. $\int (\ln x)^2\ \frac{1}{x}\ dx = \frac{(\ln x)^3}{3} + c$
4. $\int \frac{\cos x}{1+\sin^2 x}\ dx$

   Let $u = \sin x \Rightarrow du = \cos x\ dx \Rightarrow \frac{du}{\cos x} = dx$
   By substitution:

$$\int \frac{\cos x}{1+u^2}\ \frac{du}{\cos x}$$

Integration By Substitution

Example

Evaluate the integral

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$$\int \frac{\cos x}{1+u^2}\ \frac{du}{\cos x} = \int \frac{1}{1+u^2}\ du = \tan^{-1} u + c$$

Integration By Substitution

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Evaluate the integral

1. $\int 5x(x^2+3)^7 \; dx$
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$$\int \frac{\cos x}{1+u^2} \frac{du}{\cos x} = \int \frac{1}{1+u^2} \; du = \tan^{-1} u + c = \tan^{-1}(\sin x) + c$$

Exercise: Evaluate the integral.

(1) $\int x\ e^{x^2}\ dx$ Remember: $\int u'\ e^{u}\ dx = e^{u} + c$

Integration by Parts

Exercise: Evaluate the integral.

(1) $\int x\ e^{x^2}\ dx$ Remember: $\int u'\ e^u\ dx = e^u + c$

Solution: $\int x\ e^{x^2}\ dx = \frac{1}{2} \int 2\ x\ e^{x^2}\ dx = \frac{1}{2}\ e^{x^2} + c$

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(2) $\int x\ e^x\ dx$ We cannot use the previous method to calculate this integral.

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Integration by Parts

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(1) $\int x\ e^{x^2}\ dx$ Remember: $\int u'\ e^u\ dx = e^u + c$

Solution: $\int x\ e^{x^2}\ dx = \frac{1}{2} \int 2\ x\ e^{x^2}\ dx = \frac{1}{2}\ e^{x^2} + c$

(2) $\int x\ e^x\ dx$ We cannot use the previous method to calculate this integral.

(3) $\int x^2\ \cos\ x^3\ dx$

Solution: $\int x^2 \cos\ x^3\ dx = \frac{1}{3} \int 3\ x^2 \cos\ x^3\ dx = \frac{1}{3}\ \sin\ x^3 + c$

Exercise: Evaluate the integral.

(1) $\int x\ e^{x^2}\ dx$ Remember: $\int u'\ e^u\ dx = e^u + c$

Solution: $\int x\ e^{x^2}\ dx = \frac{1}{2} \int 2\ x\ e^{x^2}\ dx = \frac{1}{2}\ e^{x^2} + c$

(2) $\int x\ e^x\ dx$ We cannot use the previous method to calculate this integral.

(3) $\int x^2\ \cos\ x^3\ dx$

Solution: $\int x^2 \cos\ x^3\ dx = \frac{1}{3} \int 3\ x^2 \cos\ x^3\ dx = \frac{1}{3}\ \sin\ x^3 + c$

(4) $\int x\ \cos\ x\ dx$ We cannot use the previous method to calculate this integral.

Integration by Parts

Let $u = f(x)$ and $v = g(x)$, we know that

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x) \ .$$

Thus,

$$f(x)g'(x) = \frac{d}{dx}\left(f(x)g(x)\right) - f'(x)g(x) \ .$$

By integrating both sides, we have

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int \frac{d}{dx}\left(f(x)g(x)\right)\ dx - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx.\end{aligned}$$

Since $u = f(x) \Rightarrow du = f'(x)\ dx$ and $v = g(x) \Rightarrow dv = g'(x)\ dx$. Therefore,

$$\int u\ dv = uv - \int v\ du \ .$$

Theorem

If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u\ dv = uv - \int v\ du.$$

$$u = f(x) \Rightarrow u' = f'(x)\ dx$$

$$dv = g'(x) \Rightarrow \int dv = \int g'(x)\ dx \Rightarrow v = \int g'(x)\ dx$$

Example

Evaluate the integral $\int x\, e^{x}\, dx$.

Example

Evaluate the integral $\int x\ e^x\ dx$.

Solution: The integrand $x\ e^x$ is a product of two functions x and e^x: $\int x\ e^x\ dx$

Integration by Parts

Example

Evaluate the integral $\int x\, e^x \, dx$.

Solution: The integrand $x\, e^x$ is a product of two functions x and e^x: $\int x\, e^x \, dx$

Choose $u = x$, and $dv = e^x \, dx$. Then,

$$u = x \Rightarrow du = dx \ ,$$
$$dv = e^x \, dx \Rightarrow v = \int e^x \, dx = e^x .$$

Example

Evaluate the integral $\int x\ e^x\ dx$.

Solution: The integrand $x\ e^x$ is a product of two functions x and e^x: $\int x\ e^x\ dx$

Choose $u = x$, and $dv = e^x\ dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = e^x\ dx \Rightarrow v = \int e^x\ dx = e^x.$$

From the theorem

$$\int u\ dv = uv - \int v\ du$$
$$\int x\ e^x\ dx = x\ e^x - \int e^x\ dx$$
$$= x\ e^x - e^x + c\ .$$

Integration by Parts

Example

Evaluate the integral $\int x\, e^x\, dx$.

Solution: The integrand $x\ e^x$ is a product of two functions x and e^x: $\int x\ e^x\ dx$

Choose $u = x$, and $dv = e^x\ dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = e^x\ dx \Rightarrow v = \int e^x\ dx = e^x.$$

From the theorem

$$\int u\ dv = uv - \int v\ du$$
$$\int x\ e^x\ dx = x\ e^x - \int e^x\ dx$$
$$= x\ e^x - e^x + c\ .$$

Note:

- We choose $u = x$ because it can be differentiated to a constant. Thus the new product integral will not involve a product anymore.
- Try to choose

$$u = e^x \text{ and } dv = x\ dx$$

You will obtain

$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x\ dx.$$

However, the integral $\int \frac{x^2}{2} e^x\ dx$ is more difficult than the original one $\int x e^x\ dx$.

Example

Evaluate the integral $\int x \cos x \, dx$.

Integration by Parts

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \ ,$$
$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$\begin{aligned} u &= x \Rightarrow du = dx \,, \\ dv = \cos x \, dx &\Rightarrow v = \int \cos x \, dx = \sin x. \end{aligned}$$

From the theorem,

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + c \,. \end{aligned}$$

Example

Evaluate the integral $\int x \cos x \, dx$.

Solution: In the same manner as in the preceding example, set $u = x$ and $dv = \cos x \, dx$. Hence,

$$u = x \Rightarrow du = dx \; ,$$
$$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x.$$

From the theorem,

$$\int u \, dv = uv - \int v \, du$$
$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + c \; .$$

Note:

- We choose $u = x$ because it can be differentiated to a constant. Thus the new product integral will not involve a product anymore.
- Try to choose

$$u = \cos x \text{ and } dv = x \, dx$$

Do you have the same result?

Example

Evaluate the integral $\int \ln x \; dx$.

Integration by Parts

Example

Evaluate the integral $\int \ln x \; dx$.

Solution: Choose $u = \ln x$, and $dv = dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x}\; dx \;,$$

$$dv = dx \Rightarrow v = \int 1 \; dx = x.$$

Remember:
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

Integration by Parts

Example

Evaluate the integral $\int \ln x \, dx$.

Solution: Choose $u = \ln x$, and $dv = dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x}\, dx\ ,$$

$$dv = dx \Rightarrow v = \int 1\, dx = x.$$

Remember:
If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}$$

Apply the theorem

$$\int u\, dv = uv - \int v\, du$$

$$\int \ln x\, dx = x \ln x - \int x \frac{1}{x}\, dx$$

$$= x \ln x - \int 1\, dx$$

$$= x \ln x - x + c$$

Integration by Parts

Example

Evaluate the integral $\int x^3 \ln x \, dx$.

Example

Evaluate the integral $\int x^3 \ln x \, dx$.

Solution: Choose $u = \ln x$, and $dv = x^3 \, dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \ ,$$

$$dv = x^3 \, dx \Rightarrow v = \int x^3 \, dx = \frac{x^4}{4}.$$

Example

Evaluate the integral $\int x^3 \ \ln x \ dx$.

Solution: Choose $u = \ln x$, and $dv = x^3 \ dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \ ,$$
$$dv = x^3 \ dx \Rightarrow v = \int x^3 \ dx = \frac{x^4}{4}.$$

From the theorem,

$$\int u \ dv = uv - \int v \ du$$
$$\int x^3 \ \ln x \ dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \ \frac{1}{x} \ dx$$
$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \ dx$$
$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + c \ .$$

Integration by Parts

Example

Evaluate the integral $\int x^3 \ln x \, dx$.

Solution: Choose $u = \ln x$, and $dv = x^3 \, dx$. Then,

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \ ,$$

$$dv = x^3 \, dx \Rightarrow v = \int x^3 \, dx = \frac{x^4}{4}.$$

From the theorem,

$$\int u \, dv = uv - \int v \, du$$

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + c \ .$$

Rule:

To evaluate $\int x^n \ln x \, dx$, let

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$dv = x^n \, dx \Rightarrow v = \int x^n \, dx = \frac{x^{n+1}}{n+1}$$

Hence,

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

Integration by Parts

Example

Evaluate the integral $\int \sin\ x\ \ln(\cos\ x)\ dx$.

Integration by Parts

Example

Evaluate the integral $\int \sin x \ln(\cos x)\, dx$.

Solution: Let $u = \ln(\cos x)$ for $\cos x > 0$, and $dv = \sin x\, dx$. Then,

$$u = \ln(\cos x) \Rightarrow du = \frac{-\sin x}{\cos x} dx,$$
$$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x.$$

Integration by Parts

Example

Evaluate the integral $\int \sin\ x\ \ln(\cos\ x)\ dx$.

Solution: Let $u = \ln(\cos\ x)$ for $\cos\ x > 0$, and $dv = \sin\ x\ dx$. Then,

$$u = \ln(\cos\ x) \Rightarrow du = \frac{-\sin\ x}{\cos\ x} dx,$$
$$dv = \sin\ x\ dx \Rightarrow v = \int \sin\ x\ dx = -\cos\ x.$$

Hence,

$$\int u\ dv = uv - \int v\ du$$
$$\int \sin\ x\ \ln(\cos\ x)\ dx = -\cos\ x\ \ln(\cos\ x) - \int \cos\ x\ \frac{\sin\ x}{\cos\ x}\ dx$$
$$= -\cos\ x\ \ln(\cos\ x) - \int \sin\ x\ dx$$
$$= -\cos\ x\ \ln(\cos\ x) + \cos\ x + c\ .$$

Example

Evaluate the integral $\int x \sec^2 x \, dx$.

Integration by Parts

Example

Evaluate the integral $\int x \sec^2 x \, dx$.

Solution: Let $u = x$ and $dv = \sec^2 x \, dx$. Then,

$$u = x \Rightarrow du = dx,$$
$$dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x.$$

Integration by Parts

Example

Evaluate the integral $\int x \sec^2 x \, dx$.

Solution: Let $u = x$ and $dv = \sec^2 x \, dx$. Then,

$$u = x \Rightarrow du = dx,$$

$$dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x.$$

Hence,

$$\int u \, dv = uv - \int v \, du$$

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x} \, dx$$

$$= x \tan x + \int \frac{-\sin x}{\cos x} \, dx$$

$$= x \tan x + \ln|\cos x| + c \, .$$

Exercise: **Evaluate the integral.**

1. $\int \frac{x}{x^2+1}\, dx$

Integrals of Rational Functions

Exercise: Evaluate the integral.

1. $\displaystyle\int \frac{x}{x^2+1}\,dx$

   Solution: $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

Integrals of Rational Functions

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   Solution: $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2\,x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

2. $\displaystyle\int \frac{x+1}{x^2+2x-8}\,dx$

Integrals of Rational Functions

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   Solution: $\displaystyle\int \frac{x+1}{x^2+2x-8}\,dx = \frac{1}{2}\int \frac{2\,(x+1)}{x^2+2x-8}\,dx = \frac{1}{2}\ln|x^2+2x-8|+c$

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2. $\displaystyle\int \frac{x+1}{x^2+2x-8}\,dx$

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3. $\displaystyle\int \frac{x+1}{x^2-2x-8}\,dx$

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

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Example: A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.

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Steps of integrals of rational functions:

Step 1: Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.

Example 1:

$$x^2 + 3x + 2$$

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Example 1:

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$$1 + 2 = 3 \quad \text{and} \quad 1 \times 2 = 2$$

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Example 1:

$$x^2 + 3x + 2$$

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$$1 + 2 = 3 \quad \text{and} \quad 1 \times 2 = 2$$

$$(x + 1)(x + 2)$$

Example 2:

$$x^2 + x - 12$$

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

Example: A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.

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Example 1:

$$x^2 + 3x + 2$$

$$x^2 {+3}x{+2}$$

$$1 + 2 = 3 \quad \text{and} \quad 1 \times 2 = 2$$

$$(x + 1)(x + 2)$$

Example 2:

$$x^2 + x - 12$$

$$x^2 {+1}x{-12}$$

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

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$$x^2 + 3x + 2$$

$$x^2{+3}x{+2}$$

$$1 + 2 = 3 \quad \text{and} \quad 1 \times 2 = 2$$

$$(x+1)(x+2)$$

Example 2:

$$x^2 + x - 12$$

$$x^2{+1}x{-12}$$

$$-3 + 4 = 1 \quad \text{and} \quad -3 \times 4 = -12$$

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

Example: A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.

Steps of integrals of rational functions:

Step 1: Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.

Example 1:

$$x^2 + 3x + 2$$

$$x^2 + 3x + 2$$

$$1 + 2 = 3 \quad \text{and} \quad 1 \times 2 = 2$$

$$(x + 1)(x + 2)$$

Example 2:

$$x^2 + x - 12$$

$$x^2 + 1x - 12$$

$$-3 + 4 = 1 \quad \text{and} \quad -3 \times 4 = -12$$

$$(x - 3)(x + 4)$$

Example 3:

$$x^3 + x^2 - 12x$$

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

Example: A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.

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Example 2:

$$x^2 + x - 12$$

$$x^2 + 1x - 12$$

$$-3 + 4 = 1 \quad \text{and} \quad -3 \times 4 = -12$$

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$$x^3 + x^2 - 12x$$

$x(x^2 + x - 12)$ common factor

Integrals of Rational Functions

A rational function is a quotient of two polynomials of the form $q(x) = \frac{f(x)}{g(x)}$.

Example: A Polynomial $f(x)$ is a linear sum of powers of x, for example $f(x) = 5x^3 + x^2 + x + 1$ or $g(x) = x(x^3 - 1)$.

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Example 4:

$$x^2 - 16$$

Remember: $a^2 - b^2 = (a - b)(a + b)$

$$\Rightarrow x^2 - 16 = (x - 4)(x + 4)$$

Example 5:

$$x^2 - 2x - 8$$

$$a = 1,\ b = -2,\ c = -8$$

$$ax^2 + bx + c$$

$$x = \frac{-b \pm \sqrt{b^2 - 4\ .a\ .c}}{2a}$$

Integrals of Rational Functions

Example 5:

$$x^2 - 2x - 8$$

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Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$
4. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
5. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
7. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
8. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$

Integrals of Rational Functions

Step 2: Find the partial fractions. This step depends on the result of step 1 where the fraction $q(x) = \frac{f(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x) = \frac{f(x)}{g(x)} = P_1(x) + P_2(x) + P_3(x) + \ldots + P_n(x) ,$$

$$\text{each} \quad P(x) = \frac{A}{(ax+b)^n}, n \in \mathbb{N} \quad \text{or} \quad P(x) = \frac{Ax+B}{(ax^2+bx+c)^n} \quad \text{if} \quad b^2 - 4ac < 0$$

The constants A_k and B_k are real numbers and computed later.

Note that the denominators of the fractions $P_k(x)$ are the factors of the original denominator obtained in step 1.

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$$= \frac{x^2-10+25}{(x+1)(x-5)^2} + \frac{3(x^2-4x-5)}{(x-5)(x-5)(x+1)} + \frac{4x+4}{(x-5)^2(x+1)} = \frac{4x^2-18x+14}{(x+1)(x-5)^2}$$

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Solution: **Step 1:** Factor the denominator $g(x)$ into irreducible polynomials: $g(x) = x^2 - 2x - 8$

Integrals of Rational Functions

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Evaluate the integral $\int \frac{x+1}{x^2-2x-8}\,dx$.

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Integrals of Rational Functions

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$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax - 4A + Bx + 2B}{(x+2)(x-4)}.$$

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Find the constants A and B by equating the coefficients of like powers of x in the two sides of the equation:

$$x + 1 = (A+B)x - 4A + 2B$$

Coefficients of the numerators:

coefficients of x: $A + B = 1 \rightarrow 1$

constants: $-4A + 2B = 1 \rightarrow 2$

By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$. Thus,

$$\frac{x+1}{x^2-2x-8} = \frac{1/6}{x+2} + \frac{5/6}{x-4}.$$

Multiply equation 1 by 4, then add the result to equation 2

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$$-----------$$
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Integrals of Rational Functions

Example

Evaluate the integral $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\,dx.$

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$$\begin{aligned}
\frac{2x^2 - 25x - 33}{(x+1)^2(x-5)} &= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-5)} \\
&= \frac{A(x+1)(x-5)}{(x+1)(x+1)(x-5)} + \frac{B(x-5)}{(x+1)^2(x-5)} + \frac{C(x+1)^2}{(x-5)(x+1)^2} \\
&= \frac{A(x^2 - 4x - 5) + B(x-5) + C(x^2 + 2x + 1)}{(x+1)^2(x-5)}
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&= \frac{A(x^2 - 4x - 5) + B(x-5) + C(x^2 + 2x + 1)}{(x+1)^2(x-5)}
\end{aligned}$$

Coefficients of the numerators:

coefficients of x^2: $A + C = 2 \rightarrow 1$

coefficients of x: $-4A + B + 2C = -25 \rightarrow 2$

constants: $-5A - 5B + C = -33 \rightarrow 3$

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.

$5\times$ equation $2+$ equation 3

$-25A + 11C = -158 \rightarrow 4$

$25\times$ equation $1+$ equation 4

$36C = -108 \Rightarrow C = -3$

Integrals of Rational Functions

Step 3:

$$\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\, dx = \int \frac{5}{x+1}\, dx + \int \frac{1}{(x+1)^2}\, dx + \int \frac{-3}{x-5}\, dx$$

$$= 5\int \frac{1}{x+1}\, dx + \int (x+1)^{-2}\, dx - 3\int \frac{1}{x-5}\, dx$$

$$= 5\ln|x+1| - (x+1)^{-1} - 3\ln|x-5| + c \qquad \int f(g(x))\, g'(x)\, dx = F(g(x)) + c$$

$$= 5\ln|x+1| - \frac{1}{(x+1)} - 3\ln|x-5| + c.$$

Example

Evaluate the integral $\int \frac{x+1}{x(x^2+1)}\,dx$.

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Integrals of Rational Functions

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We have

$$A=1,\, B=-1 \quad \text{and} \quad C=1$$

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Step 3:

$$\begin{aligned} \int \frac{x+1}{x(x^2+1)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx \\ &= \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + c. \end{aligned}$$