GENERAL MATHEMATICS 2

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Definition

A matrix A of order $m \times n$ is a set of numbers or expressions arranged in a rectangular array of m rows and n columns.

The matrix is a rectangular table that takes the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix}.$$

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Notes:

1. The horizontal arrays of a matrix are called its **rows** and the vertical arrays are called its **columns**.
2. a_{ij} represents the element of the matrix A that lies in row i and column j.
3. The matrix A of order $m \times n$ can also be written as $A = [a_{ij}]_{m \times n}$.

Example

Find the order of each matrix, then find the given elements.

1. $A = \begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$, a_{11} *and* a_{22}
2. $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \end{bmatrix}$, a_{12}, a_{21} *and* a_{23}

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Definition

Two matrices $A = [a_{ij}]_{m \times n}$ *and* $B = [b_{ij}]_{m \times n}$ *are equal if* $a_{ij} = b_{ij}$ *for each* $i = 1, 2, \dots, m$ *and* $j = 1, 2, \dots, n$.

Example

Find the value of x *if the matrices* $A = B$.

$A = \begin{bmatrix} 1 & 2 \\ -1 & 4x - 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 11 \end{bmatrix}$

Example

Find the order of each matrix, then find the given elements.

1. $A = \begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}$, a_{11} and a_{22}
2. $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \end{bmatrix}$, a_{12}, a_{21} and a_{23}

Solution:

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Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal if $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example

Find the value of x if the matrices $A = B$.
$A = \begin{bmatrix} 1 & 2 \\ -1 & 4x - 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 11 \end{bmatrix}$

Solution:
Since the matrices $A = B$, then from Definition .2, we have $4x - 1 = 11$. By doing some calculation, we have $x = 3$.

Section 2: Special Types of Matrices

(1) **Row Vector.** A row vector of order n is a matrix of order $1 \times n$ written as $A = [a_1 \;\; a_2 \;\; \dots \;\; a_n]$. For example, $A = [2 \;\; 7 \;\; 0 \;\; -1 \;\; 9]$ is a row vector of order 5.

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(2) **Column Vector.** A column vector of order n is a matrix of order $n \times 1$ written as $\text{A} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$. For example, $\text{A} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$ is a column vector of order 3.

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(3) **Null Matrix.** The matrix $A = [a_{ij}]_{m \times n}$ is called a null matrix if $a_{ij} = 0$ for all i and j i.e.

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For example, A $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2×3.

Section 2: Special Types of Matrices

(4) **Square Matrix.** If the number of rows equals the number of columns ($m = n$), then the matrix is called a square matrix of order n. If $A = [a_{ij}]$ is a square matrix, the set of elements of the form a_{ii} is called the diagonal of the matrix. For example, the diagonal of the following square matrix is highlighted in red $\begin{bmatrix} 2 & -7 & 3 \\ 1 & 0 & 9 \\ -1 & 6 & 8 \end{bmatrix}$.

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(5) **Upper Triangular Matrix.** The square matrix $A = [a_{ij}]$ of order n is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

For example, $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ is an upper triangular matrix of order 3.

(6) **Lower Triangular Matrix.** The square matrix $A = [a_{ij}]$ of order n is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

(6) **Lower Triangular Matrix.** The square matrix $A = [a_{ij}]$ of order n is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

(7) **Diagonal Matrix.** The square matrix $A = [a_{ij}]$ of order n is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

(8) **Identity Matrix.** The square matrix $A = [a_{ij}]$ of order n is called an identity matrix if $a_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$.
An identity matrix of order n can be represents by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3. Note that the identity matrix is a diagonal matrix where each number in diagonal equals 1.

(1) Addition and Subtraction of Matrices.

Definition

Let $A = [a_{ij}]_{m\times n}$ *and* $B = [b_{ij}]_{m\times n}$ *be two matrices, then*

1. $A + B = C$ *with* $c_{ij} = a_{ij} + b_{ij}$.
2. $A - B = C$ *with* $c_{ij} = a_{ij} - b_{ij}$.

From the definition, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices, then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

Also,

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}.$$

Example

If $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 0 & 8 \\ 1 & 4 & -1 \\ 10 & 11 & -2 \end{bmatrix}$, *find* **(1)** $A + B$ *and* **(2)** $A - B$.

Example

If $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 0 & 8 \\ 1 & 4 & -1 \\ 10 & 11 & -2 \end{bmatrix}$, *find* **(1)** $A + B$ *and* **(2)** $A - B$.

Solution:

(1) A+B $= \begin{bmatrix} 1+5 & 3+0 & 2+8 \\ 5+1 & -4+4 & 6+(-1) \\ 0+10 & 9+11 & 2+(-2) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 10 \\ 6 & 0 & 5 \\ 10 & 20 & 0 \end{bmatrix}$.

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If $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & -4 & 6 \\ 0 & 9 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 0 & 8 \\ 1 & 4 & -1 \\ 10 & 11 & -2 \end{bmatrix}$, *find* **(1)** $A + B$ *and* **(2)** $A - B$.

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(2) A-B $= \begin{bmatrix} 1-5 & 3-0 & 2-8 \\ 5-1 & -4-4 & 6-(-1) \\ 0-10 & 9-11 & 2-(-2) \end{bmatrix} = \begin{bmatrix} -4 & 3 & -6 \\ 4 & -8 & 7 \\ -10 & -2 & 4 \end{bmatrix}$

(2) Multiplication of Matrices by Scalars.

Definition

Let $A = [a_{ij}]_{m\times n}$ be a matrix, then for any $k \in \mathbb{R}$,

$$kA = C \quad with \quad c_{ij} = ka_{ij} \ .$$

From Definition .4, if $A = [a_{ij}]_{m\times n}$ is a matrix and $k \in \mathbb{R}$ then $k\ A = [k\ a_{ij}]$:

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} .$$

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If $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 9 & 2 \end{bmatrix}$*, find* $3A$*.*

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Example

If $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 9 & 2 \end{bmatrix}$, *find* $3A$.

Solution:

$$3A = \begin{bmatrix} 3\times 1 & 3\times 3 & 3\times 2 \\ 3\times 0 & 3\times 9 & 3\times 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 6 \\ 0 & 27 & 6 \end{bmatrix} .$$

Example

If $A = \begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}$, *then find* $-2A + 3B$.

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If $A = \begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}$, *then find* $-2A + 3B$.

Solution:

$$-2A + 3B = -2\begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix} + 3\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 4 & 16 \end{bmatrix}.$$

Theorem

Let A, B and C be matrices of order $m \times n$, and let $k, \ell \in \mathbb{R}$. Then

1. *The addition of matrices is commutative:* $A + B = B + A$.
2. *The addition of matrices is associative:* $(A + B) + C = A + (B + C)$.
3. *The null matrix is the identity matrix of addition:* $A + 0 = A$.
4. $(k + \ell)A = kA + \ell A$.
5. $k(\ell A) = (k\ell)A$.
6. *Let* $A = [a_{ij}]_{m\times n}$ *be a matrix, then there exists a matrix B such that* $A + B = 0$. *This matrix B is called the additive inverse of the matrix A and it is denoted by* $-A = (-1)A$.

(3) Multiplication of Matrices.

Definition

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices, then the multiplication of the two matrices AB is a matrix C of order $m \times p$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj} \ .$$

Note that the multiplication AB is defined if and only if the number of columns of A equals the number of rows of B; otherwise, we say the multiplication is undefined.

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$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1p} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mp} \end{bmatrix} .$$

$$c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} + \ldots + a_{1n} \times b_{n1}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{bmatrix}$$

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$$c_{12} = a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32} + \ldots + a_{1n} \times b_{n2}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{bmatrix}$$

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$$c_{1p} = a_{11} \times b_{1p} + a_{12} \times b_{2p} + a_{13} \times b_{3p} + \ldots + a_{1n} \times b_{np}$$

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$$c_{1p} = a_{11} \times b_{1p} + a_{12} \times b_{2p} + a_{13} \times b_{3p} + \ldots + a_{1n} \times b_{np}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{np} \end{bmatrix}$$

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$$c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31} + \ldots + a_{2n} \times b_{n1}$$

Example

If $A = \begin{bmatrix} 1 & 6 \\ -2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 8 & 2 \end{bmatrix}$, *find* AB.

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Section 3: Operations on Matrices

Note: A special case of multiplication of matrices is multiplying a row vector by a column vector.

Let $A = [a_1 \ a_2 \ \dots \ a_n]$ be a row vector of order n and B $= \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$ be a column vector of order n.

Then the multiplication AB is a matrix $C = [c]$ of order 1×1, where

$$c = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \ .$$

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Example

If $A = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$ *,* $B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$ *, compute (if possible)* **(1)** AB **(2)** BC.

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Solution:

1. $AB = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = [-1].$

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Solution:

1. $AB = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = [-1]$.
2. The multiplication BC is not possible since the matrix B is of order 3×1 and the matrix C is of order 3×2.

Theorem

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{p\times q}$ be three matrices, then

1. *The multiplication of matrices is not commutative:* $AB \neq BA$.
2. *The multiplication of matrices is associative:* $(AB)C = A(BC)$.
3. *The matrix I_n is the identity matrix of the multiplication:* $AI_n = A$.
4. *For any* $k \in \mathbb{R}$, $(kA)B = k(AB) = A(kB)$.

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Theorem

Let A *and* B *be any two matrices of order* $m \times n$, *then*

1. $(A + B)C = AC + BC$, *where* C *is a matrix of order* $n \times p$.
2. $C(A + B) = CA + CB$, *where* C *is a matrix of order* $p \times m$.

(4) Transpose of Matrices.

Definition

Let $A = [a_{ij}]_{m\times n}$ *be a matrix, then the transpose of* A *is* $A^t = [a_{ji}]_{n\times m}$.

Example

If $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$, *find* A^t.

Solution:

$$A^t = \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

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Theorem

Let A *and* B *be any two matrices of order* $m \times n$ *and* $k \in \mathbb{R}$, *then*

1. $(A^t)^t = A$.
2. $(A+B)^t = A^t + B^t$.
3. $(kA)^t = kA^t$.
4. $(AB)^t = B^t A^t$.