GENERAL MATHEMATICS 2

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Main Contents

Definition

Let $A = [a_{ij}]$ be a square matrix of order n. Then, the determinant of A can be defined as follows:

$$|A| = det(A) = \begin{cases} a & : A = [a] \\ \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \ (i = 1, \dots, n) & : otherwise, \end{cases}$$

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(1) The determinant of 2×2 Matrices

Let A be a square matrix of order 2:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \ .$$

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Example

Find the determinant of the matrix.

1. $A = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$
2. $B = \begin{bmatrix} 4 & -1 \\ 2 & 9 \end{bmatrix}$

Section 4: Determinants of Matrices

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2. $det(B) = \begin{vmatrix} 4 & -1 \\ 2 & 9 \end{vmatrix} = 4 \times 9 - (-1) \times 2 = 36 + 2 = 38.$

(2) The determinant of 3×3 Matrices

Let A be a square matrix of order 3:

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To calculate the determinant, choose the first row of A and multiply each of its elements by the corresponding cofactor:

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$$det(A) = 1\ (-1 \times 7 - 4 \times 9) - 6(5 \times 7 - 4 \times (-2)) + 3(5 \times 9 - (-1) \times (-2))$$

$$det(A) = 1\ (-43) - 6(43) + 3(43) = 43(-1 - 6 + 3) = 43(-4) = -172$$

(3) The determinant of 4×4 Matrices
Let A be a square matrix of order 4:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \text{ then } det(A) = a_{11}det(A_1) - a_{12}det(A_2) + a_{13}det(A_3) - a_{14}det(A_4)$$

where

$$\mathrm{A}_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} A_2 = \begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$

$$\mathrm{A}_3 = \begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} A_4 = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

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Example

Find the determinant of the matrix. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 4 & 3 \\ 7 & 1 & 2 & 2 \\ 3 & 2 & 1 & 5 \end{bmatrix}$

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Solution: $det(A) = 1det(A_1) - 0det(A_2) + 0det(A_3) - 0det(A_4)$

$$\mathrm{A}_1 = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & 2 \\ 2 & 1 & 5 \end{pmatrix}$$

$det(A_1) = 2\,(2 \times 5 - 2 \times 1) - 4(1 \times 5 - 2 \times 2) + 3(1 \times 1 - 2 \times 2) = 3$

Notes:
(1) If A is a square matrix having a zero row (or a zero column), then $det(A) = 0$.

Example: $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 3 & 4 & 7 \end{bmatrix}$
The matrix A contains a zero row, so $det(A) = 0$.

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(2) If A is a square matrix having two equal rows (or two equal columns), then $det(A) = 0$.

Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 5 & 6 \\ 3 & 4 & 3 \end{bmatrix}$
The matrix A contains two equal columns, so $det(A) = 0$.

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The matrix A contains two equal columns, so $det(A) = 0$.

(3) If A is a square matrix having a row which is a multiple of another row (or a column which is a multiple of another column), then $det(A) = 0$.

Example: $A = \begin{bmatrix} 1 & 2 & -2 \\ 4 & 7 & 5 \\ 3 & 6 & -6 \end{bmatrix}$
The third row in matrix A is a multiple of the first row by 3, so $det(A) = 0$.

(4) If A is a **diagonal matrix** or an **upper triangular matrix** or a **lower triangular matrix**, then $det(A)$ is the product of the elements of the main diagonal.

Example: $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

The matrix A is an upper triangular matrix, so $det(A) = -15$.

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The matrix A is an upper triangular matrix, so $det(A) = -15$.

(5) The determinant of the null matrix is 0 and the determinant of the identity matrix is 1.

(6) If B is obtained from A by multiplying a row (or column) by λ, then $det(B) = \lambda\ det(A)$.

Example: $A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$

$$det(A) = 1(4-3) - 3(8-0) + 1(-12-0) = -35$$

The matrix B is obtained from A by multiplying the first row by 2, then $det(B) = 2det(A) = -70$.

(7) If B is obtained from A by interchanging two rows (or two columns), then $det(B) = -det(A)$.

Example: A = $\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$, det(A)=-35

B = $\begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

The matrix B is obtained from A by interchanging the first and second rows, then $det(B) = -det(A) = 35$.

(7) If B is obtained from A by interchanging two rows (or two columns), then $det(B) = -det(A)$.

Example: A = $\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$, det(A)=-35

B = $\begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

The matrix B is obtained from A by interchanging the first and second rows, then $det(B) = -det(A) = 35$.

(8) If B is obtained from A by multiplying a row by a non-zero constant and adding the result to another row (or multiplying a column by a non-zero constant and adding the result to another column), then $det(B) = det(A)$.

Example:

A = $\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & -1 \\ 0 & -3 & 2 \end{bmatrix}$

multiplying the first row by 2 and adding the result to the second row

$\xrightarrow{2R_1 + R_2}$

B = $\begin{bmatrix} 1 & 3 & 1 \\ 6 & 8 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

The matrix B is obtained from A by multiplying the first row by 2 and adding the result to the second row. Therefore, $det(B) = det(A) = -35$.