GENERAL MATHEMATICS 2

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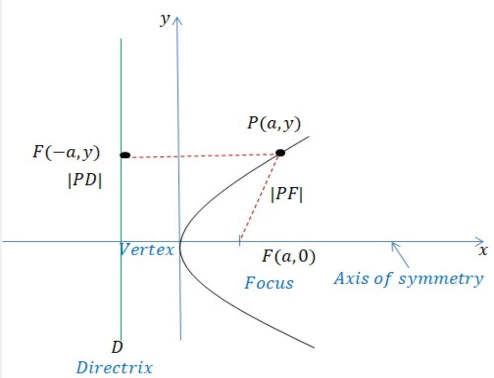
Main Contents

Definition

A parabola is a set of all points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix) in the same plane.

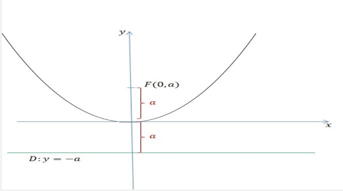


(1) Vertical Parabolas

(A) Parabolas with the Vertex at the Origin $x^2 = \pm 4ay$, where $a > 0$.

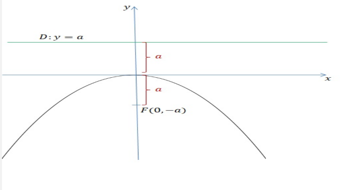
(A.1) The equation $x^2 = 4\ a\ y$ has the following features:

- The vertex is at the origin $V(0,0)$.
- The parabola opens upwards.
- The axis of symmetry is y-axis.
- The focus is $F(0, a)$.
- The directrix is $D : y = -a$.



(A.2) The equation $x^2 = -4\ a\ y$ has the following features:

- The vertex is at the origin $V(0,0)$.
- The parabola opens downwards.
- The axis of symmetry is y-axis.
- The focus is $F(0, -a)$.
- The directrix is $D : y = a$.



Example

Find the focus and the directrix of the parabola $x^2 = 4y$*, and sketch its graph.*

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Solution:
The equation

$$x^2 = 4y$$

takes the form

$$x^2 = 4\ a\ y$$

$$\Rightarrow 4a = 4 \Rightarrow a = 1$$

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The parabola has the following features:

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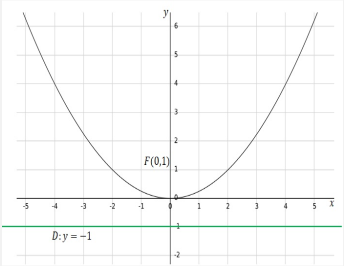
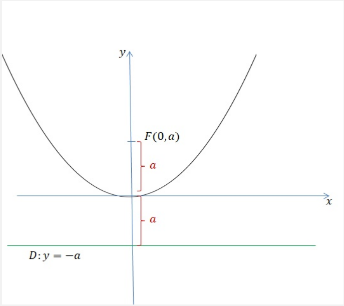


Section 1: Parabola

Special case

$$x^2 = 4\ a\ y$$



- The vertex is at the origin $V(0, 0)$.
- The parabola opens upwards.
- The axis of symmetry is y-axis.
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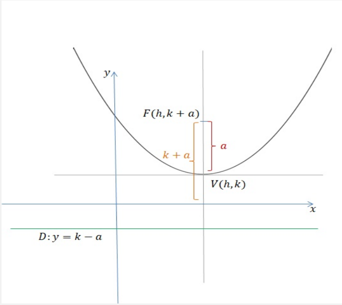
General case:

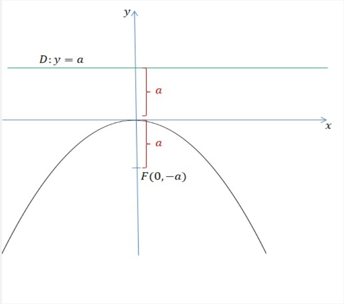
$$(x - h)^2 = 4\ a\ (y - k)$$



- The vertex is the point $V(h, k)$.
- The parabola opens upwards.
- The axis of symmetry is parallel to y-axis.
- The focus is $F(h, k + a)$.
- The directrix is $D : y = k - a$.

Special case

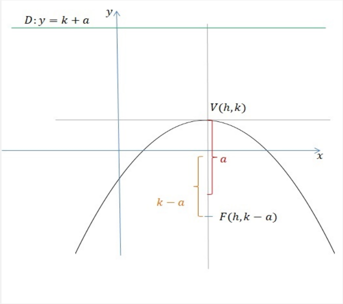
$$x^2 = -4\,a\,y$$



- The vertex is at the origin $V(0,0)$.
- The parabola opens downwards.
- The axis of symmetry is y-axis.
- The focus is $F(0,-a)$.
- The directrix is $D: y = a$.

General case:

$$(x-h)^2 = -4\,a\,(y-k)$$



- The vertex is the point $V(h,k)$.
- The parabola opens downwards.
- The axis of symmetry is parallel to y-axis.
- The focus is $F(h, k-a)$.
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Find the focus and the directrix of the parabola $(x+1)^2 = -4(y-1)$, *and sketch its graph.*

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$$-h = 1 \Rightarrow h = -1\ ,\ \ k = 1 \ \text{ and } \ 4a = 4 \Rightarrow a = 1$$

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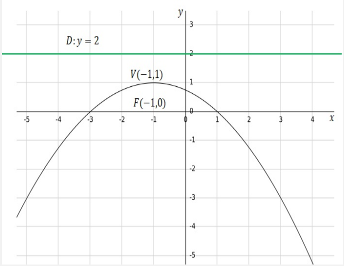
The parabola has the following features:

- The vertex is $V(h,k) = V(-1,1)$.
- The parabola opens downwards.
- The axis of symmetry is parallel to y-axis.
- The focus is $F(h, k-a) = F(-1,0)$.
- The directrix is
  $D: y = k + a \Rightarrow D: y = 2$.



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Find the equation of the parabola with vertex $(2, 1)$ *and focus* $F(2, 3)$. *Then, sketch the graph.*

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Solution:

1. The vertex and the focus are on the same line $x = 2$ (the x-term of the two points is constant), so the axis of symmetry of the parabola is parallel to y-axis.
2. From the y-term of the vertex and the focus, the parabola opens upwards.

Thus, the parabola equation takes the form

$$(x - h)^2 = 4a(y - k) \ .$$

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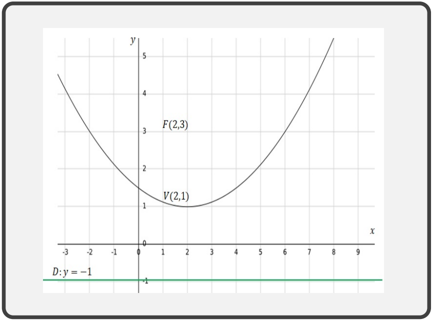
From the vertex and focus, we have

$$V(h, k) = (2, 1) \Rightarrow h = 2 \text{ and } k = 1$$

$$F(h, k + a) = (2, 3) \Rightarrow 1 + a = 3 \Rightarrow a = 2$$

By substituting the values of a, h and k, the equation of the parabola becomes

$$(x - 2)^2 = 8(y - 1) \ .$$

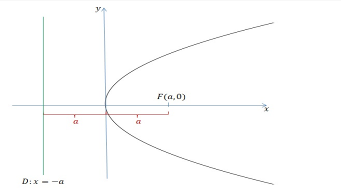


(2) Horizontal Parabolas

(A) Parabolas with the Vertex at the Origin $y^2 = \pm 4ax$, where $a > 0$.

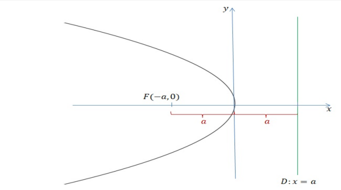
(A.1) The equation $y^2 = 4\ a\ x$ has the following features:

- The vertex is at the origin $V(0, 0)$.
- The parabola opens to the right.
- The axis of symmetry is x-axis.
- The focus is $F(a, 0)$.
- The directrix is $D : x = -a$.



(A.2) The equation $y^2 = -4\ a\ x$ has the following properties:

- The vertex is at the origin $V(0, 0)$.
- The parabola opens to the left.
- The axis of symmetry is x-axis.
- The focus is $F(-a, 0)$.
- The directrix is $D : x = a$.



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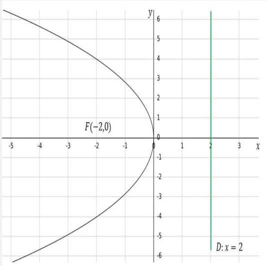

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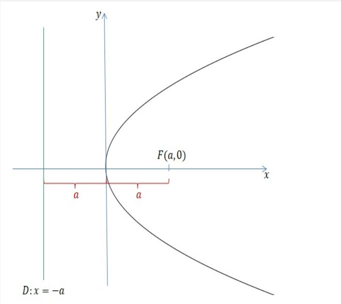
Special case

$$y^2 = 4\,a\,x$$



- The vertex is the point $V(0,0)$.
- The parabola opens to the right.
- The axis of symmetry is x-axis.
- The focus is $F(a,0)$.
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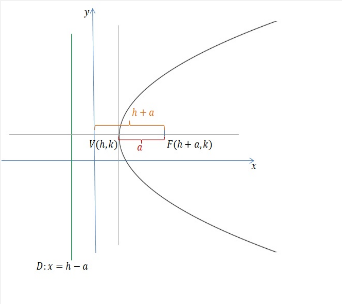
General case:

$$(y-k)^2 = 4\,a\,(x-h)$$



- The vertex is the point $V(h,k)$.
- The parabola opens to the right.
- The axis of symmetry is parallel to x-axis.
- The focus is $F(h+a,k)$.
- The directrix is $D : x = h - a$.

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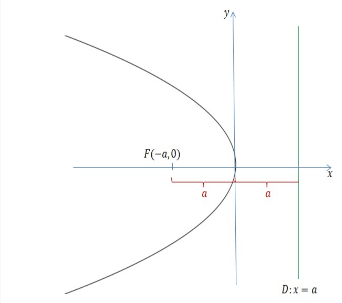

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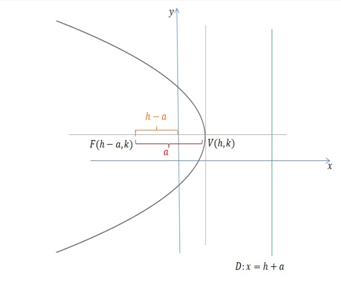

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- The focus is $F(h - a, k)$.
- The directrix is $D : x = h + a$.

Example

Find the focus and the directrix of the parabola $2y^2 - 4y + 8x + 10 = 0$, *and sketch its graph.*

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Solution: Since the quadrature is on the y-term, the parabola takes the form $(y - k)^2 = \pm 4\, a\, (x - h)$.

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$$2y^2 - 4y + 8x + 10 = 0, \qquad \text{divide all terms by 2}$$

$$y^2 - 2y + 4x + 5 = 0,$$

$$y^2 - 2y = -4x - 5, \qquad \text{isolate } y\text{-terms}$$

$$\underbrace{(y^2 - 2y + 1)}_{\text{completing square}} = -4x - 5 + 1 \qquad (u \pm v)^2 = u^2 \pm 2uv + v^2$$

$$(y-1)^2 = -4x - 4$$

$$(y-1)^2 = -4(x+1) \qquad (y-k)^2 = -4a(x-h)$$

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$$h = -1, k = 1, a = 1$$

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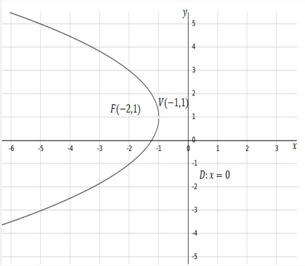

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- The focus is $F(h-a, k) = F(-2, 1)$.
- The directrix is $D : x = h + a \Rightarrow D : x = 0$.