# GENERAL MATHEMATICS 2

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## Main Contents

1. Parabola
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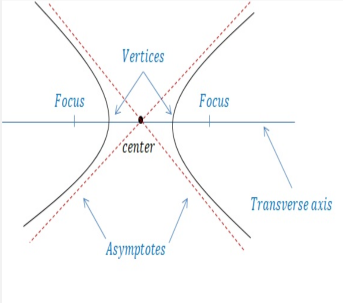
# Section 3: Hyperbola

## Definition

*A hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances of each point from two fixed points (called foci) is constant.*

- Each fixed point mentioned in the previous definition is called a **focus**.
- The point midway between the foci is called the center. The line containing the foci is the **transverse axis**.
- The graph of the hyperbola is made up of two parts called **branches**. Each branch intersects the transverse axis at a point called the **vertex**.



**(1) Hyperbolas with Centers Located at the Origin Point**

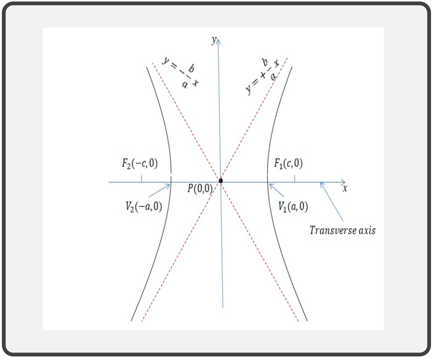
**(A)** The equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The hyperbola has the following features:

- The center is $P(0, 0)$.
- The vertices are $V_1(a, 0)$, $V_2(-a, 0)$.
- The foci are $F_1(c, 0)$, $F_2(-c, 0)$, where

$$c = \sqrt{a^2 + b^2} \ .$$

- The transverse axis is $x$-axis with length $2a$.
- The asymptotes are $y = \pm \frac{b}{a} x$.



**(B)** The equation of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

The hyperbola has the following features:

- The center is $P(0,0)$.
- The vertices are $V_1(0,b)$, $V_2(0,-b)$.
- The foci are $F_1(0,c)$, $F_2(0,-c)$, where

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- The transverse axis is $x$-axis with length $2b$.
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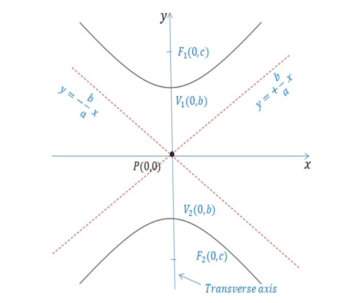

## Example

*Identify the features of the hyperbola* $4x^2 - 16y^2 = 64$ *and sketch its graph.*

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Solution:

$$4x^2 - 16y^2 = 64 \overbrace{\Rightarrow}^{\text{divide by } 64} \frac{4x^2}{64} - \frac{16y^2}{64} = \frac{64}{64} \Rightarrow \frac{x^2}{16} - \frac{y^2}{4} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

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$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$a^2 = 16 \Rightarrow a = 4,\ b^2 = 4 \Rightarrow b = 2 \Rightarrow\ c = \sqrt{a^2 + b^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$$

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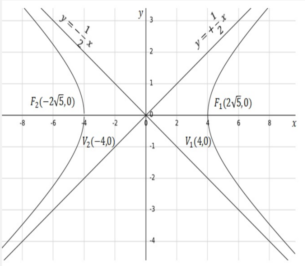
The hyperbola has the following features:

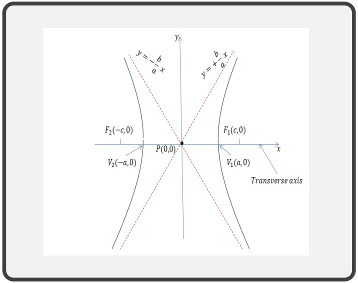
- The center is $P(0, 0)$
- The vertices are $V_1(4, 0)$, $V_2(-4, 0)$.
- The foci are $F_1(2\sqrt{5}, 0)$, $F_2(-2\sqrt{5}, 0)$.
- The transverse axis is $x$-axis with length 8.
- The asymptotes are $y = \pm\frac{2}{4}x = \pm\frac{1}{2}x \Rightarrow y = \pm\frac{1}{2}$.



# Section 3: Hyperbola

**(1) Hyperbolas with Centers Located at the Origin**

**Point (A)** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$



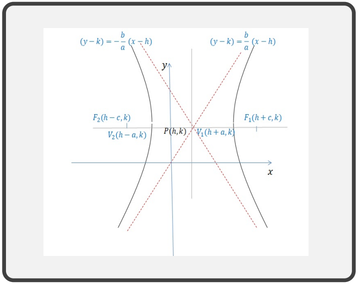

- The center is $P(0,0)$.
- The vertices are $V_1(a,0)$, $V_2(-a,0)$.
- The foci are $F_1(c,0)$, $F_2(-c,0)$, where

$$c = \sqrt{a^2 + b^2}\ .$$

- The transverse axis is $x$-axis with length $2a$.
- The asymptotes are $y = \pm \frac{b}{a}x$.

**(2) Hyperbolas with Centers Not at the Origin**

**(A)** $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$



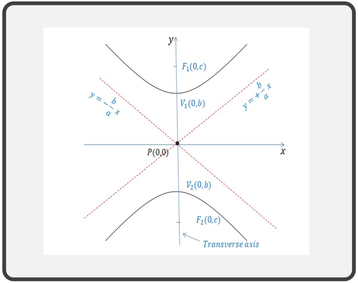

- The center is $P(h,k)$
- The vertices are $V_1(h+a,k)$, $V_2(h-a,k)$.
- The foci are $F_1(h+c,k)$, $F_2(h-c,k)$, where

$$c = \sqrt{a^2 + b^2}\ .$$

- The transverse axis is parallel to $x$-axis with length $2a$.
- The asymptotes are $(y-k) = \pm \frac{b}{a}(x-h)$.

**(1) Hyperbolas with Centers Located at the Origin Point**

**(B)** $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$




- The center is $P(0,0)$.
- The vertices are $V_1(0,b)$, $V_2(0,-b)$.
- The foci are $F_1(0,c)$, $F_2(0,-c)$, where

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- The transverse axis is $x$-axis with length $2b$.
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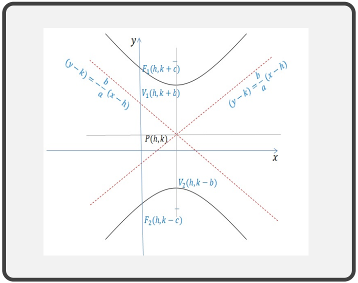


- The center is $P(h,k)$
- The vertices are $V_1(h,k+b)$, $V_2(h,k-b)$.
- The foci are $F_1(h,k+c)$, $F_2(h,k-c)$, where

$$c = \sqrt{a^2 + b^2}\ .$$

- The transverse axis is $x$-axis with length $2b$.
- The asymptotes are $(y-k) = \pm\frac{b}{a}(x-h)$.

## Example

*Find the equation of the hyperbola with foci at* $(-2, 2)$, $(6, 2)$ *and one of its vertices is* $(5, 2)$, *then sketch its graph.*

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Since the $y-$term in the foci is constant, then the equation of the hyperbola takes the form

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$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

From the given foci, we have

$$F_1(h+c, k) = (6, 2) \Rightarrow h+c = 6, k = 2$$

$$F_2(h-c, k) = (-2, 2) \Rightarrow h-c = -2, k = 2$$

By doing some calculation, we obtain $h = 2$ and $c = 4$.

**Illustration:** equation (1) + equation (2):

h+c =6 $\rightarrow$ 1

$h - c = -2 \rightarrow 2$

$- - - - - - - - -$

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h+c =6 $\rightarrow$ 1
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From

$$V_1(h+a, k) = (5, 2) \Rightarrow h + a = 5 \Rightarrow 2 + a = 5 \Rightarrow a = 3$$

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h+c =6 $\rightarrow$ 1
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$$V_1(h+a, k) = (5, 2) \Rightarrow h+a = 5 \Rightarrow 2+a = 5 \Rightarrow a = 3$$

From the formula

$$c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b^2 = 16 - 9 = 7 \Rightarrow b = \sqrt{7}$$

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$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

From the given foci, we have

$$F_1(h+c, k) = (6, 2) \Rightarrow h+c = 6, k = 2$$

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**Illustration:** equation (1) + equation (2):
h+c =6 $\rightarrow$ 1
$h - c = -2 \rightarrow 2$
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$2h = 4 \Rightarrow h = 2$

By doing some calculation, we obtain $h = 2$ and $c = 4$.

From

$$V_1(h+a, k) = (5, 2) \Rightarrow h+a = 5 \Rightarrow 2+a = 5 \Rightarrow a = 3$$

From the formula

$$c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b^2 = 16 - 9 = 7 \Rightarrow b = \sqrt{7}$$

Thus, the equation of the hyperbola is

$$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{7} = 1$$

$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{7} = 1$$

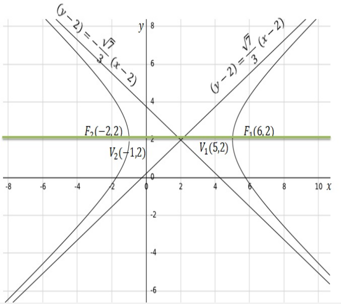
$a^2 = 9 \Rightarrow a = 3,\ b^2 = 7 \Rightarrow b = \sqrt{7},\ c = 4,\ h = 2,\ k = 2$

Remember:

- Vertices: $V(h \pm a, k)$
- Foci: $F(h \pm c, k)$
- The asymptotes: $(y-k) = \pm \frac{b}{a}(x-h)$.

The hyperbola has the following features:

- The center is $P(2, 2)$
- The vertices are $V_1(5, 2)$, $V_2(-1, 2)$.
- The foci are $F_1(6, 2)$, $F_2(-4, 2)$.
- The transverse axis is parallel to $x$-axis with length 6.
- The asymptotes are $(y-2) = \pm \frac{\sqrt{7}}{3}(x-2)$.



## Example

*Identify the features of the hyperbola* $2y^2 - 4x^2 - 4y - 8x - 34 = 0$, *then sketch its graph.*

Solution:

$$\begin{aligned}
2y^2 - 4x^2 - 4y - 8x - 34 &= 0, \\
2y^2 - 4y - 4x^2 - 8x &= 34 \\
2(y^2 - 2y) - 4(x^2 + 2x) &= 34 \qquad \text{rearrange } x\text{-terms and } y\text{-terms} \\
2(y^2 - 2y + 1) - 4(x^2 + 2x + 1) &= 34 + 2 - 4 \qquad \text{complete the square} \\
2(y - 1)^2 - 4(x + 1)^2 &= 32 \qquad (u \pm v)^2 = u^2 \pm 2uv + v^2 \\
\frac{(y-1)^2}{16} - \frac{(x+1)^2}{8} &= 1 \qquad \text{divide both sides by } 40 \ .
\end{aligned}$$

From the standard form

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

$$h = -1, \ k = 1, \ a^2 = 8 \Rightarrow a = \sqrt{8} = 2\sqrt{2}, \ b^2 = 16 \Rightarrow b = 4$$

From the formula

$$c^2 = a^2 + b^2 \Rightarrow c^2 = 8 + 16 = 24 \Rightarrow c = 2\sqrt{6}$$