GENERAL MATHEMATICS 2

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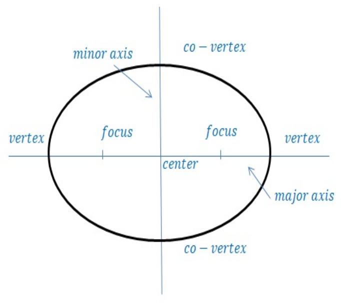
Main Contents

1. Parabola
2. Ellipse
3. Hyperbola

Definition

An ellipse is a set of all points in a plane such that the sum of the distances from each point to two fixed points (called foci) is constant.

- Each of the two fixed points mentioned in the previous definition is called a **focus**. The line containing the foci intersects the ellipse at points called **vertices**.
- The line segment between the vertices is called the **major axis**, and its midpoint is the center of the ellipse.
- A line perpendicular to the major axis through the center intersects the ellipse is called the **minor axis** and its endpoints called **co-vertices**.



(1) Ellipses with Centers Located at the Origin Point

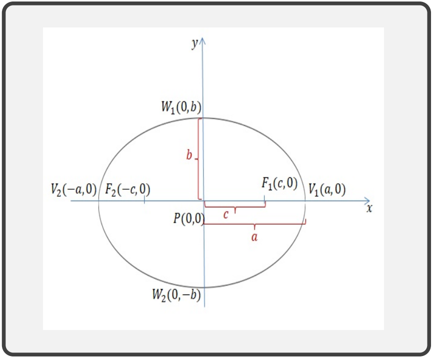
The ellipse equation with a center located at the point of origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(A) If $a > b$, the ellipse has the following properties:

- The center is $P(0, 0)$.
- The vertices are $V_1(a, 0)$, $V_2(-a, 0)$.
- The foci are $F_1(c, 0)$, $F_2(-c, 0)$, where

$$c^2 = a^2 - b^2 \text{ OR } c = \sqrt{a^2 - b^2}\ .$$

- The major axis is x-axis with length $2a$.
- The minor axis endpoints (co-vertices) are $W_1(0, b)$, $W_2(0, -b)$.
- The minor axis is y-axis with length $2b$.

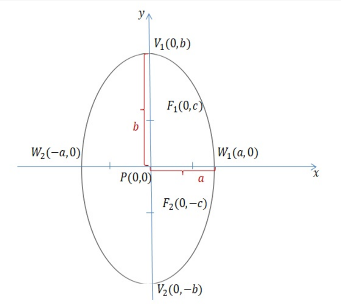


(B) If $a < b$, the ellipse has the following properties:

- The center is $P(0,0)$.
- The vertices are $V_1(0,b)$, $V_2(0,-b)$.
- The foci are $F_1(0,c)$, $F_2(0,-c)$, where

$$c^2 = b^2 - a^2 \text{ OR } c = \sqrt{b^2 - a^2}\ .$$

- The major axis is y-axis with length $2b$.
- The minor axis endpoints (co-vertices) are $W_1(a,0)$, $W_2(-a,0)$.
- The minor axis is x-axis with length $2a$.



Example

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$$9x^2 + 25y^2 = 225 \Rightarrow \frac{9x^2}{225} + \frac{25y^2}{225} = \frac{225}{225}$$

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Solution:

$$9x^2 + 25y^2 = 225 \Rightarrow \frac{9x^2}{225} + \frac{25y^2}{225} = \frac{225}{225} \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$$

Example

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Solution:

$$9x^2 + 25y^2 = 225 \Rightarrow \frac{9x^2}{225} + \frac{25y^2}{225} = \frac{225}{225} \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \text{from the form:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, .$$

we have

$$a^2 = 25 \Rightarrow a = 5 \text{ and } b = 3$$

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we have

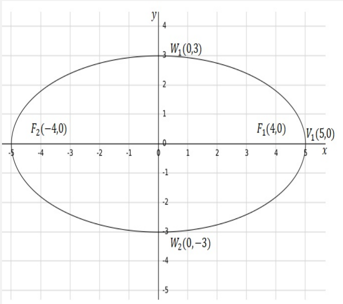
$$a^2 = 25 \Rightarrow a = 5 \ \text{ and } \ b = 3$$

Since $a > b$, the ellipse has the following features:

- The center is $P(0, 0)$.
- The vertices are $V_1(5, 0)$, $V_2(-5, 0)$.
- The foci are $F_1(4, 0)$, $F_2(-4, 0)$, where

$$c = \sqrt{25 - 9} = \sqrt{16} = 4 \ .$$

- The major axis is x-axis with length 10.
- The minor axis endpoints (co-vertices) are $W_1(0, 3)$, $W_2(0, -3)$.
- The minor axis is y-axis with length 6.



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If the center of an ellipse is at the origin, find the equation for the following properties:
1. Major axis is on the x-axis
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If the center of an ellipse is at the origin, find the equation for the following properties:
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2. Major axis length is 14
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Solution:
Since the major axis is on the x-axis, then the equation of the ellipse takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } \ a > b \ .$$

From the major axis length, we have

$$2a = 14 \Rightarrow a = 7$$

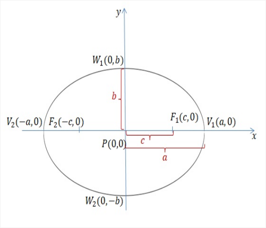
From the minor axis length, we have

$$2b = 10 \Rightarrow b = 5$$

By substituting the values of a and b into the main equation, we obtain

$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

(A) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \;\; a > b$

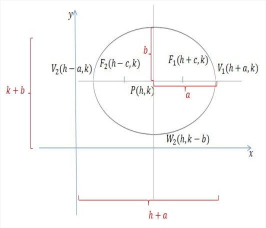


- The center is $P(0, 0)$.
- The vertices are $V_1(a, 0)$, $V_2(-a, 0)$.
- The foci are $F_1(c, 0)$, $F_2(-c, 0)$, where

$$c^2 = a^2 - b^2 \text{ OR } c = \sqrt{a^2 - b^2} \,.$$

- The major axis is x-axis with length $2a$.
- The co-vertices are $W_1(0, b)$, $W_2(0, -b)$.
- The minor axis is y-axis with length $2b$.

(A) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \;\; a > b$

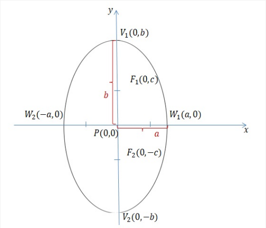


- The center is $P(h, k)$.
- The vertices are $V_1(h + a, k)$, $V_2(h - a, k)$.
- The foci are $F_1(h + c, k)$, $F_2(h - c, k)$, where

$$c^2 = a^2 - b^2 \text{ OR } c = \sqrt{a^2 - b^2} \,.$$

- The major axis is parallel to x-axis with length $2a$.
- The co-vertices are $W_1(h, k + b)$, $W_2(h, k - b)$.
- The minor axis is parallel to y-axis with length $2b$.

(B) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \;\; b > a$



- The center is $P(0, 0)$.
- The vertices are $V_1(0, b)$, $V_2(0, -b)$.
- The foci are $F_1(0, c)$, $F_2(0, -c)$, where

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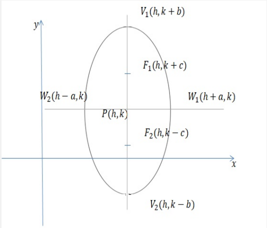

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- The major axis is parallel to y-axis with length $2b$.
- The co-vertices are $W_1(h + a, k)$, $W_2(h - a, k)$.
- The minor axis is parallel to x-axis with length $2a$.

Example

Find the equation of the ellipse with foci at $(-3, 1)$, $(5, 1)$ *and one of its vertice is* $(7, 1)$, *then sketch its graph.*

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Solution:
Since the y-term in the foci is constant, the equation of the ellipse is of the form

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where $a > b$.

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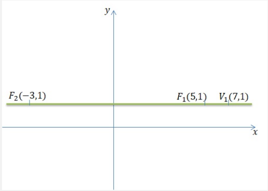

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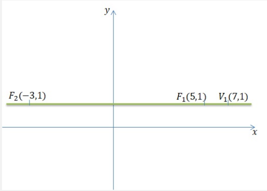

From the given foci, we have

$F_1(h+c,k) = (5,1) \Rightarrow h+c = 5,\ k = 1$

$F_2(h-c,k) = (-3,1) \Rightarrow h-c = -3,\ k = 1$

By doing some calculation, we obtain $h = 1$ and $c = 4$.

Illustration: By adding equation 1 to equation 2:

$h + c = 5 \rightarrow \boxed{1}$

$h - c = -3 \rightarrow \boxed{2}$

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$2h = 2 \Rightarrow h = 1$

From the given vertex, we have $V_1(h + a, k) = (7, 1) \Rightarrow h + a = 7 \Rightarrow 1 + a = 7 \Rightarrow a = 6$.

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After substitution, the ellipse equation becomes

$$\frac{(x-1)^2}{36} + \frac{(y-1)^2}{20} = 1$$

$h = 1, k = 1, a = 6, b = \sqrt{20} = 2\sqrt{5}, c = 4$

Remember:

- Vertices: $V(h \pm a, k)$
- Foci: $F(h \pm c, k)$
- co-vertices: $W(h, k \pm b)$.

Section 2: Ellipse

From the given vertex, we have $V_1(h+a, k) = (7, 1) \Rightarrow h + a = 7 \Rightarrow 1 + a = 7 \Rightarrow a = 6$.

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After substitution, the ellipse equation becomes

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$h = 1,\ k = 1,\ a = 6,\ b = \sqrt{20} = 2\sqrt{5},\ c = 4$

The ellipse has the following features:

- The center is $P(1, 1)$.
- The vertices are $V_1(7, 1)$, $V_2(-5, 1)$.
- The foci are $F_1(5, 1)$, $F_2(-3, 1)$.
- The major axis is parallel to x-axis with length 12.
- The co-vertices are $W_1(1, 1 + 2\sqrt{5})$ and $W_2(1, 1 - 2\sqrt{5})$.
- The minor axis of the ellipse is parallel to y-axis with length $4\sqrt{5}$.

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- Foci: $F(h \pm c, k)$
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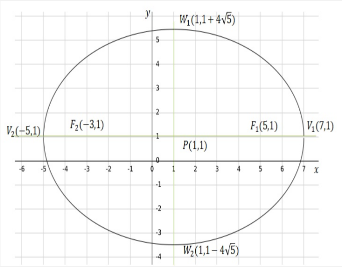

Example

Identify the features of the ellipse $4x^2 + 2y^2 - 8x - 8y - 20 = 0$*, then sketch its graph.*

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Identify the features of the ellipse $4x^2 + 2y^2 - 8x - 8y - 20 = 0$, *then sketch its graph.*

Solution:

$$\begin{aligned}
4x^2 + 2y^2 - 8x - 8y - 20 &= 0 \\
2x^2 + y^2 - 4x - 4y - 10 &= 0 \qquad \text{divide by 2} \\
2x^2 - 4x + y^2 - 4y &= 10 \qquad \text{isolate } x \text{ any } y \text{ terms} \\
2(x^2 - 2x) + (y^2 - 4y) &= 10 \\
2(x^2 - 2x + 1) + (y^2 - 4x + 4) &= 10 + 2 + 4 \\
2(x-1)^2 + (y-2)^2 &= 16 \qquad \text{completing square: } (u \pm v)^2 = u^2 \pm 2uv + v^2 \\
\frac{(x-1)^2}{8} + \frac{(y-2)^2}{16} &= 1\ . \qquad \text{divide by 16}
\end{aligned}$$

The result takes the standard form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\ ,$$

where

$$h = 1,\ k = 2,\ a^2 = 8 \Rightarrow a = 2\sqrt{2},\ \text{ and } \ b^2 = 16 \Rightarrow b = 4,\ \text{ then } \ c = \sqrt{16 - 8} = \sqrt{8} = 2\sqrt{2}\ .$$

Section 2: Ellipse

The result takes the standard form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \ ,$$

where

$$h = 1, \ k = 2, \ a^2 = 8 \Rightarrow a = \sqrt{8} = 2\sqrt{2}$$

$b^2 = 16 \Rightarrow b = 4$, then $c = \sqrt{16-8} = \sqrt{8} = 2\sqrt{2}$.
The ellipse has the following features:

Remember:

- Vertices: $V(h, k \pm b)$
- Foci: $F(h, k \pm c)$
- co-vertices: $W(h \pm a, k)$.

- The center is $P(1, 2)$.
- The Vertices are $V_1(1, 6)$, $V_2(1, -2)$.
- The foci are $F_1(1, 2 + 2\sqrt{2})$, $F_2(1, 2 - 2\sqrt{2})$.
- The co-vertices are $W_1(1 + 2\sqrt{2}, 2)$ and $W_2(1 - 2\sqrt{2}, 2)$.
- The major axis is parallel to y-axis with length 8.
- The minor axis is parallel to x-axis with length $4\sqrt{2}$