

33.6 Power in an AC Circuit

In chapter 28, we found that the power delivered by a battery to a DC circuit equals:

$$P = I^2 R = \Delta V \cdot I$$

Similarly, the power delivered by an AC generator to RLC circuit can be calculated as:

$$P = I \cdot \Delta V = (I_{max} \sin(\omega t - \phi)) \cdot (\Delta V_{max} \sin(\omega t))$$

Also, we can express **the average power** $P_{av.}$ as follows:

$$P_{av.} = \frac{1}{2} I_{max} \Delta V_{max} \cos(\phi) = I_{rms} \Delta V_{rms} \cos(\phi)$$

where $\cos(\phi)$ is called the **power factor**, an indicator of how effectively power is being used.

This Equation shows that the power delivered by an AC source to any circuit **depends on the phase**, a result that has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, **technicians introduce capacitance in the circuits to shift the phase. (More explanations are shown below)**

Special case:

When $\phi = 0 \rightarrow \cos(\phi) = 1$,

Then: $P_{av.} = \frac{1}{2} I_{max} \Delta V_{max} = I_{rms} \Delta V_{rms}$

Moreover, we can write the average power as:

$$P_{av.} = I_{rms}^2 R$$

We can conclude that:

- ✓ No power losses are associated with **pure capacitors and pure inductors** in an AC circuit. Why?
 - In an **ideal capacitor** or **ideal inductor**, **no energy is permanently consumed or lost**.
- ✓ The average power delivered by the source is converted to internal energy in the **resistor**.
 - A **resistor converts electrical energy into heat** continuously.

33.7 Resonance in a series RLC circuit

In an RLC circuit, resonance occurs when the driving frequency is such that the rms current is at its maximum.

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{Z} = \frac{\Delta V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}} \quad 33.14$$

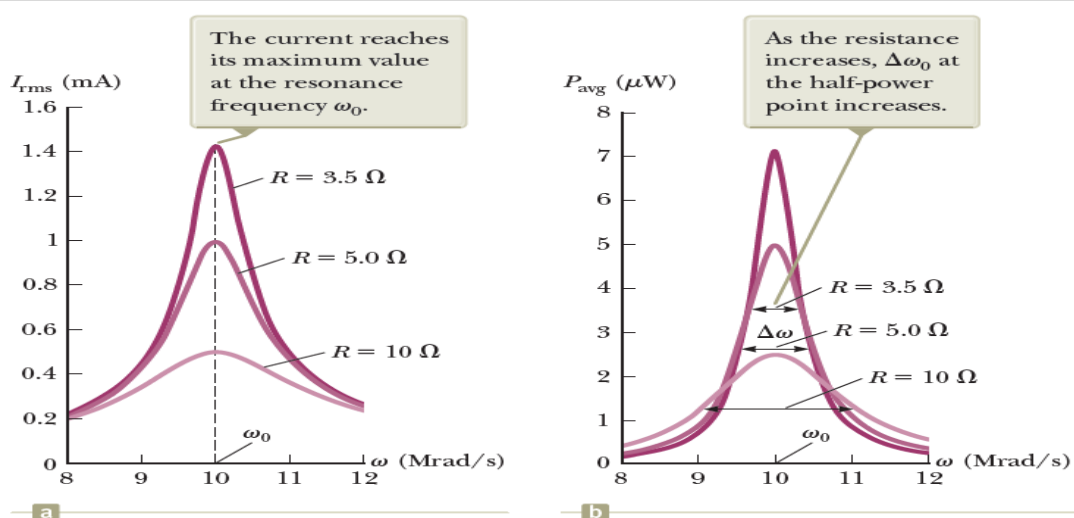
The angular frequency ω_0 at which $(X_L - X_C) = 0$ is called the resonance frequency of the circuit.

$$\begin{aligned} X_L &= X_C \\ \omega_0 L &= \frac{1}{\omega_0 C} \\ \omega_0^2 &= \frac{1}{LC} \\ \omega_0 &= \frac{1}{\sqrt{LC}} \end{aligned} \quad 33.15$$

❖ **At resonance**, the circuit behaves like a **pure resistor**.

- ✓ The current is **maximum** because impedance is **minimum**.
- ✓ The current is **in phase** with the voltage because the inductor and capacitor cancel each other's effects.
- ✓ $Z = \sqrt{R^2 + (X_L - X_C)^2}$; since $X_L = X_C$; Then, $Z = R$

A plot of I_{rms} current versus angular frequency (ω) for a series RLC circuit is shown in the Figure below. The data assume a constant $\Delta V_{\text{rms}} = 5.0$ mV, $L = 5.0$ mH, and $C = 2.0$ nF.



- ✓ All curves peak sharply at ω_0 , confirming that **maximum current occurs at resonance**.
- ✓ As **R** increases, the **maximum current decreases**.
- ✓ Lower resistance (e.g., $R=3.5\ \Omega$) results in a **sharper and taller peak**, indicating a **higher Q factor** (better resonance).
- ✓ Higher resistance makes the curve flatter (broader resonance).

➤ The *average power* as a function of frequency for a series RLC circuit as:

$$P_{av.} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R = \frac{(\Delta V_{rms})^2}{R^2 + (X_L - X_C)^2} R$$

Since: $X_L = \omega L$, $X_C = \frac{1}{\omega C}$ and $\omega_0^2 = \frac{1}{LC}$

Then, $(X_L - X_C)^2 = (\omega L - \frac{1}{\omega C})^2 = \frac{L^2}{\omega^2} (\omega^2 - \omega_0^2)^2$

These give us:
$$P_{av.} = \frac{(\Delta V_{rms})^2 R \omega^2}{R^2 \omega^2 + L^2 (\omega^2 - \omega_0^2)^2}$$

- When $\omega = \omega_0$, the average power is **maximum**.

$$P_{av.} = \frac{(\Delta V_{rms})^2}{R}$$

The quality factor Q can also be computed as follows:

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}$$

Where $\Delta\omega$ is the width of the curve measured between the two values of ω for which P_{avg} has one-half its maximum value, called **the half-power points**.

Right Plot (b):

- ✓ **Power Peaks at Resonance ω_0 :**
 - Just like current, **power is maximized** when the driving frequency equals the resonance frequency.
- ✓ **Sharper Peak with Lower Resistance:**
 - Lower resistance produces a **narrower, taller peak**, which means **more efficient power transfer** near ω_0 , but less outside that frequency.

- The quantity $\Delta\omega$ (bandwidth) is **smaller** for smaller R, indicating a higher selectivity or **sharper resonance**.
- ✓ **Power Decreases with Higher Resistance:**
- For R=10, the peak power is lower and more spread out, showing that **resistance reduces power output** and **broadens the response**.

Exercise-I:

A series RLC circuit is connected to an AC source with the voltage:

$$v(t) = 200 \sin(314t) \quad , \quad i(t) = 8 \sin(314t - 0.643)$$

Given:

- $L=0.5 \text{ H}$
- $R=20 \Omega$

Q-1: What is the total impedance of the circuit?

- a) 1600Ω b) 157Ω c) 142Ω d) 25Ω
-

Q-2: What is the power factor of the circuit?

- a) 1 b) 0.25 c) 0.8 d) Zero
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Q-3: Based on the given values, which of the following is true about the phase relationship in the circuit?

- a) Voltage leads the current
 b) Current and voltage are in phase
 c) Current leads the voltage
 d) Current is perpendicular to the voltage
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Q-4: At resonance, what happens to the resonance frequency if the resistance R in the circuit is decreased?

- a) Remains constant
 b) Increases
 c) Decreases
 d) Changes as a sinusoidal wave
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Q-5: What is the capacitive reactance X_C of the circuit?

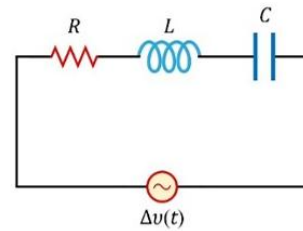
- a) 1600Ω b) 157Ω c) 142Ω d) 25Ω

Exercise-II:

A sinusoidal voltage source is described by:

$$\Delta v(t) = 100\sin(1000t)$$

where t is in seconds and Δv is in volts. This voltage is applied to a series RLC circuit with the following parameters:



- Resistance: $R=400\Omega$
- Capacitance: $C=5\mu\text{F}$
- Inductance: $L=0.5\text{H}$

Q-1: What is the total impedance Z of the circuit (in ohms)?

- a) 50 b) 100 **c) 500** d) 1000
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Q-2: In reference to Q-1, by how many degrees does the voltage lead the current in the RLC circuit?

- a) 36.9°** b) 43.2° c) 64.5° d) 85.3°
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Q-3: Based on the same RLC circuit, what is the resonance angular frequency ω_0 of the circuit (in rad/s)?

- a) 59.3 b) 264.3 c) 417.5 **d) 632.5**
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Q-4: In a series RLC AC circuit, the instantaneous voltage and current are given as:

$$\Delta v(t) = 100\sin(\omega t), \quad i(t) = 100\sin(\omega t + 3\pi)$$

where t is in seconds, Δv is in volts, and i is in amperes.

What is the average power consumed by the circuit (in kilowatts)?

- a) 1.5 **b) 2.5** c) 5.5 d) 10.5

Extra Problem:

[RLC Solutions](#)

⚡ Average Power in AC Circuits

The formula:

$$P_{av} = 1/2 I_{max} \Delta V_{max} \cos(\phi) = I_{rms} \Delta V_{rms} \cos(\phi)$$

✓ Explanation of Each Term:

- P_{av} : **Average power** delivered to the circuit (in watts)
 - I_{max} , ΔV_{max} : Maximum current and voltage
 - I_{rms} , ΔV_{rms} : RMS (root mean square) values of current and voltage
 - ϕ : **Phase angle** between voltage and current
 - $\cos(\phi)$: **Power factor** — a crucial indicator of how effectively power is being used
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⚡ What the Equation Tells Us:

- Power delivery in an AC circuit **depends on the phase difference** ϕ between voltage and current.
 - If voltage and current are **in phase** ($\phi=0^\circ$), $\cos(\phi)=1$, and **maximum power is delivered**.
 - If voltage and current are **out of phase** (like in inductive or capacitive loads), $\cos(\phi)<1$, and **less real power is delivered**, even though current and voltage are still present.
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🏭 Real-World Example — Industrial Settings:

Factories using:

- **Large motors**
- **Transformers**
- **Generators**

... often deal with **inductive loads** due to the coils in these machines. Inductance causes the **current to lag** behind the voltage, resulting in a **low power factor**.

To **improve the power factor** and make the system more efficient:

- **Capacitors are added** to the circuit.
- Capacitors cause **current to lead voltage**, which **offsets** the lag from the inductors.

- This **reduces the phase angle ϕ** and increases $\cos(\phi)$, allowing **more real power** to be delivered **without dangerously increasing the voltage or current**.
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🔧 Why This Matters:

Improving the power factor means:

- Less energy wasted
- Lower electricity bills
- Smaller voltage drops
- Better performance of electrical systems

Think about it !!

Radio Receiver (RLC Circuit)

- A radio uses a **resonant RLC circuit** to receive signals.
- Many radio stations send signals, but the circuit responds **only to the one with the matching frequency**.
- You change the resonance frequency by **turning the tuning knob**, which adjusts the **capacitance**.
- The selected signal is then sent to the **amplifier and speakers**.
- To avoid picking up unwanted stations, the circuit must have a **high Q-factor**.
- A high Q-factor means the circuit responds **strongly to one frequency** and **weakly to nearby frequencies**.
- As a result, the radio receives **only the station you tune to**.



Exercise:

Plotting I_{rms} and P_{av} vs Frequency for a Series RLC Circuit using **Python**

Given:

- $L=5.0\ \mu\text{H}$
- $C=2.0\ \text{nF}$
- $\Delta V_{\text{rms}}=5.0\ \text{mV}$
- Test three resistance values: $R=3.5\ \Omega$, $5.0\ \Omega$, and $10.0\ \Omega$
- Angular frequency range: 8×10^6 to $12\times 10^6\ \text{rad/s}$

Starter Python Code:

```
import numpy as np
import matplotlib.pyplot as plt

# Constants
L = 5e-6          # H
C = 2e-9          # F
Vrms = 5e-3       # V
omega = np.linspace(8e6, 12e6, 1000) # Angular frequency range

# Resistance values
R_values = [0.50, 5.0, 50.0]

# Set up plots
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5))

for R in R_values:
    # Reactances
    XL = omega * L
    XC = 1 / (omega * C)

    # Impedance
    Z = np.sqrt(R**2 + (XL - XC)**2)

    # RMS Current
    Irms = Vrms / Z

    # Average Power
    P_avg = Irms**2 * R

    # Plotting
    ax1.plot(omega / 1e6, Irms * 1e3, label=f"R = {R}  $\Omega$ ")
    ax2.plot(omega / 1e6, P_avg * 1e6, label=f"R = {R}  $\Omega$ ")

# Labels and titles
ax1.set_title("RMS Current vs Frequency")
ax1.set_xlabel("Angular Frequency (Mrad/s)")
```



```
ax1.set_ylabel("I_rms (mA)")
ax1.grid(True)
ax1.legend()

ax2.set_title("Average Power vs Frequency")
ax2.set_xlabel("Angular Frequency (Mrad/s)")
ax2.set_ylabel("P_avg (μW)")
ax2.grid(True)
ax2.legend()

plt.tight_layout()
plt.show()
```
