32.6 Power in an AC Circuit

In chapter 26, we found that the power delivered by a battery to a DC circuit equals:

$$P=I^2R=\Delta V.I$$

Similarly, the power delivered by an AC generator to RLC circuit can be calculated as:

$$P = I.\Delta V = \left(I_{max}sin(\omega t-\phi)\right).(\Delta V_{max}sin(\omega t))$$

Also, we can express **the average power $P_{av.}$** as follows:

$$P_{av.} = \frac{1}{2}I_{max}\Delta V_{max}\cos(\phi) = I_{rms}\Delta V_{rms}\cos(\phi)$$

where $\cos(\phi)$ is called the **power factor**, an indicator of how effectively power is being used.

This Equation shows that the power delivered by an AC source to any circuit **depends on the phase**, a result that has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, technicians introduce capacitance in the circuits to shift the phase. (More explanations are shown below)

Special case:

When $\phi = 0 \rightarrow cos_{(0)} = 1$,

Then: $$P_{av.} = \frac{1}{2}I_{max}\Delta V_{max} = I_{rms}\Delta V_{rms}$$

Moreover, we can write the average power as:

$$P_{av.} = I_{rms}^2R$$

We can conclude that:

- ✓ No power losses are associated with **pure capacitors and pure inductors** in an AC circuit. Why?
  - In an **ideal capacitor** or **ideal inductor**, **no energy is permanently consumed or lost**.
- ✓ The average power delivered by the source is converted to internal energy in the **resistor**.
  - A **resistor converts electrical energy into heat** continuously.

32.7 Resonance in a series RLC circuit

In an RLC circuit, resonance occurs when the driving frequency is such that the rms current is at its maximum.

$$I_{rms} = \frac{\Delta V_{rms}}{Z} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad 32.14$$

The angular frequency ω₀ at which (X_L-X_C)=0 is called the resonance frequency of the circuit.

$$X_L = X_C$$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0^2 = \frac{1}{LC}$$

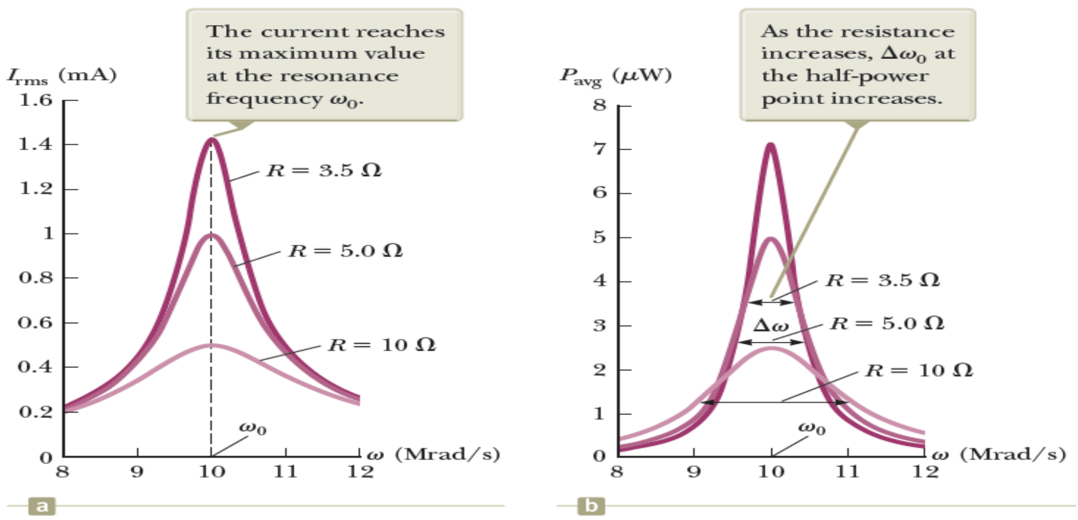
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad 32.15$$

- ❖ **At resonance**, the circuit behaves like a **pure resistor**.

- ✓ The current is **maximum** because impedance is **minimum**.
- ✓ The current is **in phase** with the voltage because the inductor and capacitor cancel each other's effects.
- ✓ $Z = \sqrt{R^2 - (X_L - X_C)^2}$; $since: X_L = X_C;$ $Then, Z = R$

A plot of I_{rms} current versus angular frequency (ω) for a series *RLC* circuit is shown in the Figure below. The data assume a constant ΔV_{rms} = 5.0 mV, *L* = 5.0 *m*H, and *C* =2.0 nF.



- ✓ All curves peak sharply at ω_0, confirming that **maximum current occurs at resonance**.
- ✓ As **R** increases, the **maximum current decreases**.
- ✓ Lower resistance (e.g., R=3.5 Ω) results in a **sharper and taller peak**, indicating a **higher Q factor** (better resonance).
- ✓ Higher resistance makes the curve flatter (broader resonance).

➢ The *average power* as a function of frequency for a series RLC circuit is:

$$P_{\text{av.}} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R = \frac{(\Delta V_{rms})^2}{R^2+(X_L-X_C)^2} R$$

Since: $X_L = \omega L$, $X_C = \frac{1}{\omega C}$ and $\omega_0^2 = \frac{1}{LC}$

Then, $(X_L - X_C)^2 = (\omega L - \frac{1}{\omega C})^2 = \frac{L^2}{\omega^2}(\omega^2 - \omega_0^2)^2$

These give us: $$P_{\text{av.}} = \frac{(\Delta V_{rms})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2-\omega_0^2)^2}$$

- When $\omega=\omega_0$, the average power is **maximum.**

$$P_{\text{av.}} = \frac{(\Delta V_{rms})^2}{R}$$

The **quality factor Q** can also be computed as follows:

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}$$

Where $\Delta\omega$ is the width of the curve measured between the two values of ω for which P_{avg} has one-half its maximum value, called the *half-power points.*

Right Plot (b):

✓ **Power Peaks at Resonance ω_0**:

- Just like current, **power is maximized** when the driving frequency equals the resonance frequency.

✓ **Sharper Peak with Lower Resistance**:

- Lower resistance produces a **narrower, taller peak**, which means **more efficient power transfer** near ω_0, but less outside that frequency.

- The quantity $\Delta\omega$ (bandwidth) is **smaller** for smaller R, indicating a higher selectivity or **sharper resonance**.

✓ **Power Decreases with Higher Resistance**:

- For R=10, the peak power is lower and more spread out, showing that **resistance reduces power output** and **broadens the response**.

Exercise-I:

A series RLC circuit is connected to an AC source with the voltage:

v(t) = 200 sin(314t) , i(t) = 8 sin(314t - 0.643)

Given:

- L=0.5 H
- R=20 Ω

Q-1: What is the total impedance of the circuit?
a) 1600 Ω b) 157 Ω c) 142 Ω d) 25 Ω

Q-2: What is the power factor of the circuit?
a) 1 b) 0.25 c) 0.8 d) Zero

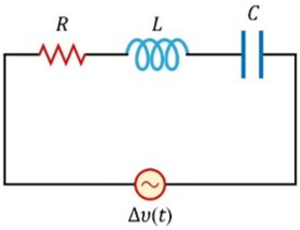
Q-3: Based on the given values, which of the following is true about the phase relationship in the circuit?
a) Voltage leads the current
b) Current and voltage are in phase
c) Current leads the voltage
d) Current is perpendicular to the voltage

Q-4: At resonance, what happens to the resonance frequency if the resistance R in the circuit is decreased?
a) Remains constant
b) Increases
c) Decreases
d) Changes as a sinusoidal wave

Q-5: What is the capacitive reactance X_C of the circuit?
a) 1600 Ω b) 157 Ω c) 142 Ω d) 25 Ω

Exercise-II:

A sinusoidal voltage source is described by:



$$\Delta v(t)=100\sin(1000t)$$

where t is in seconds and Δv is in volts. This voltage is applied to a series RLC circuit with the following parameters:

- **Resistance: R=400Ω**
- **Capacitance: C=5μF**
- **Inductance: L=0.5H**

Q-1: What is the total impedance Z of the circuit (in ohms)?
a) 50 b) 100 c) 500 d) 1000

Q-2: In reference to Q-1, by how many degrees does the **voltage lead the current** in the RLC circuit?
a) 36.9° b) 43.2° c) 64.5° d) 85.3°

Q-3: Based on the same RLC circuit, what is the **resonance angular frequency ω_0** of the circuit (in rad/s)?
a) 59.3 b) 264.3 c) 417.5 d) 632.5

Q-4: In a series RLC AC circuit, the instantaneous voltage and current are given as:

$\Delta v(t)=100\sin(\omega t)$, $i(t)=100\sin(\omega t+3\pi)$

where t is in seconds, Δv is in volts, and i is in amperes.

What is the average power consumed by the circuit (in kilowatts)?
a) 1.5 b) 2.5 c) 5.5 d) 10.5

Extra Problem:

RLC_Solutions

🔌 Average Power in AC Circuits

The formula:

$P_{av} = 1/2\ I_{max} \Delta V_{max} \cos(\phi) = I_{rms} \Delta V_{rms} \cos(\phi)$

✅ Explanation of Each Term:

- P_{av}: **Average power** delivered to the circuit (in watts)
- I_{max}, ΔV_{max}: Maximum current and voltage
- I_{rms}, ΔV_{rms}: RMS (root mean square) values of current and voltage
- ϕ: **Phase angle** between voltage and current
- $\cos(\phi)$: **Power factor** — a crucial indicator of how effectively power is being used

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⚡ What the Equation Tells Us:

- Power delivery in an AC circuit **depends on the phase difference** ϕ between voltage and current.
- If voltage and current are **in phase** ($\phi = 0°$), $\cos(\phi) = 1$, and **maximum power is delivered**.
- If voltage and current are **out of phase** (like in inductive or capacitive loads), $\cos(\phi) < 1$, and **less real power is delivered**, even though current and voltage are still present.

🏭 Real-World Example — Industrial Settings:

Factories using:

- **Large motors**
- **Transformers**
- **Generators**

... often deal with **inductive loads** due to the coils in these machines. Inductance causes the **current to lag** behind the voltage, resulting in a **low power factor**.

To **improve the power factor** and make the system more efficient:

- **Capacitors are added** to the circuit.
- Capacitors cause **current to lead voltage**, which **offsets** the lag from the inductors.
- This **reduces the phase angle ϕ** and increases $\cos(\phi)$, allowing **more real power** to be delivered **without dangerously increasing the voltage or current**.

🎯 Why This Matters:

Improving the power factor means:

- Less energy wasted
- Lower electricity bills
- Smaller voltage drops
- Better performance of electrical systems

Think about it !!

Radio Receiver (RLC Circuit)

- A radio uses a **resonant RLC circuit** to receive signals.
- Many radio stations send signals, but the circuit responds **only to the one with the matching frequency**.
- You change the resonance frequency by **turning the tuning knob**, which adjusts the **capacitance**.
- The selected signal is then sent to the **amplifier and speakers**.
- To avoid picking up unwanted stations, the circuit must have a **high Q-factor**.
- A high Q-factor means the circuit responds **strongly to one frequency** and **weakly to nearby frequencies**.
- As a result, the radio receives **only the station you tune to**.

Exercise:

Plotting I_{rms} and P_{av} vs Frequency for a Series RLC Circuit using **Python**

Given:

- L=5.0 μH
- C=2.0 nF
- ΔV_{rms}=5.0 mV
- Test three resistance values: R=3.5 Ω, 5.0 Ω, and 10.0 Ω
- Angular frequency range: 8×10^6 to 12×10^6 rad/s

Starter Python Code:

```
import numpy as np
import matplotlib.pyplot as plt

# Constants
L = 5e-6         # H
C = 2e-9         # F
Vrms = 5e-3      # V
omega = np.linspace(8e6, 12e6, 1000)  # Angular frequency range

# Resistance values
R_values = [0.50, 5.0, 50.0]

# Set up plots
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5))

for R in R_values:
    # Reactances
    XL = omega * L
    XC = 1 / (omega * C)

    # Impedance
    Z = np.sqrt(R**2 + (XL - XC)**2)

    # RMS Current
    Irms = Vrms / Z

    # Average Power
    P_avg = Irms**2 * R

    # Plotting
    ax1.plot(omega / 1e6, Irms * 1e3, label=f"R = {R} Ω")
    ax2.plot(omega / 1e6, P_avg * 1e6, label=f"R = {R} Ω")

# Labels and titles
ax1.set_title("RMS Current vs Frequency")
ax1.set_xlabel("Angular Frequency (Mrad/s)")
ax1.set_ylabel("I_rms (mA)")
ax1.grid(True)
ax1.legend()

ax2.set_title("Average Power vs Frequency")
```

```
ax2.set_xlabel("Angular Frequency (Mrad/s)")
ax2.set_ylabel("P_avg (µW)")
ax2.grid(True)
ax2.legend()

plt.tight_layout()
plt.show()
```