# MATH203 Calculus

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# Outline

- Definition of sequences.
- Definition of convergent sequence.
- Definition of divergent sequence.
- Definition of constant sequence.
- Theorem 1.
- L' Hopital's rule.
- Theorem 2 (Properties of limits of sequences).
- Theorem 3 (Absolute value).

## Definition of sequences

A sequence is a function whose domain is the set of positive integers. It is denoted by $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$ (entire seq) and $\{a_n\} = a_1, a_2, a_3, \ldots, a_n$ (finite seq).

**Example:** Find the first four terms and $n$th term of each:

(a) $\{\frac{n}{n+1}\}$ (b) $\{2 + (0.1)^n\}$ (c) $\{(-1)^{n+1}\frac{n^2}{3n-1}\}$

(d) $\{4\}$ (e) $a_1 = 3$ and $a_{k+1} = 2a_k$ for $k \geqslant 1$.

### Definition of convergent sequence (c′gt)

A sequence is $\{a_n\}$ has a limit $L$, or converges to $L$ denoted by either $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ .

### Definition of divergent sequence (d′gt)

A sequence $\{a_n\}$ is called if

- $\lim_{n\to\infty} a_n$ does not exist.
- $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$.

### Definition of constant sequence

A $\{a_n\}$ is constant if $a_n = c$ for every $n$, $c \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$.

### Theorem 1

Let $\{a_n\}$ be a sequence and $f$ be a function such that

- $f(n) = a_n$
- $f(x)$ exists for every real number $x \geqslant 1$

then

1. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$
2. If $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then $\lim_{n\to\infty} f(n) = \infty$ (or $-\infty$).

**Examples:**

(1) If $a_n = 1 + (\frac{1}{n})$, determine whether $\{a_n\}$ converges or diverges.

(2) Determine whether $\{a_n\}$ converges or diverges

(a) $\{\frac{1}{4}n^2 - 1\}$ (b) $\{(-1)^{n-1}\}$

### L' Hopital's rule

It is a method for computing a limit of form $\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)}$ if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)} = \dfrac{\infty}{\infty}$, then we can use L' Hopital's rule which is defined as $\lim\limits_{n\to\infty}\dfrac{f'(n)}{g'(n)}$.

### Theorem 2 (properties)

Let $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = L \pm K$.
- $\lim\limits_{n\to\infty}(a_n.b_n) = L.K$.
- $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{L}{K},\ K \neq 0$.
- $\lim\limits_{n\to\infty} Ca_n = CL$.

### Theorem 3 (Absolute value)

For a seq $\{a_n\}$, $\lim_{n\to\infty} |a_n| = 0 \Leftrightarrow \lim_{n\to\infty} a_n = 0$.

### Theorem 4 (Geometric seq)

- $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$
- $\lim_{n\to\infty} r^n = \infty$ if $|r| > 1$

**Example:** Determine whether the following sequences converge or diverge

(1) $\{\frac{5n}{e^{2n}}\}$, (2) $\{(\frac{-2}{3})^n\}$ (3) $\{(1.01)^n\}$ (4) $\{\frac{2n^2}{5n^2-3}\}$
(5) $\{6(\frac{-5}{6})^n\}$ (6) $\{8-(\frac{7}{8})^n\}$ (7) $\{1000-n\}$ (8) $\{\frac{4n^4+1}{2n^2-1}\}$
(9) $\{\frac{e^n}{4}\}$.

## Theorem 5 (Sandwich)

If $a_n$, $b_n$ and $c_n$ are sequences such that

- $a_n \leqslant b_n \leqslant c_n$ for every n
- $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} b_n$, then $\lim\limits_{n\to\infty} c_n = L$.

## Theorem 6

A bounded, monotonic sequence has limit.

## Notations

1. $-1 \leqslant \sin(\theta) \leqslant 1$
2. $-1 \leqslant \cos(\theta) \leqslant 1$
3. $0 \leqslant \cos^2(\theta) \leqslant 1$
4. $-\frac{\pi}{2} \leqslant \tan^{-1}(\theta) \leqslant \frac{\pi}{2}$
5. $\cos(\pi n) = (-1)^n$

# Examples

▷ Determine whether the following sequences converge or diverge, if they converge find its limits.

(1) $\{\frac{\ln n}{n}\}$ (2) $\{\frac{\tan^{-1} n}{n}\}$ (3) $\{e^{-n}\ln n\}$ (4) $\{\frac{\cos^2 n}{3^n}\}$

(5) $\{(-1)^{n+1}\frac{1}{n}\}$ (6) $\{\frac{\cos n}{n}\}$ (7) $\{\frac{n^2}{2n-1} - \frac{n^2}{2n+1}\}$

(8) $\{(1+\frac{1}{n})^2\}$ (9) $\{n^{1/n}\}$

**Solution**:

## Definition of infinite series

series are the sum of the terms of an infinite sequence. It is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \dots \tag{1}$$

## Partial sums of series in (1)

First partial sum: $S_1 = a_1$
Second partial sum: $S_2 = a_1 + a_2$
Third partial sum: $S_3 = a_1 + a_2 + a_3$
$\vdots$
$n$th partial sum: $S_n = a_1 + a_2 + \cdots + a_n$
Seq of partial sum: $S_n = S_1 + S_2 + \cdots + S_n + \cdots = \{S_n\}$

If this seq $\{S_n\}$ is convergent, let say equal to $s$, if $\lim\limits_{n\to\infty} S_n$ exists, then the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

# Examples

Find (a) $S_1, S_2, S_3$ and $S_n$ (b) the sum of the series, if it converges

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}$ (2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

**Solution**:

## Definition of Harmonic series

the harmonic series is $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$

## Definition of Geometric series

a series of the type $\sum_{n=0}^{\infty} ar^n$, where $a$ and $r$ are real numbers, with $a \neq 0$.

## Theorem 1

The geometric series $\sum_{n=0}^{\infty} ar^n$

- convergent if $|r| < 1$ and its $S = \frac{a}{1-r}$
- divergent if $|r| > 1$

# Examples

Discuss the convergence of the following series

(1): $0.6 + 0.06 + 0.006 + \cdots + \frac{6}{(10)^n} + \ldots$
(2): $0.628 + 0.000628 + \cdots + \frac{628}{(1000)^n} + \ldots$
(3): $2 + \frac{2}{3} + \frac{2}{3^2} \cdots + \frac{2}{3^{n-1}} + \ldots$
**Solution**: