Lecture Notes in Actuarial Mathematics

A Probability Course for the Actuaries
A Preparation for Exam P/1

Marcel B. Finan

May 2018 Syllabus

In memory of my parents

August 1, 2008
January 7, 2009

Preface

The present manuscript is designed mainly to help students prepare for the Probability Exam (known as Exam P/1), the first actuarial examination administered by the Society of Actuaries. This examination tests a student's knowledge of the fundamental probability tools for quantitatively assessing risk. A thorough command of calculus is assumed.
More information about the exam can be found on the webpage of the Society of Actuaries www.soa.org.
Problems taken from previous exams provided by the Society of Actuaries will be indicated by the symbol ‡.
This manuscript can be used for personal use or class use, but not for commercial purposes. If you find any errors, I would appreciate hearing from you: mfinan@atu.edu
This manuscript is also suitable for a one semester course in an undergraduate course in probability theory. Answer keys to text problems are found at the end of the book.

Marcel B. Finan
Russellville, AR
May, 2017

Contents

Preface **i**

Set Theory Prerequisite **5**
1 Some Basic Definitions 6
2 Set Operations 16

Counting and Combinatorics **31**
3 The Fundamental Principle of Counting 31
4 Permutations 38
5 Combinations 44

Probability: Definitions and Properties **55**
6 Sample Space, Events, Probability Measure 55
7 Probability of Intersection, Union, and Complementary Event . . 67
8 Probability and Counting Techniques 78

Conditional Probability and Independence **87**
9 Conditional Probabilities 87
10 Posterior Probabilities: Bayes' Formula 97
11 Independent Events 110
12 Odds and Conditional Probability 121

Discrete Random Variables **125**
13 Random Variables 125
14 Probability Mass Function and Cumulative Distribution Function 133
15 Expected Value of a Discrete Random Variable 143
16 Expected Value of a Function of a Discrete Random Variable . . 153
17 Variance and Standard Deviation of a Discrete Random Variable 162

Commonly Used Discrete Random Variables **171**
18 Uniform Discrete Random Variable 171
19 Bernoulli Trials and Binomial Distributions 177
20 The Expected Value and Variance of the Binomial Distribution . 185
21 Poisson Random Variable 194
22 Geometric Random Variable 207
23 Negative Binomial Random Variable 216
24 Hyper-geometric Random Variable 224

Cumulative and Survival Distribution Functions **233**
25 The Cumulative Distribution Function 233
26 The Survival Distribution Function 248

Calculus Prerequisite I **255**
27 Improper Integrals 255

Continuous Random Variables **267**
28 Distribution Functions 267
29 Expectation and Variance 282
30 Median, Mode, and Percentiles 296
31 The Continuous Uniform Distribution Function 304
32 Normal Random Variables 312
33 The Normal Approximation to the Binomial Distribution 323
34 Exponential Random Variables 328
35 Gamma Distribution 338
36 The Distribution of a Function of a Continuous Random Variable 345

Calculus Prerequisite II **353**
37 Graphing Systems of Inequalities in Two Variables 353
38 Iterated Double Integrals 359

Joint Distributions **369**
39 Jointly Distributed Random Variables 369
40 Independent Random Variables 385
41 Sum of Two Independent Random Variables: Discrete Case . . . 398
42 Sum of Two Independent Random Variables: Continuous Case . 405
43 Conditional Distributions: Discrete Case 415
44 Conditional Distributions: Continuous Case 425

45 Joint Probability Distributions of Functions of Random Variables 435

Properties of Expectation **443**
46 Expected Value of a Function of Two Random Variables 443
47 Covariance and Variance of Sums 456
48 The Coefficient of Correlation 464
49 Conditional Expectation 471
50 Double Expectation 480
51 Conditional Variance 490

Moment Generating Functions and the Central Limit Theorem **499**
52 Moment Generating Functions 499
53 Moment Generating Functions of Sums of Independent RVs . . . 507
54 The Central Limit Theorem 517

Sample Exam 1 **525**

Sample Exam 2 **539**

Sample Exam 3 **555**

Sample Exam 4 **569**

Sample Exam 5 **583**

Sample Exam 6 **597**

Sample Exam 7 **611**

Sample Exam 8 **625**

Sample Exam 9 **639**

Sample Exam 10 **653**

Answer Keys **669**

Bibliography **757**

Index **759**

Set Theory Prerequisite

Two approaches of the concept of probability will be introduced later in the book: The classical probability and the experimental probability. The former approach is developed using the foundation of set theory, and a quick review of the theory is in order. Readers familiar with the basics of set theory such as set builder notation, Venn diagrams, and the basic operations on sets, (unions, intersections, and complements) can skip this chapter. However, going through the content of this chapter will provide a good start on what we will need right away from set theory.
Set is the most basic term in mathematics. Some synonyms of a set are *class* or *collection*. In this chapter, we introduce the concept of a set and its various operations and then study the properties of these operations.

Throughout this book, we assume that the reader is familiar with the following number systems and the algebraic operations and properties of such systems:

- The set of all positive integers

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

- The set of all integers

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

- The set of all rational numbers

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \; with \; b \neq 0\}.$$

- The set $\mathbb{R}$ of all real numbers.

1 Some Basic Definitions

We define a **set** as a collection of *well-defined* objects (called **elements** or **members** of A) such that for any given object one can assert without dispute that either the object is in the set or not but not both. Sets are usually will be represented by upper case letters. When an object x belongs to a set A, we write $x \in A$, otherwise, we use the notation $x \notin A$. Also, we mention here that the members of a set can be sets themselves.

Example 1.1
Which of the following is a set.
(a) The collection of good movies.
(b) The collection of right-handed individuals in Dallas. Exclude mixed-handedness.

Solution.
(a) Answering a question about whether a movie is good or not may be subject to dispute, the collection of good movies is not a well-defined set.
(b) This collection is a well-defined set since a person is either left-handed or right-handed ■

Next, we introduce a couple of set representations. The first one is to list, without repetition, the elements of the set. For example, if A is the solution set to the equation $x^2 - 4 = 0$ then $A = \{-2, 2\}$. The other way to represent a set is to describe a property that characterizes the elements of the set. This is known as the **set-builder** representation of a set. For example, the set A above can be written as $A = \{x | x \text{ is an integer satisfying } x^2 - 4 = 0\}$.
We define the **empty** set, denoted by $\emptyset$, to be the set with no elements. A set which is not empty is called a **non-empty** set.

Example 1.2
List the elements of the following sets.
(a) $\{x | x \text{ is a real number such that } x^2 = 1\}$.
(b) $\{x | x \text{ is an integer such that } x^2 - 3 = 0\}$.

Solution.
(a) $\{-1, 1\}$.
(b) Since the only solutions to the given equation are $-\sqrt{3}$ and $\sqrt{3}$ and both are not integers, the set in question is the empty set ■

Example 1.3
Use a property characterizing the members of the following sets.
(a) $\{a, e, i, o, u\}$.
(b) $\{1, 3, 5, 7, 9\}$.

Solution.
(a) $\{x|x \text{ is a vowel}\}$.
(b) $\{n|n \in \mathbb{N} \text{ is odd and less than 10 }\}$ ■

The first arithmetic operation involving sets that we consider is the equality of two sets. Two sets A and B are said to be **equal** if and only if they contain the same elements. We write $A = B$. For non-equal sets we write $A \neq B$. In this case, the two sets do not contain the same elements.

Example 1.4
Determine whether each of the following pairs of sets are equal.
(a) $\{1, 3, 5\}$ and $\{5, 3, 1\}$.
(b) $\{\{1\}\}$ and $\{1, \{1\}\}$.

Solution.
(a) Since the order of listing elements in a set is irrelevant, $\{1, 3, 5\} = \{5, 3, 1\}$.
(b) Since one of the sets has exactly one member and the other has two, $\{\{1\}\} \neq \{1, \{1\}\}$ ■

In set theory, the number of elements in a set has a special name. It is called the **cardinality** of the set. We write $\#(A)$ to denote the cardinality of the set A. If A has a finite cardinality we say that A is a **finite** set. Otherwise, it is called **infinite**. For example, $\mathbb{N}$ is an infinite set.
Can two infinite sets have the same cardinality? The answer is yes. If A and B are two sets (finite or infinite) and there is a **bijection** from A to B(i.e., a **one-to-one**[1] and **onto**[2] function) then the two sets are said to have the **same cardinality** and we write $\#(A) = \#(B)$.
If $\#(A)$ is either finite or has the same cardinality as $\mathbb{N}$ then we say that A is **countable**. A set that is not countable is said to be **uncountable**.

[1]A function $f : A \longmapsto B$ is a **one-to-one** function if $f(m) = f(n)$ implies $m = n$, where $m, n \in A$.

[2]A function $f : A \longmapsto B$ is an **onto** function if for every $b \in B$, there is an $a \in A$ such that $b = f(a)$.

Example 1.5
What is the cardinality of each of the following sets?
(a) $\emptyset$.
(b) $\{\emptyset\}$.
(c) $A = \{a, \{a\}, \{a, \{a\}\}\}$.

Solution.
(a) $\#(\emptyset) = 0$.
(b) This is a set consisting of one element $\emptyset$. Thus, $\#(\{\emptyset\}) = 1$.
(c) $\#(A) = 3$ ■

Example 1.6
(a) Show that the set $A = \{a_1, a_2. \cdots, a_n, \cdots\}$ is countable.
(b) Let A be the set of all infinite sequences of the digits 0 and 1. Show that A is uncountable.

Solution.
(a) We first show that the map $f : \mathbb{N} \longmapsto A$ defined by $f(n) = a_n$ is one-to-one. Indeed, if $f(n) = f(m)$ then $a_n = a_m$ and this implies $n = m$. Now, any member in A is of the form $a_n = f(n)$. Thus, A is onto. It follows that A is countable.
(b) We will argue by contradiction. Suppose that A is countable with elements $a_1, a_2, \cdots$. where each a_i is an infinite sequence of the digits 0 and 1. Let a be the infinite sequence with the first digit of 0 or 1 different from the first digit of a_1, the second digit of 0 or 1 different from the second digit of $a_2, \cdots$, the n^{th} digit is different from the n^{th} digit of a_n, etc. Thus, a is an infinite sequence of the digits 0 and 1 which is not in A, a contradiction. Hence, A is uncountable ■

Now, let A and B be two sets. We say that A is a **subset** of B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B. If there exists an element of A which is not in B, then we write $A \nsubseteq B$. A **universal set** U is the collection of all objects in a particular context or theory. All other sets in that framework constitute subsets of the universal set.
For any set A we have $\emptyset \subseteq A \subseteq A$. That is, every set has at least two subsets. Also, keep in mind that the empty set is a subset of any set.

Example 1.7
Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, and $C = \{4, 6\}$. Determine which of these sets are subsets of which other of these sets.

Solution.
Clearly, $B \subseteq A$ and $C \subseteq A$ ■

Subsets of a universal set can be represented by circles or closed curves with a rectangle that represents the universal set. We refer to such pictorial representation as a **Venn diagram.**

Example 1.8
Represent $A \subseteq B \subseteq C$ using Venn diagram.

Solution.
The Venn diagram is given in Figure 1.1 ■

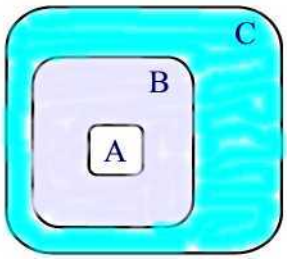


Figure 1.1

Next, let A and B be two sets. We say that A is a **proper** subset of B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$. Thus, to show that A is a proper subset of B, we must show that every element of A is an element of B and there is an element of B which is not in A.

Example 1.9
Order the sets of numbers: $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{N}$ using $\subset$

Solution.
We have, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ ■

Example 1.10
Determine whether each of the following statements is true or false.
(a) $x \in \{x\}$ (b) $\{x\} \subseteq \{x\}$ (c) $\{x\} \in \{x\}$
(d) $\{x\} \in \{\{x\}\}$ (e) $\emptyset \subseteq \{x\}$ (f) $\emptyset \in \{x\}$

Solution.
(a) True (b) True (c) False since $\{x\}$ is a set consisting of a single element x and so $\{x\}$ is not a member of this set (d) True (e) True (f) False since $\{x\}$ does not have $\emptyset$ as a listed member ■

Now, the collection of all subsets of a set A is of importance. We denote this set by $\mathcal{P}(\mathrm{A})$ and we call it the **power set** of A.

Example 1.11
Find the power set of $A = \{a, b, c\}$.

Solution.

$$\mathcal{P}(\mathrm{A}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$ ■

We conclude this section, by introducing the concept of **mathematical induction**: We want to prove that some statement $P(n)$ is true for any non-negative integer $n \geq n_0$. The steps of mathematical induction are as follows:

(i) (Basis of induction) Show that $P(n_0)$ is true.
(ii) (Induction hypothesis) Assume $P(n_0), P(n_0 + 1), \cdots, P(n)$ are true.
(iii) (Induction step) Show that $P(n + 1)$ is true.

Example 1.12
(a) Use induction to show that if $\#(A) = n$ then $\#(\mathcal{P}(\mathrm{A})) = 2^n$, where $n \geq 0$ and $n \in \mathbb{N}$.
(b) If $\mathcal{P}(\mathrm{A})$ has 256 elements, how many elements are there in A?

Solution.
(a) We apply induction to prove the claim. If $n = 0$ then $A = \emptyset$ and in this case $\mathcal{P}(\mathrm{A}) = \{\emptyset\}$. Thus, $\#(\mathcal{P}(\mathrm{A})) = 1 = 2^0$. As induction hypothesis, suppose that if $\#(A) = k$, where $k = n_0, n_0 + 1, \cdots, n$, then $n(\mathcal{P}(\mathrm{A})) = 2^k$. Let $B = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$. Then $\mathcal{P}(\mathrm{B})$ consists of all subsets of $\{a_1, a_2, \cdots, a_n\}$ together with all subsets of $\{a_1, a_2, \cdots, a_n\}$ with the element a_{n+1} added to them. Hence, $\#(\mathcal{P}(\mathrm{B})) = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$.
(b) Since $\#(\mathcal{P}(\mathrm{A})) = 256 = 2^8$, by (a) we have $\#(A) = 8$ ■

Example 1.13
Use induction to show that $\displaystyle\sum_{i=1}^{n}(2i - 1) = n^2,\ n \in \mathbb{N}$.

Solution.

If $n = 1$ we have $1^2 = 2(1) - 1 = \sum_{i=1}^{1}(2i-1)$. Suppose that the result is true for $k = 1, 2, \cdots, n$. We will show that it is true for $n+1$. Indeed, $\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 = n^2 + 2n + 2 - 1 = (n+1)^2$ ■

Practice Problems

Problem 1.1
Consider the experiment of rolling a die. List the elements of the set $A = \{x | x$ shows a face with prime number$\}$. Recall that a **prime** number is a positive integer greater than 1 with only two different divisors: 1 and the number itself.

Problem 1.2
Consider the random experiment of tossing a coin three times.
(a) Let S be the collection of all outcomes of this experiment. List the elements of S. Use H for head and T for tail.
(b) Let E be the subset of S with more than one tail. List the elements of E.
(c) Suppose $F = \{THH, HTH, HHT, HHH\}$. Write F in set-builder notation.

Problem 1.3
Consider the experiment of tossing a coin three times. Let E be the collection of outcomes with at least one head and F the collection of outcomes of more than one head. Compare the two sets E and F.

Problem 1.4
A hand of 5 cards is dealt from a deck of 52 cards. Let E be the event that the hand contains 5 aces. List the elements of E.

Problem 1.5
Prove the following properties:
(a) Reflexive Property: $A \subseteq A$.
(b) Antisymmetric Property: If $A \subseteq B$ and $B \subseteq A$ then $A = B$.
(c) Transitive Property: If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Problem 1.6
Prove by using mathematical induction that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}.$$

Problem 1.7
Prove by using mathematical induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \in \mathbb{N}.$$

Problem 1.8
Use induction to show that $(1+x)^n \geq 1 + nx$ for all non-negative integer n, where $x > -1$.

Problem 1.9
Use induction to show that

$$1 + a + a^2 + \cdots + a^{n-1} = \frac{1 - a^n}{1 - a}, \ a \neq 1.$$

Problem 1.10
Subway prepared 60 4-inch sandwiches for a birthday party. Among these sandwiches, 45 of them had tomatoes, 30 had both tomatoes and onions, and 5 had neither tomatoes nor onions. Using a Venn diagram, how many sandwiches did he make with
(a) tomatoes or onions?
(b) onions?
(c) onions but not tomatoes?

Problem 1.11
A camp of international students has 110 students. Among these students,

75	speak English,
52	speak Spanish,
50	speak French,
33	speak English and Spanish,
30	speak English and French,
22	speak Spanish and French,
13	speak all three languages.

How many students speak
(a) English and Spanish, but not French,
(b) neither English, Spanish, nor French,
(c) French, but neither English nor Spanish,
(d) English, but not Spanish,
(e) only one of the three languages,
(f) exactly two of the three languages.

Problem 1.12
An experiment consists of the following two stages:
(1) a fair coin is tossed
(2) if the coin shows a head, then a fair die is rolled; otherwise, the coin is flipped again.
An outcome of this experiment is a pair of the form (outcome from stage 1, outcome from stage 2). Let S be the collection of all outcomes. List the elements of S and then find the cardinality of S.

Problem 1.13
Show that the function $f : \mathbb{R} \longmapsto \mathbb{R}$ defined by $f(x) = 3x + 5$ is one-to-one and onto.

Problem 1.14
Find $\#(A)$ if $\#(\mathcal{P}(A)) = 32$.

Problem 1.15
Consider the function $f : \mathbb{N} \longmapsto \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

(a) Show that $f(n) = f(m)$ only if n and m have the same parity, i.e., either both are even or both are odd..
(b) Show that $\mathbb{Z}$ is countable.

Problem 1.16
Let A be a non-empty set and $f : A \longmapsto \mathcal{P}(A)$ be any function. Let $B = \{a \in A | a \notin f(a)\}$. Clearly, $B \in \mathcal{P}(A)$. Show that there is no $b \in A$ such that $f(b) = B$. Hence, there is no onto map from A to $\mathcal{P}(A)$.

Problem 1.17
Use the previous problem to show that $\mathcal{P}(\mathbb{N})$ is uncountable.

Problem 1.18
Show that the sets $(0, \infty)$ and $\mathbb{R}$ have the same cardinality.

Problem 1.19
Show that the function $f : \mathbb{N} \times \mathbb{N} \longmapsto \mathbb{N}$ defined by $f(n, m) = 2^n 3^m$ is one-to-one.

Problem 1.20
A marketing survey indicates that 60% of the population owns a laptop, 30% owns a desktop computer, and 20% owns both a laptop and a desktop computers. What percent of the population owns a laptop or a desktop, but not both?

2 Set Operations

In this section we introduce various operations on sets and study the properties of these operations.

Complements
If U is a given set whose subsets are under consideration, then we call U a **universal set.** Let U be a universal set and A, B be two subsets of U. The **absolute complement** of A (See Figure 2.1(I)) is the set

$$A^c = \{x \in U | x \notin A\}.$$

Example 2.1
Find the complement of $A = \{1, 2, 3\}$ if $U = \{1, 2, 3, 4, 5, 6\}$.

Solution.
From the definition, $A^c = \{4, 5, 6\}$ ■

The **relative complement** of A with respect to B (See Figure 2.1(II)) is the set

$$B - A = \{x \in U | x \in B \text{ and } x \notin A\}.$$

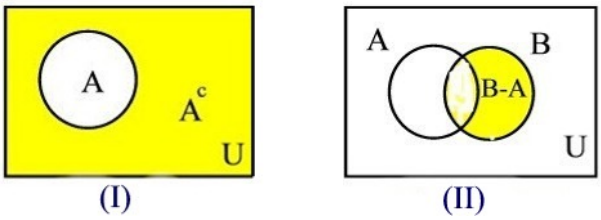


Figure 2.1

Example 2.2
Let $A = \{1, 2, 3\}$ and $B = \{\{1, 2\}, 3\}$. Find $A - B$.

Solution.
The elements of A that are not in B are 1 and 2. Thus, $A - B = \{1, 2\}$ ■

Union and Intersection
In the remaining of this book, all sets are assumed to be subsets of a universal set. Given two sets A and B. The **union** of A and B is the set

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

where the 'or' is inclusive.(See Figure 2.2(a))

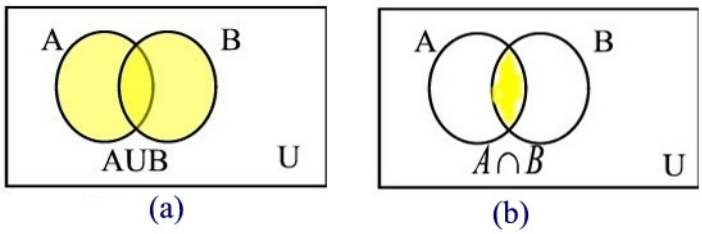


Figure 2.2

The above definition can be extended to more than two sets. More precisely, if $A_1, A_2, \cdots$, are sets then

$$\bigcup_{n=1}^{\infty} A_n = \{x | x \in A_i \text{ for some } i \in \mathbb{N}\}.$$

The **intersection** of A and B is the set (See Figure 2.2(b))

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

If $A \cap B = \emptyset$ we say that A and B are **disjoint** sets.

Example 2.3
Express each of the following events in terms of the events A, B, and C as well as the operations of absolute complement, union and intersection:
(a) at least one of the events A, B, C occurs;
(b) at most one of the events A, B, C occurs;
(c) none of the events A, B, C occurs;
(d) all three events A, B, C occur;
(e) exactly one of the events A, B, C occurs;
(f) events A and B occur, but not C;
(g) either event A occurs or, if not, then B also does not occur.
In each case draw the corresponding Venn diagram.

Solution.
The venn diagrams are shown in Figure 2.3.
(a) $A \cup B \cup C$
(b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
(c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
(d) $A \cap B \cap C$
(e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
(f) $A \cap B \cap C^c$
(g) $A \cup (A^c \cap B^c)$ ■

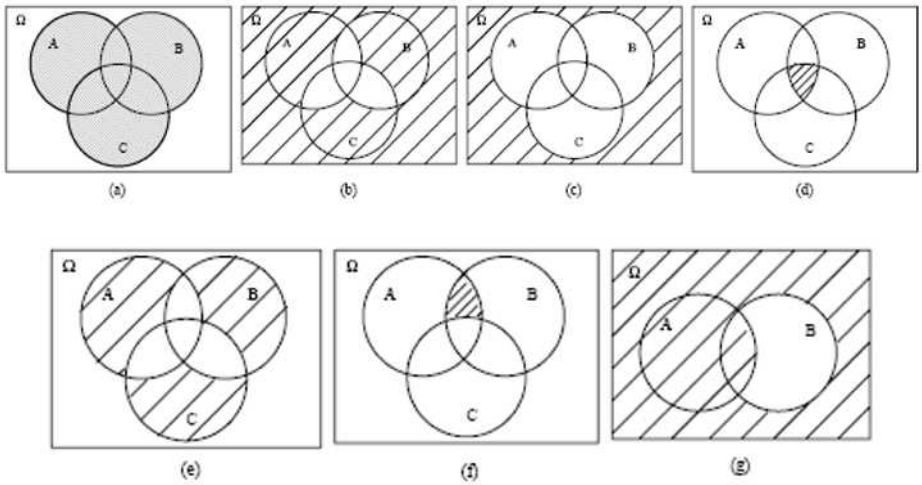


Figure 2.3

Example 2.4
Translate the following set-theoretic notation into event language. For example, "$A \cup B$" means "A or B occurs".
(a) $A \cap B$
(b) $A - B$
(c) $A \cup B - A \cap B$
(d) $A - (B \cup C)$
(e) $A \subset B$
(f) $A \cap B = \emptyset$

Solution.
(a) A and B occur

(b) A occurs and B does not occur
(c) A or B, but not both, occurs
(d) A occurs, and B and C do not occur
(e) if A occurs, then B occurs but if B occurs then A need not occur.
(f) if A occurs, then B does not occur or if B occurs then A does not occur ■

Example 2.5
Find a simpler expression of $[(A \cup B) \cap (A \cup C) \cap (B^c \cap C^c)]$ assuming all three sets A, B, C intersect.

Solution.
Using the Venn diagram in Figure 2.4, one can easily see that $[(A \cup B) \cap (A \cup C) \cap (B^c \cap C^c)] = A - [A \cap (B \cup C)] = A - B \cup C$ ■

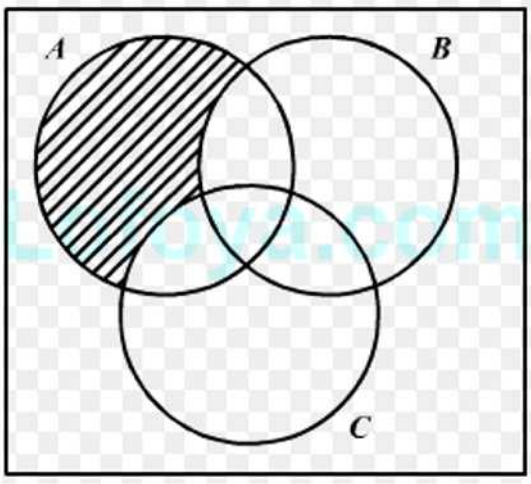


Figure 2.4

Example 2.6
Let A and B be two non-empty sets. Write A as the union of two disjoint sets.

Solution.
Using a Venn diagram one can easily see that $A \cap B$ and $A \cap B^c$ are disjoint sets such that $A = (A \cap B) \cup (A \cap B^c)$ ■

Example 2.7
In a junior league tennis tournament, teams play 20 games. Let A denote

the event that Team Blazers wins 15 or more games in the tournament. Let B be the event that the Blazers win less than 10 games and C be the event that they win between 8 to 16 games. The Blazers can win at most 20 games. Using words, what do the following events represent?
(a) $A \cup B$ and $A \cap B$.
(b) $A \cup C$ and $A \cap C$.
(c) $B \cup C$ and $B \cap C$.
(d) $A^c,\ B^c$, and C^c.

Solution.
(a) $A \cup B$ is the event that the Blazers win 15 or more games or win 9 or less games. $A \cap B$ is the empty set, since the Blazers cannot win 15 or more games and have less than 10 wins at the same time. Therefore, event A and event B are disjoint.
(b) $A \cup C$ is the event that the Blazers win at least 8 games. $A \cap C$ is the event that the Blazers win 15 or 16 games.
(c) $B \cup C$ is the event that the Blazers win at most 16 games. $B \cap C$ is the event that the Blazers win 8 or 9 games.
(d) A^c is the event that the Blazers win 14 or fewer games. B^c is the event that the Blazers win 10 or more games. C^c is the event that the Blazers win fewer than 8 or more than 16 games ■

The concept of intersection can be extended to a countable number of sets. Given the sets $A_1, A_2, \cdots$, we define

$$\bigcap_{n=1}^{\infty} A_n = \{x | x \in A_i \ for\ all\ i \in \mathbb{N}\}.$$

Example 2.8

For each positive integer n, we define $A_n = \{n\}$. Find $\bigcap_{n=1}^{\infty} A_n$.

Solution.
Clearly, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ ■

Theorem 2.1
Let A, B and C be subsets of U. We have
(a) Commutative law: $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
(b) Associative law: $A \cap (B \cap C) = (A \cap B) \cap C$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof.
(a) We have: $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B \Leftrightarrow x \in B$ and $x \in A \Leftrightarrow x \in B \cap A$. Similar proof holds for the union where the "and" is replaced by "or".
(b) We have: $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in B \cap C \Leftrightarrow x \in A$ and $(x \in B$ and $x \in C) \Leftrightarrow (x \in A$ and $x \in B)$ and $x \in C \Leftrightarrow x \in (A \cap B) \cap C$. Similar proof holds for the union ■

The following theorem establishes the distributive laws of sets.

Theorem 2.2
If A, B, and C are subsets of U then
(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.
See Problem 2.15 ■

Remark 2.1
Note that since $\cap$ and $\cup$ are commutative operations, we have $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

The following theorem presents the relationships between $(A \cup B)^c, (A \cap B)^c, A^c$ and B^c.

Theorem 2.3 *(De Morgan's Laws)*
Let A and B be subsets of U. We have
(a) $(A \cup B)^c = A^c \cap B^c$.
(b) $(A \cap B)^c = A^c \cup B^c$.

Proof.
We prove part (a) leaving part(b) as an exercise for the reader.
(a) Let $x \in (A \cup B)^c$. Then $x \in U$ and $x \notin A \cup B$. Hence, $x \in U$ and $(x \notin A$ and $x \notin B)$. This implies that $(x \in U$ and $x \notin A)$ and $(x \in U$ and $x \notin B)$. It follows that $x \in A^c \cap B^c$.
Conversely, let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. Hence, $x \notin A$ and $x \notin B$ which implies that $x \notin (A \cup B)$. Hence, $x \in (A \cup B)^c$ ■

Remark 2.2
Using mathematical induction, De Morgan's laws are valid for any countable number of sets. That is

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

and

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

Example 2.9
An assisted living agency advertises its program through videos and booklets. Let U be the set of people solicited for the agency program. All participants were given a chance to watch a video and to read a booklet describing the program. Let V be the set of people who watched the video, B the set of people who read the booklet, and C the set of people who decided to enroll in the program.
(a) Describe with set notation: "The set of people who did not see the video or read the booklet but who still enrolled in the program"
(b) Rewrite your answer using De Morgan's law and and then restate the above.

Solution.
(a) $(V \cup B)^c \cap C$.
(b) $(V \cup B)^c \cap C = V^c \cap B^c \cap C =$ the set of people who did not watch the video, did not read the booklet, but did enroll ■

If $A_i \cap A_j = \emptyset$ for all $i \neq j$ then we say that the sets in the collection $\{A_n\}_{n=1}^{\infty}$ are **pairwise disjoint.**

Example 2.10
Find three sets A, B, and C that are not pairwise disjoint but $A \cap B \cap C = \emptyset$.

Solution.
One example is $A = B = \{1\}$ and $C = \emptyset$ ■

Example 2.11

Find sets $A_1, A_2, \cdots$ that are pairwise disjoint and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Solution.
For each positive integer n, let $A_n = \{n\}$ ■

Example 2.12
Throw a pair of fair dice. Let A be the event the total is 5, B the event the total is even, and C the event the total is divisible by 9. Show that A, B, and C are pairwise disjoint.

Solution.
We have

$$\begin{aligned} A =&\{(1,4),(2,3),(3,2),(4,1)\} \\ B =&\{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6)(3,1),(3,3),(3,5),(4,2), \\ &(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\} \\ C =&\{(3,6),(4,5),(5,4),(6,3)\}. \end{aligned}$$

Clearly, $A \cap B = A \cap C = B \cap C = \emptyset$ ■

Next, we establish the following rule of counting.

Theorem 2.4 *(Inclusion-Exclusion Principle)*
Suppose A and B are finite sets. Then
(a) $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$.
(b) If $A \cap B = \emptyset$, then $\#(A \cup B) = \#(A) + \#(B)$.
(c) If $A \subseteq B$, then $\#(A) \leq \#(B)$.

Proof.
(a) Indeed, $\#(A)$ gives the number of elements in A including those that are common to A and B. The same holds for $\#(B)$. Hence, $\#(A) + \#(B)$ includes twice the number of common elements. Therefore, to get an accurate count of the elements of $A \cup B$, it is necessary to subtract $\#(A \cap B)$ from $\#(A) + \#(B)$. This establishes the result.
(b) If A and B are disjoint then $\#(A \cap B) = 0$ and by (a) we have $\#(A \cup B) = \#(A) + \#(B)$.
(c) If A is a subset of B then the number of elements of A cannot exceed the number of elements of B. That is, $\#(A) \leq \#(B)$ ■

Example 2.13
The State Department interviewed 35 candidates for a diplomatic post in Algeria; 25 speak arabic, 28 speak french, and 2 speak neither languages. How many speak both languages?

Solution.
Let F be the group of applicants that speak french, A those who speak arabic. Then $F \cap A$ consists if those who speak both languages. By the Inclusion-Exclusion Principle, we have $\#(F \cup A) = \#(F) + \#(A) - \#(F \cap A)$. That is, $33 = 28 + 25 - \#(F \cap A)$. Solving for $\#(F \cap A)$ we find $\#(F \cap A) = 20$ ■

Cartesian Product
The notation (a, b) is known as an **ordered pair** of elements and is defined by $(a, b) = \{\{a\}, \{a, b\}\}$.
The **Cartesian product** of two sets A and B is the set

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

The idea can be extended to products of any number of sets. Given n sets $A_1, A_2, \cdots, A_n$ the Cartesian product of these sets is the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\}$$

where, we define

$$(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n), \ n \geq 2.$$

Example 2.14
Show that $(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \cdots, n$.

Solution.
The proof is by induction on $n \geq 2$. For the basis of induction, we have $(a_1, a_2) = (b_1, b_2)$ if and only if $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$ and this is equivalent to $a_1 = b_1$ and $\{a_1, a_2\} = \{b_1, b_2\}$. Thus, $a_1 = b_1$ and $a_2 = b_2$. For the induction hypothesis, suppose that the result holds for $k = 2, 3, \cdots, n$. For the induction step, we must show that $(a_1, a_2, \cdots, a_{n+1}) = (b_1, b_2, \cdots, b_{n+1})$ is equivalent to $a_i = b_i$ for $i = 1, 2, \cdots, n + 1$. Indeed,

$$\begin{aligned}(a_1, a_2, \cdots, a_{n+1}) =& (b_1, b_2, \cdots, b_{n+1}) \\ \Leftrightarrow& ((a_1, a_2, \cdots, a_n), a_{n+1}) = ((b_1, b_2, \cdots, b_n), b_{n+1}) \\ \Leftrightarrow& (a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n), \ a_{n+1} = b_{n+1} \\ \Leftrightarrow& a_i = b_i, \ i = 1, 2, \cdots, n + 1 \ ■\end{aligned}$$

Example 2.15
Consider the experiment of tossing a fair coin n times. Represent the sample space as a Cartesian product.

Solution.
If S is the sample space then $S = S_1 \times S_2 \times \cdots \times S_n$, where $S_i,\ 1 \leq i \leq n$, is the set consisting of the two outcomes H=head and T = tail ■

The following theorem is a tool for finding the cardinality of the Cartesian product of two finite sets.

Theorem 2.5
Given two finite sets A and B. Then

$$\#(A \times B) = \#(A) \cdot \#(B).$$

Proof.
Suppose that $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_m\}$. Then

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_m), \\ &(a_2, b_1), (a_2, b_2), \cdots, (a_2, b_m), \\ &(a_3, b_1), (a_3, b_2), \cdots, (a_3, b_m), \\ &\vdots \\ &(a_n, b_1), (a_n, b_2), \cdots, (a_n, b_m)\} \end{aligned}$$

Thus, $\#(A \times B) = n \cdot m = \#(A) \cdot \#(B)$ ■

Remark 2.3
By induction, the previous result can be extended to any finite number of sets. See Problem 2.18.

Example 2.16
What is the total number of outcomes of tossing a fair coin n times.

Solution.
If S is the sample space then $S = S_1 \times S_2 \times \cdots \times S_n$ where $S_i,\ 1 \leq i \leq n$, is the set consisting of the two outcomes H=head and T = tail. By the previous theorem, $\#(S) = 2^n$ ■

Practice Problems

Problem 2.1
Let A and B be any two sets. Use Venn diagrams to show that $B = (A \cap B) \cup (A^c \cap B)$ and $A \cup B = A \cup (A^c \cap B)$.

Problem 2.2
Show that if $A \subseteq B$ then $B = A \cup (A^c \cap B)$. Thus, B can be written as the union of two disjoint sets.

Problem 2.3 ‡
A survey of a group's viewing habits over the last year revealed the following information

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Represent the statement "the group that watched none of the three sports during the last year" using operations on sets.

Problem 2.4
An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. For $i = 1, 2$, let R_i denote the event that a red ball is drawn from urn i and B_i the event that a blue ball is drawn from urn i. Show that the sets $R_1 \cap R_2$ and $B_1 \cap B_2$ are disjoint.

Problem 2.5 ‡
An auto insurance has 10,000 policyholders. Each policyholder is classified as

(i) young or old;
(ii) male or female;
(iii) married or single.

Of these policyholders, 3,000 are young, 4,600 are male, and 7,000 are married. The policyholders can also be classified as 1,320 young males, 3,010 married males, and 1,400 young married persons. Finally, 600 of the policyholders are young married males.
How many of the company's policyholders are young, female, and single?

Problem 2.6 ‡
A marketing survey indicates that 60% of the population owns an automobile, 30% owns a house, and 20% owns both an automobile and a house. What percentage of the population owns an automobile or a house, but not both?

Problem 2.7 ‡
35% of visits to a primary care physicians (PCP) office results in neither lab work nor referral to a specialist. Of those coming to a PCPs office, 30% are referred to specialists and 40% require lab work.
What percentage of visit to a PCPs office results in both lab work and referral to a specialist?

Problem 2.8
In a universe U of 100, let A and B be subsets of U such that $\#(A \cup B) = 70$ and $\#(A \cup B^c) = 90$. Determine $\#(A)$.

Problem 2.9 ‡
An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

Problem 2.10
Show that if A, B, and C are subsets of a universe U then

$$\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C).$$

Problem 2.11
In a survey on popsicle flavor preferences of kids aged 3-5, it was found that
• 22 like strawberry.
• 25 like blueberry.
• 39 like grape.
• 9 like blueberry and strawberry.
• 17 like strawberry and grape.
• 20 like blueberry and grape.
• 6 like all flavors.
• 4 like none.

How many kids were surveyed?

Problem 2.12
Let A, B, and C be three subsets of a universe U with the following properties: $\#(A) = 63, \#(B) = 91, \#(C) = 44, \#(A \cap B) = 25, \#(A \cap C) = 23, \#(C \cap B) = 21, \#(A \cup B \cup C) = 139$. Find $\#(A \cap B \cap C)$.

Problem 2.13
Fifty students living in a college dormitory were registering for classes for the fall semester. The following were observed:
• 30 registered in a math class,
• 18 registered in a history class,
• 26 registered in a computer class,
• 9 registered in both math and history classes,
• 16 registered in both math and computer classes,
• 8 registered in both history and computer classes,
• 47 registered in at least one of the three classes.
(a) How many students did not register in any of these classes ?
(b) How many students registered in all three classes?

Problem 2.14 ‡
A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.
(ii) 22% have low blood pressure.
(iii) 15% have an irregular heartbeat.
(iv) Of those with an irregular heartbeat, one-third have high blood pressure.
(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

Problem 2.15
Prove: If A, B, and C are subsets of U then
(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 2.16
Translate the following verbal description of events into set theoretic notation. For example, "A or B occurs, but not both" corresponds to the set $A \cup B - A \cap B$.
(a) A occurs whenever B occurs.
(b) If A occurs, then B does not occur.
(c) Exactly one of the events A and B occurs.
(d) Neither A nor B occur.

Problem 2.17 ‡
A survey of 100 TV watchers revealed that over the last year:
i) 34 watched CBS.
ii) 15 watched NBC.
iii) 10 watched ABC.
iv) 7 watched CBS and NBC.
v) 6 watched CBS and ABC.
vi) 5 watched NBC and ABC.
vii) 4 watched CBS, NBC, and ABC.
viii) 18 watched HGTV and of these, none watched CBS, NBC, or ABC.
Calculate how many of the 100 TV watchers did not watch any of the four channels (CBS, NBC, ABC or HGTV).

Problem 2.18
Let $S_1, S_2, \cdots, S_n$ be non-empty sets. Show that the function $f : S_1 \times S_2 \times \cdots \times S_n \longrightarrow (S_1 \times S_2 \times \cdots \times S_{n-1}) \times S_n$ defined by $f((s_1, s_2, \cdots, s_n)) = ((s_1, s_2, \cdots, s_{n-1}), s_n)$ is one-to-one and onto.

Problem 2.19
A room contains n people. Let i_k be the birthday date of person k, $k = 1, 2, \cdots, n$. Let S be the collection of all $n-$tuples of the form $(i_1, i_2, \cdots, i_n)$. Find $\#(S)$.

Problem 2.20
An insurance company offers three types of insurance: life insurance, auto insurance, and home insurance. Of all the customers, 55% have life insurance, 60% have auto insurance, 30% have home insurance, 25% have both life insurance and auto insurance, 15% have life insurance and home insurance, and 15% have auto insurance and home insurance. What percentage of customers have all three types of insurance?

Counting and Combinatorics

The major goal of this chapter is to establish several (combinatorial) techniques for counting large finite sets without actually listing their elements. These techniques provide effective methods for counting the size of events, an important concept in probability theory.

3 The Fundamental Principle of Counting

Sometimes one encounters the question of listing all the outcomes of a certain experiment. One way for doing that is by constructing a so-called **tree diagram.**

Example 3.1
Create a tree diagram that lists all the sequences of heads and tails obtained by tossing a coin three times.

Solution.
The tree diagrams along its branches is shown in Figure 3.1 ■

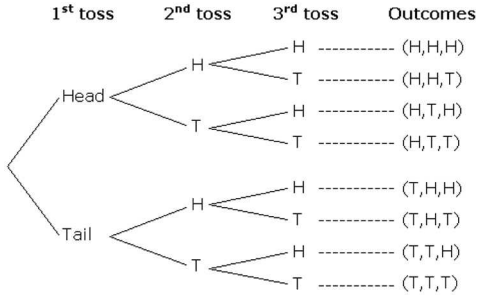


Figure 3.1

Example 3.2
List all two-digit numbers that can be constructed from the digits 1,2, and 3.

Solution.
The tree diagram is shown in Figure 3.2.

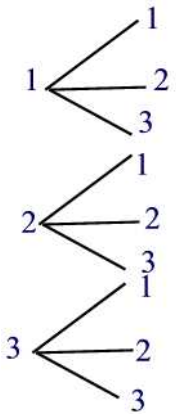


Figure 3.2

The different numbers are $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ ■

Of course, trees are manageable as long as the number of outcomes is not large. If there are many stages to an experiment and several possibilities at each stage, the tree diagram associated with the experiment would become too large to be manageable. For such problems the counting of the outcomes is simplified by means of algebraic formulas. The commonly used formula is the **Fundamental Principle of Counting**, also known as the **multiplication rule of counting**, which states:

Theorem 3.1
If a choice consists of k steps, of which the first can be made in n_1 ways, for each of these the second can be made in n_2 ways,$\cdots$, and for each of these the k^{th} can be made in n_k ways, then the whole choice can be made in $n_1 \cdot n_2 \cdot \cdots n_k$ ways.

Proof.
In set-theoretic term, we let S_i denote the set of outcomes for the i^{th} step, $i = 1, 2, \cdots, k$. Then $\#(S_i) = n_i$. The set of outcomes for the entire job is the

Cartesian product $S_1 \times S_2 \times \cdots \times S_k = \{(s_1, s_2, \cdots, s_k) : s_i \in S_i, 1 \leq i \leq k\}$. Thus, we just need to show that

$$\#(S_1 \times S_2 \times \cdots \times S_k) = \#(S_1) \cdot \#(S_2) \cdots \#(S_k).$$

The proof is by induction on $k \geq 2$.

Basis of Induction
By Theorem 2.4, we have $\#(S_1 \times S_2) = \#(S_1) \times \#(S_2))$, Thus, the property is true for $n = 2$.
Induction Hypothesis
Suppose

$$\#(S_1 \times S_2 \times \cdots \times S_k) = \#(S_1) \cdot \#(S_2) \cdots \#(S_k)$$

for $k = 2, 3, \cdots, n$.
Induction Conclusion
We must show

$$\#(S_1 \times S_2 \times \cdots \times S_{n+1}) = \#(S_1) \cdot \#(S_2) \cdots \#(S_{n+1}).$$

To see this, note that there is a one-to-one correspondence between the sets $S_1 \times S_2 \times \cdots \times S_{n+1}$ and $(S_1 \times S_2 \times \cdots S_n) \times S_{n+1}$ given by $f(s_1, s_2, \cdots, s_n, s_{n+1}) = ((s_1, s_2, \cdots, s_n), s_{n+1})$. See Problem 2.18. Thus, $\#(S_1 \times S_2 \times \cdots \times S_{n+1}) = \#((S_1 \times S_2 \times \cdots S_n) \times S_{n+1}) = \#(S_1 \times S_2 \times \cdots S_n)\#(S_{n+1})$ (by Theorem 2.5). Now, applying the induction hypothesis gives

$$\#(S_1 \times S_2 \times \cdots S_n \times S_{n+1}) = \#(S_1) \cdot \#(S_2) \cdots \#(S_{n+1}) \blacksquare$$

Example 3.3
The following three factors were considered in the study of the effectiveness of a certain cancer treatment:

(i) Medicine $(A_1, A_2, A_3, A_4, A_5)$
(ii) Dosage Level (Low, Medium, High)
(iii) Dosage Frequency (1,2,3,4 times/day)

Find the number of ways that a cancer patient can be given the medecine?

Solution.
The choice here consists of three stages, that is, $k = 3$. The first stage, can be made in $n_1 = 5$ different ways, the second in $n_2 = 3$ different ways, and the third in $n_3 = 4$ ways. Hence, the number of possible ways a cancer patient can be given medecine is $n_1 \cdot n_2 \cdot n_3 = 5 \cdot 3 \cdot 4 = 60$ different ways $\blacksquare$

Example 3.4
How many license-plates with 3 letters followed by 3 digits exist? Repetition of either letter or digit is permitted.

Solution.
A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third letter, (4) choose the first digit, (5) choose the second digit, and (6) choose the third digit. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ ways ■

Example 3.5
How many numbers in the range 1000 - 9999 have no repeated digits?

Solution.
A 4-step process: (1) Choose first digit, (2) choose second digit, (3) choose third digit, (4) choose fourth digit. Every step can be done in a number of ways that does not depend on previous choices, and each number can be specified in this manner. So there are $9 \cdot 9 \cdot 8 \cdot 7 = 4,536$ ways ■

Example 3.6
How many license plates with 3 letters followed by 3 digits exist if exactly one of the digits is 1? Repetition of either letter or digit is permitted.

Solution.
In this case, we must pick a place for the 1 digit, and then the remaining digit places must be populated from the digits $\{0, 2, \cdots 9\}$. A 6-step process: (1) Choose the first letter, (2) choose the second letter, (3) choose the third letter, (4) choose which of three positions the 1 goes, (5) choose the first of the other digits, and (6) choose the second of the other digits. Every step can be done in a number of ways that does not depend on previous choices, and each license plate can be specified in this manner. So there are $26 \cdot 26 \cdot 26 \cdot 3 \cdot 9 \cdot 9 = 4,270,968$ ways ■

Practice Problems

Problem 3.1
If each of the 10 digits 0-9 is chosen at random, how many ways can you choose the following numbers?
(a) A two-digit code number, repeated digits permitted.
(b) A three-digit identification card number, for which the first digit cannot be a 0. Repeated digits permitted.
(c) A four-digit bicycle lock number, where no digit can be used twice.
(d) A five-digit zip code number, with the first digit not zero. Repeated digits permitted.

Problem 3.2
(a) If eight cars are entered in a race and three finishing places are considered, how many finishing orders can they finish? Assume no ties.
(b) If the top three cars are Buick, Honda, and BMW, in how many possible orders can they finish?

Problem 3.3
You are taking 2 shirts(white and red) and 3 pairs of pants (black, blue, and gray) on a trip. How many different choices of outfits do you have?

Problem 3.4
A Poker club has 10 members. A president and a vice-president are to be selected. In how many ways can this be done if everyone is eligible?

Problem 3.5
In a medical study, patients are classified according to whether they have regular (RHB) or irregular heartbeat (IHB) and also according to whether their blood pressure is low (L), normal (N), or high (H). Use a tree diagram to represent the various outcomes that can occur.

Problem 3.6
If a travel agency offers special weekend trips to 12 different cities, by air, rail, bus, or sea, in how many different ways can such a trip be arranged?

Problem 3.7
If twenty different types of wine are entered in wine-tasting competition, in how many different ways can the judges award a first prize and a second prize?

Problem 3.8
In how many ways can the 24 members of a faculty senate of a college choose a president, a vice-president, a secretary, and a treasurer?

Problem 3.9
Find the number of ways in which four of ten new novels can be ranked first, second, third, and fourth according to their figure sales for the first three months.

Problem 3.10
How many ways are there to seat 8 people, consisting of 4 couples, in a row of seats (8 seats wide) if all couples are to get adjacent seats?

Problem 3.11
Consider strings of length 4 that can be formed using the letters A,B,C,D and F. How many strings that do not start with the letter B can be formed if repetitions are not allowed?

Problem 3.12
Let $A = \{a_1, a_2, \cdots, a_n\}$. Use the Fundamental Principle of Counting to show that $\#\mathcal{P}(A) = 2^n$.

Problem 3.13
Let A and B be two sets with $\#(A) = n$ and $\#(B) = m$. How many functions are there from A to B?

Problem 3.14
Let A and B be two sets with $\#(A) = n$, $\#(B) = m$ and $m \geq n$. How many one-to-one functions are there from A to B?

Problem 3.15
A university has 20,000 students registered. If each student has three initials; is it true that there must be at least two students with the same initials?

Problem 3.16
How many 4-digit odd numbers are there?

Problem 3.17
How many 4-letter words begin with AZ?

Problem 3.18
There are five pants $P_1, \cdots, P_5$ you are considering taking on your vacation. You can take them all; you can take none of them; you can take all except P_1 etc. How many possibilities are there?

Problem 3.19
Find the number of 4-letter words with at least one repeated letter.

Problem 3.20
There are 50 states and 2 U.S. Senators for each state. How many committees can be formed consisting of one Senator from each of the 50 states?

4 Permutations

Consider the following problem: In how many ways can 8 horses finish in a race (assuming there are no ties)? We can look at this problem as a decision consisting of 8 steps. The first step is the possibility of a horse to finish first in the race, the second step is the possibility of a horse to finish second, $\cdots$, the 8^{th} step is the possibility of a horse to finish 8^{th} in the race. Thus, by the Fundamental Principle of Counting there are

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320 \text{ ways.}$$

This problem exhibits an example of an ordered arrangement, that is, the order the objects are arranged is important. Such an ordered arrangement is called a **permutation**. Products such as $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written in a shorthand notation called factorial. That is, $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$ (read "8 factorial"). In general, we define **n factorial** by

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1, \ n \geq 1$$

where n is a whole number. By convention we define

$$0! = 1.$$

Example 4.1
Evaluate the following expressions: (a) 6! (b) $\frac{10!}{7!}$.

Solution.
(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
(b) $\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$ ■

Using factorials and the Fundamental Principle of Counting, we see that the number of permutations of n objects is $n!$.

Example 4.2
There are 5! permutations of the 5 letters of the word "rehab." In how many of them is h the second letter?

Solution.
Then there are 4 ways to fill the first spot. The second spot is filled by the letter h. There are 3 ways to fill the third, 2 to fill the fourth, and one way to fill the fifth. There are 4! such permutations ■

Example 4.3
Five different books are on a shelf. In how many different ways could you arrange them?

Solution.
The five books can be arranged in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways ■

Counting Permutations
We next consider the permutations of a set of *distinct* objects taken from a larger set. Suppose we have n items. How many ordered arrangements of k items can we form from these n items? The number of permutations is denoted by ${}_nP_k$. The n refers to the number of different items and the k refers to the number of them appearing in each arrangement. A formula for ${}_nP_k$ is given next.

Theorem 4.1
For any non-negative integer n and $0 \leq k \leq n$, we have

$$ {}_nP_k = \frac{n!}{(n-k)!}. $$

Proof.
We can treat a permutation as a decision with k steps. The first step can be made in n different ways, the second in $n-1$ different ways, ..., the k^{th} in $n-k+1$ different ways. Thus, by the Fundamental Principle of Counting there are $n(n-1)\cdots(n-k+1)$ $k-$permutations of n objects. That is, ${}_nP_k = n(n-1)\cdots(n-k+1) = \frac{n(n-1)\cdots(n-k+1)(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$ ■

Example 4.4
How many license plates are there that start with three letters followed by 4 digits (no repetitions)?

Solution.
The decision consists of two steps. The first is to select the letters and this can be done in ${}_{26}P_3$ ways. The second step is to select the digits and this can be done in ${}_{10}P_4$ ways. Thus, by the Fundamental Principle of Counting there are ${}_{26}P_3 \cdot {}_{10}P_4 = 78,624,000$ license plates ■

Example 4.5
How many five-digit zip codes can be made where all digits are different? The possible digits are the numbers 0 through 9.

Solution.
The answer is ${}_{10}P_5 = \frac{10!}{(10-5)!} = 30,240$ zip codes ■

Practice Problems

Problem 4.1
Find m and n so that ${}_mP_n = \frac{9!}{6!}$.

Problem 4.2
How many four-letter code words can be formed using a standard 26-letter alphabet
(a) if repetition is allowed?
(b) if repetition is not allowed?

Problem 4.3
Certain automobile license plates consist of a sequence of three letters followed by three digits.
(a) If letters can not be repeated but digits can, how many possible license plates are there?
(b) If no letters and no digits are repeated, how many license plates are possible?

Problem 4.4
A permutation lock has 40 numbers on it.
(a) How many different three-number permutation lock can be made if the numbers can be repeated?
(b) How many different permutation locks are there if the three numbers are different?

Problem 4.5
(a) 12 cabinet officials are to be seated in a row for a picture. How many different seating arrangements are there?
(b) Seven of the cabinet members are women and 5 are men. In how many different ways can the 7 women be seated together on the left, and then the 5 men together on the right?

Problem 4.6
Using the digits 1, 3, 5, 7, and 9, with no repetitions of the digits, how many
(a) one-digit numbers can be made?
(b) two-digit numbers can be made?
(c) three-digit numbers can be made?
(d) four-digit numbers can be made?

Problem 4.7
There are five members of the Math Club. In how many ways can the positions of a president, a secretary, and a treasurer, be chosen?

Problem 4.8
Find the number of ways of choosing three initials from the alphabet if none of the letters can be repeated. Name initials such as MBF and BMF are considered different.

Problem 4.9
(a) How many three-letter words can be made using the English alphabet.
(b) How many three-letter words can be made using the English alphabet where no two letters are the same.
(c) How many three-letter words have at least two letters the same?

Problem 4.10
Find n that satisfies the equation $3\,({}_{n-1}P_{n-2}) = n!$.

Problem 4.11
How many 3-digit odd numbers greater than 600 can be created using the digits 2,3,4,5,6, and 7? Repetition of digits is allowed.

Problem 4.12
How many 3-digit odd numbers greater than 600 can be created using the digits 2,3,4,5,6, and 7? Repetition of digits is not allowed.

Problem 4.13
Find the number of permutations of r objects chosen from n objects, if repetition is allowed.

Problem 4.14
Solve: ${}_{n-1}P_{n-3} = \frac{n!}{10}$.

Problem 4.15
Show that ${}_{n-1}P_r + r \cdot {}_{n-1}P_{r-1} = {}_nP_r$.

Problem 4.16
Show that $\frac{{}_nP_r}{{}_nP_{r-1}} = n - r + 1$.

Problem 4.17
How many distinct values can be represented with 5 decimal digits?

Problem 4.18
How many permutations of the letters ABCDEFGH contain the string ABC ?

Problem 4.19
There are 3 men and 3 women to be seated in a row of 10 chairs. In how many different ways can they be seated if one men must be seated at each end of the row?

Problem 4.20
How many two digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if repetition is allowed?

5 Combinations

In a permutation the order of the set of objects or people is taken into account. However, there are many problems in which we want to know the number of ways in which k objects can be selected from n distinct objects in arbitrary order. For example, when selecting a two-person committee from a club of 10 members, the order in the committee is irrelevant. That is, choosing Mr. A and Ms. B in a committee is the same as choosing Ms. B and Mr. A.
A combination is defined as a possible selection of a certain number of objects taken from a group without regard to order. More precisely, the number of $k-$element subsets of an $n-$element set is called the number of **combinations of n objects taken k at a time**. It is denoted by ${}_nC_k$ and is read "n choose k". The formula for ${}_nC_k$ is given next.

Theorem 5.1
If ${}_nC_k$ denotes the number of ways in which k objects can be selected from a set of n distinct objects then

$$ {}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!}. $$

Proof.
Since the number of groups of k elements out of n elements is ${}_nC_k$ and each group can be arranged in $k!$ ways, we have ${}_nP_k = k!{}_nC_k$. It follows that

$$ {}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!} \blacksquare $$

An alternative notation for ${}_nC_k$ is $\begin{pmatrix} n \\ k \end{pmatrix}$. We define ${}_nC_k = 0$ if $k < 0$ or $k > n$.

Example 5.1
A jury consisting of 2 women and 3 men is to be selected from a group of 5 women and 7 men. In how many different ways can this be done? Suppose that either Steve or Harry must be selected but not both, then in how many ways this jury can be formed?

Solution.
By the Fundamental Principle of Counting, there are ${}_5C_2 \cdot {}_7C_3 = 350$ possible jury combinations consisting of 2 women and 3 men. Now, if a committee selected must include either Steve or Harry but not both then by the Fundamental Principle of Counting the number of jury groups that do not include the two men at the same time is ${}_5C_{25}C_{22}C_1 = 200$ ■

The next theorem discusses some of the properties of combinations.

Theorem 5.2
Suppose that n and k are whole numbers with $0 \leq k \leq n$. Then
(a) ${}_nC_0 =_n C_n = 1$ and ${}_nC_1 =_n C_{n-1} = n$.
(b) Symmetry property: ${}_nC_k =_n C_{n-k}$.
(c) Pascal's identity: ${}_{n+1}C_k =_n C_{k-1} +_n C_k$.

Proof.
(a) From the formula of ${}_nC_k$ we have ${}_nC_0 = \frac{n!}{0!(n-0)!} = 1$ and ${}_nC_n = \frac{n!}{n!(n-n)!} = 1$. Similarly, ${}_nC_1 = \frac{n!}{1!(n-1)!} = n$ and ${}_nC_{n-1} = \frac{n!}{(n-1)!} = n$.
(b) Indeed, we have ${}_nC_{n-k} = \frac{n!}{(n-k)!(n-n+k)!} = \frac{n!}{k!(n-k)!} =_n C_k$.
(c) We have

$$\begin{aligned}
{}_nC_{k-1} +_n C_k =& \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
=& \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
=& \frac{n!}{k!(n-k+1)!}(k+n-k+1) \\
=& \frac{(n+1)!}{k!(n+1-k)!} =_{n+1} C_k \blacksquare
\end{aligned}$$

Example 5.2
The Russellville School District has six members. In how many ways
(a) can all six members line up for a picture?
(b) can they choose a president and a secretary?
(c) can they choose three members to attend a state conference with no regard to order?

Solution.
(a) ${}_6P_6 = 6! = 720$ different ways
(b) ${}_6P_2 = 30$ ways
(c) ${}_6C_3 = 20$ different ways ■

Pascal's identity allows one to construct the so-called Pascal's triangle (for $n = 10$) as shown in Figure 5.1.

											n
				1							0
			1		1						1
		1		2		1					2
	1		3		3		1				3
1		4		6		4		1			4
	1		5		10		10		5	1	5
		1		6		15		20		15	6	1	6
	1		7		21		35		35		21		7	1	7
1		8		28		56		70		56		28		8	1	8
	1		9		36		84		126		126		84		36		9	1	9
1		10		45		120		210		252		210		120		45		10		1	10

Figure 5.1

As an application of combination we have the following theorem which provides an expansion of $(x + y)^n$, where n is a non-negative integer.

Theorem 5.3 *(Binomial Theorem)*
Let x and y be variables, and let n be a non-negative integer. Then

$$(x + y)^n = \sum_{k=0}^{n} {}_nC_k x^{n-k} y^k$$

where ${}_nC_k$ will be called the **binomial coefficient**.

Proof.
The proof is by induction on n.

Basis of induction: For $n = 0$, we have

$$(x + y)^0 = \sum_{k=0}^{0} {}_0C_k x^{0-k} y^k = {}_0C_0 x^{0-0} y^0 = 1.$$

Induction hypothesis: Suppose that the theorem is true for $k = 0, 1, \cdots, n$.
Induction step: Let us show that it is still true for $n+1$. That is,

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} {}_{n+1}C_k x^{n-k+1} y^k.$$

Indeed, we have

$$\begin{aligned}
(x+y)^{n+1} =& (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n \\
=& x \sum_{k=0}^{n} {}_nC_k x^{n-k} y^k + y \sum_{k=0}^{n} {}_nC_k x^{n-k} y^k \\
=& \sum_{k=0}^{n} {}_nC_k x^{n-k+1} y^k + \sum_{k=0}^{n} {}_nC_k x^{n-k} y^{k+1} \\
=& [{}_nC_0 x^{n+1} + {}_nC_1 x^n y + {}_nC_2 x^{n-1} y^2 + \cdots + {}_nC_n x y^n] \\
&+ [{}_nC_0 x^n y + {}_nC_1 x^{n-1} y^2 + \cdots + {}_nC_{n-1} x y^n + {}_nC_n y^{n+1}] \\
=& {}_{n+1}C_0 x^{n+1} + [{}_nC_1 + {}_nC_0] x^n y + \cdots + \\
& [{}_nC_n + {}_nC_{n-1}] x y^n + {}_{n+1}C_{n+1} y^{n+1} \\
=& {}_{n+1}C_0 x^{n+1} + {}_{n+1}C_1 x^n y + {}_{n+1}C_2 x^{n-1} y^2 + \cdots \\
&+ {}_{n+1}C_n x y^n + {}_{n+1}C_{n+1} y^{n+1} \\
=& \sum_{k=0}^{n+1} {}_{n+1}C_k x^{n-k+1} y^k \ \blacksquare
\end{aligned}$$

Note that the coefficients in the expansion of $(x+y)^n$ are the entries of the $(n+1)^{\text{st}}$ row of Pascal's triangle.

Example 5.3
Expand $(x+y)^6$ using the Binomial Theorem.

Solution.
By the Binomial Theorem and Pascal's triangle we have

$$(x+y)^6 = x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6xy^5 + y^6 \ \blacksquare$$

Example 5.4
How many subsets are there of a set with n elements?

Solution.
Since there are ${}_nC_k$ subsets of k elements with $0 \leq k \leq n$, the total number of subsets of a set of n elements is

$$\sum_{k=0}^{n} {}_nC_k = (1+1)^n = 2^n \blacksquare$$

Combination with Repetition
A combination of n objects taken r at a time with repetition is a selection of r objects from the n objects, where each object can be selected more than once. This is the same as the selection of r objects from n different categories of objects with repetition allowed. We will refer to such a combination as an $r-$combination. How many such combinations are possible? The answer is provided by the following theorem.

Theorem 5.4
The number of $r-$combinations with repetition allowed that can be selected from the n objects is ${}_{r+n-1}C_r$.

Proof.
Let $A = \{a_1, a_2, \cdots, a_n\}$. An $r-$combination can be pictorially represented by $n-1$ dividers and r crosses. The $n-1$ dividers create n different cells to be filled with the crosses. To elaborate, suppose $n = 4$ and $r = 5$. Then the $5-$combination

$$\times \times || \times | \times \times$$

is the combination $a_1a_1a_3a_4a_4$. Thus, any list of $n-1$ bars and r crosses represents an $r-$combination. Hence, an $r-$combination corresponds to a choice of the r positions to place the r crosses from the $r+n-1$ positions that contains r crosses and $n-1$ bars. The number of such $r-$combinations is ${}_{r+n-1}C_r$. Alternatively, an $r-$combination corresponds to a choice of the $n-1$ bars to place in the $r+n-1$ positions. The number of such choices is ${}_{r+n-1}C_{n-1}$. One can easily check that ${}_{r+n-1}C_r = {}_{r+n-1}C_{n-1}$ $\blacksquare$

Example 5.5
Consider a cash box that contains 5 bills of each of the following types: \$1, \$5, \$10, \$20, \$50, and \$100. How many ways are there to select five bills from the cash box?

Solution.
The number of possible combination is

$$_{10}C_5 = \frac{10!}{5!5!} = 252 \blacksquare$$

Permutations with Indistinguishable objects
Suppose we have n objects, where n_1 of them are identical, n_2 of them are identical,$\cdots$, and n_r of them are identical. Thus,

$$n_1 + n_2 + \cdots + n_r = n.$$

We want to find the number of permutations of the n objects.

Theorem 5.5
The number of permutations of n objects where, n_1 of them are identical, n_2 of them are identical,$\cdots$, and n_r of them are identical, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}.$$

Proof.
The n_1 objects can be placed in $_nC_{n_1}$ ways. Once, these objects are placed, there are $n - n_1$ positions left. Then there are $_{n-n_1}C_{n_2}$ ways to place the n_2 objects, leaving $n-n_1-n_2$ positions free. Continuing this process of placing the objects, the n_r objects can be placed in $_{n-n_1-\cdots-n_{r-1}}C_{n_r}$ ways. By the Fundamental Principle of counting, the total number of different permutation is

$$\begin{aligned} {}_nC_{n_1} \times_{n-n_1} C_{n_2} \times \cdots \times_{n-n_1-\cdots-n_{r-1}} C_{n_r} =& \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \\ & \times \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{n_r!0!} \\ =& \frac{n!}{n_1!n_2!\cdots n_r!} \blacksquare \end{aligned}$$

Example 5.6
How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

Solution.
We have $n_1 = 1$ for the letter M, $n_2 = n_3 = 4$ for the letters I and S, and $n_4 = 2$ for the letter P. Thus, the number of different strings is

$$\frac{11!}{1!4!4!2!} = 34,650 \blacksquare$$

Practice Problems

Problem 5.1
Find m and n so that ${}_mC_n = 13$.

Problem 5.2
A club with 42 members has to select three representatives for a regional meeting. How many possible choices are there?

Problem 5.3
In a UN ceremony, 25 diplomats were introduced to each other. Suppose that the diplomats shook hands with each other exactly once. How many handshakes took place?

Problem 5.4
There are five members of the math club. In how many ways can the two-person Social Committee be chosen?

Problem 5.5
A medical research group plans to select 2 volunteers out of 8 for a drug experiment. In how many ways can they choose the 2 volunteers?

Problem 5.6
A consumer group has 30 members. In how many ways can the group choose 3 members to attend a national meeting?

Problem 5.7
Which is usually greater the number of combinations of a set of objects or the number of permutations?

Problem 5.8
Determine whether each problem requires a combination or a permutation:
(a) There are 10 toppings available for your ice cream and you are allowed to choose only three. How many possible 3-topping combinations can you have?
(b) Fifteen students participated in a spelling bee competition. The first place winner will receive $1,000, the second place $500, and the third place $250. In how many ways can the 3 winners be drawn?

Problem 5.9
Use the binomial theorem and Pascal's triangle to find the expansion of $(a+b)^7$.

Problem 5.10
Find the 5^{th} term in the expansion of $(2a-3b)^7$.

Problem 5.11 ‡
Thirty items are arranged in a 6-by-5 array as shown.

A_1	A_2	A_3	A_4	A_5
A_6	A_7	A_8	A_9	A_{10}
A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
A_{16}	A_{17}	A_{18}	A_{19}	A_{20}
A_{21}	A_{22}	A_{23}	A_{24}	A_{25}
A_{26}	A_{27}	A_{28}	A_{29}	A_{30}

Calculate the number of ways to form a set of three distinct items such that no two of the selected items are in the same row or same column.

Problem 5.12
Suppose $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Find the 7-combination corresponding to

$$\times|\times\times\times|\times\times||\times|.$$

Problem 5.13
How many solutions does the equation $x_1 + x_2 + x_3 = 100$ have, where the x_i are non-negative integers integers?

Problem 5.14
How many different strings can be made from the letters in HAPPINESS, using all the letters?

Problem 5.15
Solve: ${}_nC_4 = {}_{n-2}C_2$.

Problem 5.16
How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Problem 5.17
Find the largest values of m and n such that $24_m C_n =_{15} P_4$.

Problem 5.18
There are 10 boys and 13 girls in Mr. Bensons fourth-grade class and 12 boys and 11 girls in Mr. Johnson fourth-grade class. A picnic committee of six people is selected at random from the total group of students in both classes. How many committees consisting of three boys and three girls?

Problem 5.19
A store has 80 modems in its inventory, 30 coming from Source A and the remainder from Source B: Of the modems from Source A; 20% are defective. Of the modems from Source B; 8% are defective. How many groups of 5 modems will have exactly two defective modems?

Problem 5.20
A jar contains 4 red marbles, 3 green marbles, 2 white marbles, and 1 purple marble. You randomly grab 5 marbles. Of the groups of the selected 5 marbles, how many will have at least one white marble?

Problem 5.21 ‡
Each week, a subcommittee of four individuals is formed from among the members of a committee comprising seven individuals. Two subcommittee members are then assigned to lead the subcommittee, one as chair and the other as secretary.
Calculate the maximum number of consecutive weeks that can elapse without having the subcommittee contain four individuals who have previously served together with the same subcommittee chair.

Probability: Definitions and Properties

In this chapter, we discuss the fundamental concepts of probability at a level at which no previous exposure to the topic is assumed.
Probability has been used in many applications ranging from medicine to business and so the study of probability is considered an essential component of any mathematics curriculum.
So what is probability? Before answering this question we start with some basic definitions.

6 Sample Space, Events, Probability Measure

A **random experiment** or simply an **experiment** is a process whose **outcomes** cannot be predicted with certainty. Examples of an experiment include rolling a die, flipping a coin, and choosing a card from a deck of playing cards.
The **sample space** S of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $S = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die.
An **event** is a subset of the sample space. For example, the event of rolling an odd number with a die consists of three outcomes $\{1, 3, 5\}$.

Example 6.1
Consider the random experiment of tossing a coin three times.
(a) Find the sample space of this experiment.
(b) Find the outcomes of the event of obtaining more than one head.

Solution.
We will use T for tail and H for head.
(a) The sample space is composed of eight outcomes:

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

(b) The event of obtaining more than one head is the set

$$\{THH, HTH, HHT, HHH\}\blacksquare$$

Probability is the measure of occurrence of an event. Various probability concepts exist nowadays. A widely used probability concept is the **experimental** probability which uses the relative frequency[3] of an event and is defined as follows. Let $\#(E)$ denote the number of times in the first n repetitions of the experiment that the event E occurs. Then $P(E)$, the probability of the event E, is defined by

$$P(E) = \lim_{n\to\infty} \frac{\#(E)}{n}.$$

This result is a theorem called the law of large numbers[4]
In contrast to experimental probability, we have the theoretical or classical probability. When the outcome of an experiment has the same chance of being selected as any other outcome, as in the example of tossing a coin, the outcomes are said to be **equally likely**. The **theoretical** or **classical probability** concept applies *only* when all possible outcomes are equally likely, in which case the probability of an event E is given by the formula

$$P(E) = \frac{\text{number of outcomes favorable to event}}{\text{total number of outcomes}} = \frac{\#(E)}{\#(S)}.$$

The function P satisfies the following axioms, known as **Kolmogorov axioms**:
Axiom 1: For any event E, $0 \leq P(E) \leq 1$.

[3]The **relative frequency of an event** is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted. The relative frequency is not a theoretical quantity, but an experimental one. Thus, the term experimental probability.

[4]The relative frequency of an event converges to the probability of the event as the experiment is repeated a large number of times.

Proof. Since for any event E we have $\emptyset \subseteq E \subseteq S$, we can write $0 \leq \#(E) \leq \#(S)$ so that $0 \leq \frac{\#(E)}{\#(S)} \leq 1$. It follows that $0 \leq P(E) \leq 1$ ■

Axiom 2: $P(S) = 1$.
Proof. we have $P(S) = \frac{\#(S)}{\#(S)} = 1$ ■

Axiom 3: For any sequence of mutually exclusive events $\{E_n\}_{n\geq 1}$, that is $E_i \cap E_j = \emptyset$ for $i \neq j$, we have

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n).\text{(Countable additivity)}$$

Proof. We have $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\#\left(\bigcup_{n=1}^{\infty} E_n\right)}{\#(S)} = \frac{\sum_{n=1}^{\infty} \#(E_n)}{\#(S)} = \sum_{n=1}^{\infty} P(E_n)$ ■

Any function P that satisfies Axioms 1 - 3 will be called a **probability measure**.

Example 6.2
Consider the sample space $S = \{1, 2, 3\}$. Suppose that $P(\{1, 2\}) = 0.5$ and $P(\{2, 3\}) = 0.7$. Is P a valid probability measure? Justify your answer.

Solution.
In the affirmative, we must have $P(S) = P(\{1\}) + P(\{2\}) + P(\{3\}) = 1$. But $P(\{1, 2\}) = P(\{1\}) + P(\{2\}) = 0.5$. This implies that $0.5 + P(\{3\}) = 1$ or $P(\{3\}) = 0.5$. Similarly, $1 = P(\{2, 3\}) + P(\{1\}) = 0.7 + P(\{1\})$ and so $P(\{1\}) = 0.3$. It follows that $P(\{2\}) = 1 - P(\{1\}) - P(\{3\}) = 1 - 0.3 - 0.5 = 0.2$. We can easily verify Kolmogorov's axioms for this P. Hence, P is a valid (non-classical) probability measure ■

Example 6.3
If, for a given experiment, $O_1, O_2, O_3, \cdots$ is an infinite sequence of distinct outcomes such that

$$P(\{O_i\}) = \left(\frac{1}{2}\right)^i, \ i = 1, 2, 3, \cdots .$$

Show that P is a probability measure.

Solution.
If S is the sample space then

$$P(S) = \sum_{i=1}^{\infty} P(\{O_i\}) = \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = 1$$

where the infinite sum is the infinite geometric series

$$1 + a + a^2 + \cdots + a^n + \cdots = \frac{1}{1-a}, |a| < 1$$

with $a = \frac{1}{2}$. Thus, Axiom 2 is satisfied.
Let $E = \{O_{n_1}, O_{n_2}, \cdots, \cdots\}$ be an arbitrary event. Then $P(E) = \sum_{i=1}^{\infty} O_{n_i} \leq \sum_{i=1}^{\infty} O_i = P(S) = 1$. Thus, P satisfies Axiom 1. Finally, if $E_1, E_2, \cdots$ is a sequence of mutually exclusive events with

$$E_n = \{O_{n1}, O_{n2}, \cdots\}.$$

Then

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\{O_{nj}\}.$$

Thus,

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} P(\{O_{nj}\}) = \sum_{n=1}^{\infty} P(E_n).$$

Hence, P defines a probability function ■

A probability measure function satisfies the following results.

Theorem 6.1
If P is a probability measure function then
(i) $P(\emptyset) = 0$.
(ii) If $\{E_1, E_2, \cdots, E_n\}$ is a finite set of mutually exclusive events then

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k).$$

(iii) $P(E^c) = 1 - P(E)$, where E^c is the **complementary event**.

Proof.
(i) We have: $1 = P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset) = 1 + P(\emptyset)$. This implies that $P(\emptyset) = 0$.
(ii) If $\{E_1, E_2, \cdots, E_n\}$ is a finite set of mutually exclusive events, then by defining $E_k = \emptyset$ for $k > n$ and using Axiom 3 and (i) above, we find

$$P\left(\bigcup_{k=1}^{n} E_k\right) = P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k) = \sum_{k=1}^{n} P(E_k).$$

(iii) Since $E \cup E^c = S,\ E \cap E^c = \emptyset$, and $P(S) = 1$ we find

$$P(E^c) = 1 - P(E) \blacksquare$$

In the remaining of this book, P will stand for the classical probability.

Example 6.4
A hand of 5 cards is dealt from a deck. Let E be the event that the hand contains 5 aces. List the elements of E and find $P(E)$.

Solution.
Recall that a standard deck of 52 playing cards can be described as follows:

hearts (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
clubs (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
diamonds (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
spades (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King

Cards labeled Ace, Jack, Queen, or King are called **face cards.**
Since there are only 4 aces in the deck, event E is impossible, i.e. $E = \emptyset$ so that $P(E) = 0$ ■

Example 6.5
What is the probability of drawing an ace from a well-shuffled deck of 52 playing cards?

Solution.
Since there are four aces in a deck of 52 playing cards, the probability of getting an ace is

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \blacksquare$$

Example 6.6
What is the probability of rolling a 3 or a 4 with a fair die?

Solution.
The event of having a 3 or a 4 is the event $E = \{3, 4\}$. The probability of rolling a 3 or a 4 is $P(E) = \frac{2}{6} = \frac{1}{3}$ ■

Example 6.7 *(Birthday problem)*
In a room containing n people, calculate the probability that at least two of them have the same birthday. Assume all years are non-leap years.

Solution.
In a group of n randomly chosen people, the sample space S is the set

$$S = \{(i_1, i_2, \cdots, i_n) | 1 \leq i_k \leq 365,\ k = 1, 2, \cdots, n\}.$$

Hence, $\#(S) = 365^n$. Let E be the event that at least two people share the same birthday. Then the complementary event E^c is the event that no two people of the n people share the same birthday. Moreover,

$$P(E) = 1 - P(E^c).$$

The outcomes in E^c are permutations of n numbers chosen from 365 numbers without repetitions. Therefore

$$\#(E^c) = {}_{365}P_n = (365)(364)\cdots(365 - n + 1).$$

Hence,

$$P(E^c) = \frac{(365)(364)\cdots(365-n+1)}{(365)^n}$$

and

$$P(E) = 1 - \frac{(365)(364)\cdots(365 - n + 1)}{(365)^n}$$ ■

Remark 6.1
It is important to keep in mind that the classical definition of probability applies only to a sample space that has equally likely outcomes. Applying the definition to a space with outcomes that are not equally likely leads to incorrect conclusions. For example, the sample space for spinning the spinner in Figure 6.1 is given by $S = \{Red, Blue\}$, but the outcome Blue is more

likely to occur than is the outcome Red. Indeed, $P(Blue) = \frac{3}{4}$ whereas $P(Red) = \frac{1}{4}$ as opposed to $P(Blue) = P(Red) = \frac{1}{2}$

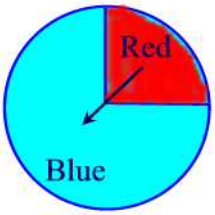


Figure 6.1

Practice Problems

Problem 6.1
Consider the random experiment of rolling a die.
(a) Find the sample space of this experiment.
(b) Find the event of rolling the die an even number.

Problem 6.2
An experiment consists of the following two stages: (1) first a coin is tossed (2) if the face appearing is a head, then a die is rolled; if the face appearing is a tail, then the coin is tossed again. An outcome of this experiment is a pair of the form (outcome from stage 1, outcome from stage 2). Let S be the collection of all outcomes.
Find the sample space of this experiment.

Problem 6.3 ‡
An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $\frac{1}{4}, \frac{1}{3}$, and , $\frac{5}{12}$ respectively.
Determine the probability that a randomly chosen employee will choose no supplementary coverage.

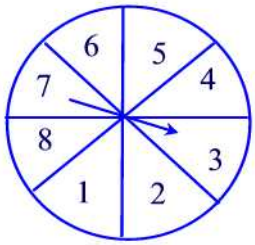
Problem 6.4
An experiment consists of throwing two dice.
(a) Write down the sample space of this experiment.
(b) If E is the event "total score is at most 10", list the outcomes belonging to E^c.
(c) Find the probability that the total score is at most 10 when the two dice are thrown.
(d) What is the probability that a double, that is,

$$\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

will not be thrown?
(e) What is the probability that a double is not thrown nor is the score greater than 10?

Problem 6.5
Let $S = \{1, 2, 3, \cdots, 10\}$. If a number is chosen at **random**, that is, with the same chance of being drawn as all other numbers in the set, calculate each of the following probabilities:
(a) The event A that an even number is drawn.
(b) The event B that a number less than 5 and greater than 9 is drawn.
(c) The event C that a number less than 11 but greater than 0 is drawn.
(d) The event D that a prime number is drawn.
(e) The event E that a number both odd and prime is drawn.

Problem 6.6
The following spinner is spun:



Find the probabilities of obtaining each of the following:
(a) P(factor of 24).
(b) P(multiple of 4).
(c) P(odd number).
(d) P($\{9\}$).
(e) P(composite number), i.e., a number greater than 1 that is not prime.
(f) P(neither prime nor composite).

Problem 6.7
A box of clothes contains 15 shirts and 10 pants. Three items are drawn from the box without replacement. What is the probability that all three are all shirts or all pants?

Problem 6.8
A coin is tossed seven times. What is the probability that the second head appears at the 7^{th} toss?

Problem 6.9
Suppose each of 40 professors in a mathematics department picks at random

one of 200 courses. What is the probability that at least two professors pick the same course?

Problem 6.10
A large classroom has 100 foreign students, 30 of whom speak Spanish. 25 of the students speak Italian, while 55 do not speak Spanish and Italian.
(a) How many of those speak both Spanish and Italian?
(b) A student who speaks Italian is chosen at random. What is the probability that he/she speaks Spanish?

Problem 6.11
A box contains 5 batteries of which 2 are defective. An inspector selects 2 batteries at random from the box. She/he tests the 2 items and observes whether the sampled items are defective.
(a) Write out the sample space of all possible outcomes of this experiment. Be very specific when identifying these.
(b) The box will not be accepted if both of the sampled items are defective. What is the probability the inspector will reject the box?

Problem 6.12
Consider the experiment of rolling two dice. How many events A are there with $P(A) = \frac{1}{3}$?

Problem 6.13 ‡
A store has 80 modems in its inventory, 30 coming from Source A and the remainder from Source B. Of the modems from Source A, 20% are defective. Of the modems from Source B, 8% are defective.
Calculate the probability that exactly two out of a random sample of five modems from the store's inventory are defective.

Problem 6.14 ‡
From 27 pieces of luggage, an airline luggage handler damages a random sample of four. The probability that exactly one of the damaged pieces of luggage is insured is twice the probability that none of the damaged pieces are insured. Calculate the probability that exactly two of the four damaged pieces are insured.

Problem 6.15
A board of trustees of a university consists of 8 men and 7 women. A committee of 3 must be selected at random and without replacement. The role

of the committee is to select a new president for the university. Calculate the probability that the number of men selected exceeds the number of women selected.

Problem 6.16 ‡
A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.
Determine the probability that a man randomly selected from this group has parents not suffering from a heart disease.

Problem 6.17
Given $P(A \cap B) = 0.18$ and $P(A \cap B^c) = 0.22$. Find $P(A)$.

Problem 6.18
A pollster surveyed 100 people about watching the TV show "The big bang theory". The results of the poll are shown in the table.

	Yes	No	Total
Male	19	41	60
Female	12	28	40
Total	31	69	100

(a) What is the probability of a randomly selected individual is a male and watching the show?
(b) What is the probability of a randomly selected individual is a male?
(c) What is the probability of a randomly selected individual watches the show?

Problem 6.19
Given $P(A) = 0.4,\ P(B) = 0.5$, and $P(A \cup B) = 0.3$. Find $P(A^c \cap B^c)$.

Problem 6.20
Ten persons are chosen at random. Find the probability that at least 2 have the same birth week.

Problem 6.21 ‡
A company issues auto insurance policies. There are 900 insured individuals. Fifty-four percent of them are male. If a female is randomly selected from the 900, the probability she is over 25 years old is 0.43. There are 395 total insured individuals over 25 years old.
A person under 25 years old is randomly selected. Calculate the probability that the person selected is male.

Problem 6.22 ‡
George and Paul play a betting game. Each chooses an integer from 1 to 20 (inclusive) at random. If the two numbers differ by more than 3, George wins the bet. Otherwise, Paul wins the bet. Calculate the probability that Paul wins the bet.

Problem 6.23 ‡
In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4.
i) 75% of the patients in the group have stage 2 or lower.
ii) 80% of the patients in the group have stage 1 or higher.
iii) 80% of the patients in the group have stage 0, 1, 3, or 4.
One patient from the group is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

7 Probability of Intersection, Union, and Complementary Event

In this section we find the probability of a complementary event, the union of two events and the intersection of two events.
We define the probability of nonoccurrence of an event E (called its **failure** or the **complementary event**) to be the number $P(E^c)$. Since $S = E \cup E^c$ and $E \cap E^c = \emptyset$, $P(S) = P(E) + P(E^c)$. Thus,

$$P(E^c) = 1 - P(E).$$

Example 7.1
The probability that a senior citizen in a nursing home without a pneumonia shot will get pneumonia is 0.45. What is the probability that a senior citizen without pneumonia shot will not get pneumonia?

Solution.
Our sample space consists of those senior citizens in the nursing home who did not get the pneumonia shot. Let E be the set of those individuals without the shot who did get the illness. Then $P(E) = 0.45$. The probability that an individual without the shot will not get the illness is then $P(E^c) = 1 - P(E) = 1 - 0.45 = 0.55$ ■

The **union** of two events A and B is the event $A \cup B$ whose outcomes are either in A or in B. The **intersection** of two events A and B is the event $A \cap B$ whose outcomes are outcomes of both events A and B. Two events A and B are said to be **mutually exclusive** if they have no outcomes in common. In this case $A \cap B = \emptyset$ and $P(A \cap B) = P(\emptyset) = 0$.

Example 7.2
Consider the sample space of rolling a die. Let A be the event of rolling a prime number, B the event of rolling a composite number, and C the event of rolling a 4. Find
(a) $A \cup B, A \cup C$, and $B \cup C$.
(b) $A \cap B, A \cap C$, and $B \cap C$.
(c) Which events are mutually exclusive?

Solution.
(a) We have

$$A \cup B = \{2, 3, 4, 5, 6\}$$
$$A \cup C = \{2, 3, 4, 5\}$$
$$B \cup C = \{4, 6\}.$$

(b)

$$A \cap B = \emptyset$$
$$A \cap C = \emptyset$$
$$B \cap C = \{4\}.$$

(c) A and B are mutually exclusive as well as A and C ■

Example 7.3
Let A be the event of drawing a "Queen" from a well-shuffled standard deck of playing cards and B the event of drawing an "ace" card. Are A and B mutually exclusive?

Solution.
Since $A = \{\text{queen of diamonds}, \text{queen of hearts}, \text{queen of clubs}, \text{queen of spades}\}$ and $B = \{\text{ace of diamonds}, \text{ace of hearts}, \text{ace of clubs}, \text{ace of spades}\}$, A and B are mutually exclusive ■

For any events A and B the probability of $A \cup B$ is given by the addition rule.

Theorem 7.1
Let A and B be two events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof.
Let $A^c \cap B$ denote the event whose outcomes are the outcomes in B that are not in A. Then using the Venn diagram in Figure 7.1 we see that $B = (A \cap B) \cup (A^c \cap B)$ and $A \cup B = A \cup (A^c \cap B)$.

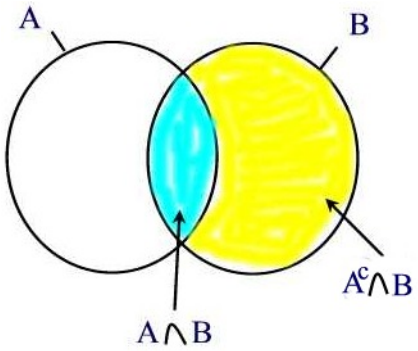


Figure 7.1

Since $(A \cap B)$ and $(A^c \cap B)$ are mutually exclusive, by Axiom 3 of Section 6, we have

$$P(B) = P(A \cap B) + P(A^c \cap B).$$

Thus,

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

Similarly, A and $A^c \cap B$ are mutually exclusive, thus we have

$$P(A \cup B) = P(A) + P(A^c \cap B) = P(A) + P(B) - P(A \cap B) \blacksquare$$

Note that in the case A and B are mutually exclusive, $P(A \cap B) = 0$ so that

$$P(A \cup B) = P(A) + P(B).$$

Example 7.4
An airport security has two checkpoints. Let A be the event that the first checkpoint is busy, and let B be the event the second checkpoint is busy. Assume that $P(A) = 0.2$, $P(B) = 0.3$ and $P(A \cap B) = 0.06$. Find the probability that neither of the two checkpoints is busy.

Solution.
The probability that neither of the checkpoints is busy is $P[(A \cup B)^c] = 1 - P(A \cup B)$. But $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0.06 = 0.44$. Hence, $P[(A \cup B)^c] = 1 - 0.44 = 0.56$ $\blacksquare$

Example 7.5
Let $P(A) = 0.9$ and $P(B) = 0.6$. Find the minimum possible value for $P(A \cap B)$.

Solution.
Since $P(A) + P(B) = 1.5$ and $0 \leq P(A \cup B) \leq 1$, by the previous theorem

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq 1.5 - 1 = 0.5.$$

So the minimum value of $P(A \cap B)$ is 0.5 ■

Example 7.6
Suppose there's 40% chance of getting a freezing rain, 10% chance of snow and freezing rain, 80% chance of snow or freezing rain. Find the chance of snow.

Solution.
By the addition rule we have

$$P(R) = P(R \cup C) - P(C) + P(R \cap C) = 0.8 - 0.4 + 0.1 = 0.5 \text{ ■}$$

Example 7.7
Let $\mathbb{N}$ be the set of all positive integers and P be a probability measure defined by $P(n) = 2\left(\frac{1}{3}\right)^n$ for all $n \in \mathbb{N}$. What is the probability that a number chosen at random from $\mathbb{N}$ will be odd?

Solution.
We have

$$\begin{aligned} P(\{1,3,5,\cdots\}) =& P(\{1\}) + P(\{3\}) + P(\{5\}) + \cdots \\ =& 2\left[\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^5 + \cdots\right] \\ =& \left(\frac{2}{3}\right)\left[1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \cdots\right] \\ =& \left(\frac{2}{3}\right) \cdot \frac{1}{1 - \left(\frac{1}{3}\right)^2} = \frac{3}{4} \text{ ■} \end{aligned}$$

Finally, if E and F are two events such that $E \subseteq F$, then F can be written as the union of two mutually exclusive events $F = E \cup (E^c \cap F)$. By Axiom 3 we obtain

$$P(F) = P(E) + P(E^c \cap F).$$

Thus, $P(F) - P(E) = P(E^c \cap F) \geq 0$ and this shows

$$E \subseteq F \Longrightarrow P(E) \leq P(F).$$

Theorem 7.2
For any three events A, B, and C we have

$$\begin{aligned}P(A\cup B\cup C) =&P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)\\ &+P(A\cap B\cap C).\end{aligned}$$

Proof.
We have

$$\begin{aligned}P(A\cup B\cup C) &= P(A)+P(B\cup C)-P(A\cap(B\cup C))\\ &= P(A)+P(B)+P(C)-P(B\cap C)\\ &- P((A\cap B)\cup(A\cap C))\\ &= P(A)+P(B)+P(C)-P(B\cap C)\\ &- [P(A\cap B)+P(A\cap C)-P((A\cap B)\cap(A\cap C))]\\ &= P(A)+P(B)+P(C)-P(B\cap C)\\ &- P(A\cap B)-P(A\cap C)+P((A\cap B)\cap(A\cap C))\\ &= P(A)+P(B)+P(C)-P(A\cap B)-\\ &P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)\ \blacksquare\end{aligned}$$

Example 7.8
If a person visits his primary care physician, suppose that the probability that he will have blood test work is 0.44, the probability that he will have an X-ray is 0.24, the probability that he will have an MRI is 0.21, the probability that he will have blood test and an X-ray is 0.08, the probability that he will have blood test and an MRI is 0.11, the probability that he will have an X-ray and an MRI is 0.07, and the probability that he will have blood test, an X-ray, and an MRI is 0.03. What is the probability that a person visiting his PCP will have at least one of these things done to him/her?

Solution.
Let B be the event that a person will have blood test, X is the event that a person will have an X-ray, and M is the event a person will have an MRI. We are given $P(B)=0.44, P(X)=0.24, P(M)=0.21, P(B\cap X)=0.08, P(B\cap M)=0.11, P(X\cap M)=0.07$ and $P(B\cap X\cap M)=0.03$. Thus,

$$P(B\cup X\cup M)=0.44+0.24+0.21-0.08-0.11-0.07+0.03=0.66\ \blacksquare$$

Practice Problems

Problem 7.1
A consumer testing service rates a given DVD player as either very good or good. Let A denote the event that the rating is very good and B the event that the rating is good. You are given: $P(A) = 0.22$, $P(B) = 0.35$. Find
(a) $P(A^c)$;
(b) $P(A \cup B)$;
(c) $P(A \cap B)$.

Problem 7.2
An entrance exam consists of two subjects: Math and english. The probability that a student fails the math test is 0.20. The probability of failing english is 0.15, and the probability of failing both subjects is 0.03. What is the probability that the student will fail at least one of these subjects?

Problem 7.3
Let A be the event of "drawing a king" from a deck of cards and B the event of "drawing a diamond". Are A and B mutually exclusive? Find $P(A \cup B)$.

Problem 7.4
An urn contains 4 red balls, 8 yellow balls, and 6 green balls. A ball is selected at random. What is the probability that the ball chosen is either red or green?

Problem 7.5
Show that for any events A and B, $P(A \cap B) \geq P(A) + P(B) - 1$.

Problem 7.6
An urn contains 2 red balls, 4 blue balls, and 5 white balls.
(a) What is the probability of the event R that a ball drawn at random is red?
(b) What is the probability of the event "not R" that is, that a ball drawn at random is not red?
(c) What is the probability of the event that a ball drawn at random is either red or blue?

Problem 7.7
In the experiment of rolling of fair pair of dice, let E denote the event of rolling a sum that is an even number and P the event of rolling a sum that is a prime number. Find the probability of rolling a sum that is even or prime?

Problem 7.8
Let S be a sample space and A and B be two events such that $P(A) = 0.8$ and $P(B) = 0.9$. Determine whether A and B are mutually exclusive or not.

Problem 7.9 ‡
A survey of a group's viewing habits over the last year revealed the following information

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Find the probability of the group that watched none of the three sports during the last year.

Problem 7.10 ‡
The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35% . Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.
Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

Problem 7.11 ‡
You are given $P(A \cup B) = 0.7$ and $P(A \cup B^c) = 0.9$. Determine $P(A)$.

Problem 7.12 ‡
Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 14% the probability that a patient visits a physical therapist.
Determine the probability that a randomly chosen member of this group visits a physical therapist.

Problem 7.13 ‡
In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying

assumption that for all integers $n \geq 0, p_{n+1} = \frac{1}{5}p_n$, where p_n represents the probability that the policyholder files n claims during the period.
Under this assumption, what is the probability that a policyholder files more than one claim during the period?

Problem 7.14 ‡
A marketing survey indicates that 60% of the population owns an automobile, 30% owns a house, and 20% owns both an automobile and a house.
Calculate the probability that a person chosen at random owns an automobile or a house, but not both.

Problem 7.15 ‡
An insurance agent offers his clients auto insurance, homeowners insurance and renters insurance. The purchase of homeowners insurance and the purchase of renters insurance are mutually exclusive. The profile of the agent's clients is as follows:
i) 17% of the clients have none of these three products.
ii) 64% of the clients have auto insurance.
iii) Twice as many of the clients have homeowners insurance as have renters insurance.
iv) 35% of the clients have two of these three products.
v) 11% of the clients have homeowners insurance, but not auto insurance.
Calculate the percentage of the agent's clients that have both auto and renters insurance.

Problem 7.16 ‡
A mattress store sells only king, queen and twin-size mattresses. Sales records at the store indicate that one-fourth as many queen-size mattresses are sold as king and twin-size mattresses combined. Records also indicate that three times as many king-size mattresses are sold as twin-size mattresses.
Calculate the probability that the next mattress sold is either king or queen-size.

Problem 7.17 ‡
The probability that a member of a certain class of homeowners with liability and property coverage will file a liability claim is 0.04, and the probability that a member of this class will file a property claim is 0.10. The probability that a member of this class will file a liability claim but not a property claim

is 0.01.
Calculate the probability that a randomly selected member of this class of homeowners will not file a claim of either type.

Problem 7.18
If you roll two dice, what is the probability of an even sum or a sum of 8 or more?

Problem 7.19
In a class of 10 students, 6 are female and 4 are male. If 3 of the students are selected at random, find the probability that at least one female is selected?

Problem 7.20
Let A and B be any two events. Show that $P(A \cup B) \leq P(A) + P(B)$.

Problem 7.21 ‡
This year, a medical insurance policyholder has probability 0.70 of having no emergency room visits, 0.85 of having no hospital stays, and 0.61 of having neither emergency room visits nor hospital stays.
Calculate the probability that the policyholder has at least one emergency room visit and at least one hospital stay this year.

Problem 7.22 ‡
A policyholder purchases automobile insurance for two years. Define the following events:

F = the policyholder has exactly one accident in year one.
G = the policyholder has one or more accidents in year two.

Define the following events:
i) The policyholder has exactly one accident in year one and has more than one accident in year two.
ii) The policyholder has at least two accidents during the two-year period.
iii) The policyholder has exactly one accident in year one and has at least one accident in year two.
iv) The policyholder has exactly one accident in year one and has a total of two or more accidents in the two-year period.
v) The policyholder has exactly one accident in year one and has more accidents in year two than in year one.
Determine the number of events from the above list of five that are the same as $F \cap G$.

Problem 7.23 ‡
An insurance agent's files reveal the following facts about his policyholders:
i) 243 own auto insurance.
ii) 207 own homeowner insurance.
iii) 55 own life insurance and homeowner insurance.
iv) 96 own auto insurance and homeowner insurance.
v) 32 own life insurance, auto insurance and homeowner insurance.
vi) 76 more clients own only auto insurance than only life insurance.
vii) 270 own only one of these three insurance products.
Calculate the total number of the agent's policyholders who own at least one of these three insurance products.

Problem 7.24 ‡
A profile of the investments owned by an agent's clients follows:
i) 228 own annuities.
ii) 220 own mutual funds.
iii) 98 own life insurance and mutual funds.
iv) 93 own annuities and mutual funds.
v) 16 own annuities, mutual funds, and life insurance.
vi) 45 more clients own only life insurance than own only annuities.
vii) 290 own only one type of investment (i.e., annuity, mutual fund, or life insurance).
Calculate the agent's total number of clients.

Problem 7.25 ‡
An actuary compiles the following information from a portfolio of 1000 homeowners insurance policies:
i) 130 policies insure three-bedroom homes.
ii) 280 policies insure one-story homes.
iii) 150 policies insure two-bath homes.
iv) 30 policies insure three-bedroom, two-bath homes.
v) 50 policies insure one-story, two-bath homes.
vi) 40 policies insure three-bedroom, one-story homes.
vii) 10 policies insure three-bedroom, one-story, two-bath homes.
Calculate the number of homeowners policies in the portfolio that insure neither one-story nor two-bath nor three-bedroom homes.

Problem 7.26 ‡
The annual numbers of thefts a homeowners insurance policyholder experi-

ences are analyzed over three years. Define the following events:
i) A = the event that the policyholder experiences no thefts in the three years.
ii) B = the event that the policyholder experiences at least one theft in the second year.
iii) C = the event that the policyholder experiences exactly one theft in the first year.
iv) D = the event that the policyholder experiences no thefts in the third year.
v) E = the event that the policyholder experiences no thefts in the second year, and at least one theft in the third year.
Determine which three events satisfy the condition that the probability of their union equals the sum of their probabilities.

8 Probability and Counting Techniques

The Fundamental Principle of Counting can be used to compute probabilities as shown in the following example.

Example 8.1
In an actuarial course in probability, an instructor has decided to give his class a weekly quiz consisting of 5 multiple-choice questions taken from a pool of previous SOA P/1 exams. Each question has 4 answer choices, of which 1 is correct and the other 3 are incorrect.
(a) How many answer choices are there?
(b) What is the probability of getting all 5 right answers?
(c) What is the probability of answering exactly 4 questions correctly?
(d) What is the probability of getting at least four answers correctly?

Solution.
(a) We can look at this question as a decision consisting of five steps. There are 4 ways to do each step so that by the Fundamental Principle of Counting there are

$$(4)(4)(4)(4)(4) = 1024 \text{ possible choices of answers.}$$

(b) There is only one way to answer each question correctly. Using the Fundamental Principle of Counting there is $(1)(1)(1)(1)(1) = 1$ way to answer all 5 questions correctly out of 1024 possible answer choices. Hence,

$$\text{P(all 5 right)} = \tfrac{1}{1024}$$

(c) The following table lists all possible responses that involve exactly 4 right answers where R stands for right and W stands for a wrong answer

Five Responses	Number of ways to fill out the test
WRRRR	$(3)(1)(1)(1)(1) = 3$
RWRRR	$(1)(3)(1)(1)(1) = 3$
RRWRR	$(1)(1)(3)(1)(1) = 3$
RRRWR	$(1)(1)(1)(3)(1) = 3$
RRRRW	$(1)(1)(1)(1)(3) = 3$

So there are 15 ways out of the 1024 possible ways that result in 4 right answers and 1 wrong answer so that

$$\text{P(4 right,1 wrong)} = \frac{15}{1024} \approx 1.5\%$$

(d) "At least 4" means you can get either 4 right and 1 wrong or all 5 right. Thus,

$$\begin{aligned} P(\text{at least 4 right}) =& P(4R, 1W) + P(5R) \\ =& \frac{15}{1024} + \frac{1}{1024} \\ =& \frac{16}{1024} \approx 0.016 \end{aligned}$$ ■

Example 8.2
Consider the experiment of rolling two dice. How many events A are there with $P(A) = \frac{1}{3}$?

Solution.
We want to find events A such that $P(A) = \frac{1}{3} = \frac{12}{36}$ where 36 is the total number of outcomes of the experiment. There are ${}_{36}C_{12}$ such distinct events ■

Probability Trees
Probability trees can be used to compute the probabilities of combined outcomes in a sequence of experiments.

Example 8.3
Construct the probability tree of the experiment of flipping a fair coin twice.

Solution.
The probability tree is shown in Figure 8.1■

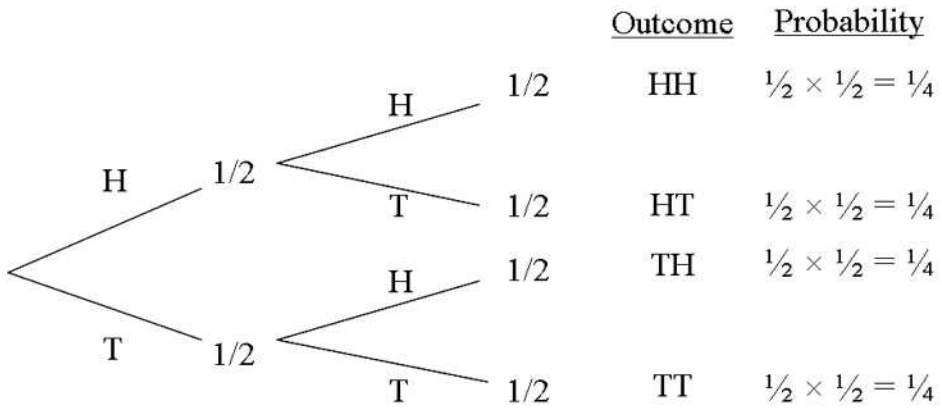


Figure 8.1

The probabilities shown in Figure 8.1 are obtained by following the paths leading to each of the four outcomes and multiplying the probabilities along the paths. This procedure is an instance of the following general property.

Multiplication Rule for Probabilities for Tree Diagrams
For all multistage experiments, the probability of the outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Example 8.4
A shipment of 500 DVD players contains 9 defective DVD players. Construct the probability tree of the experiment of sampling two of them without replacement.

Solution.
The probability tree is shown in Figure 8.2 ■

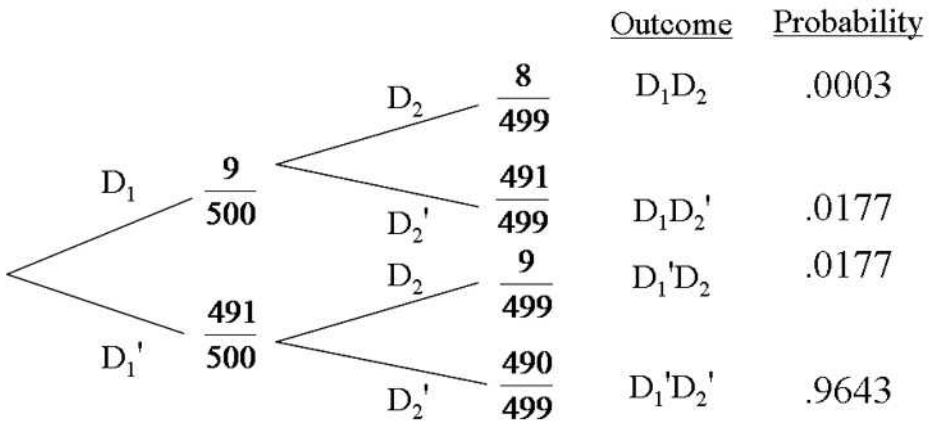


Figure 8.2

Example 8.5
The faculty of a college consists of 35 female faculty and 65 male faculty. 70% of the female faculty favor raising tuition, while only 40% of the male faculty favor the increase.
If a faculty member is selected at random from this group, what is the probability that he or she favors raising tuition?

Solution.
Figure 8.3 shows a tree diagram for this problem where F stands for female,

M for male.

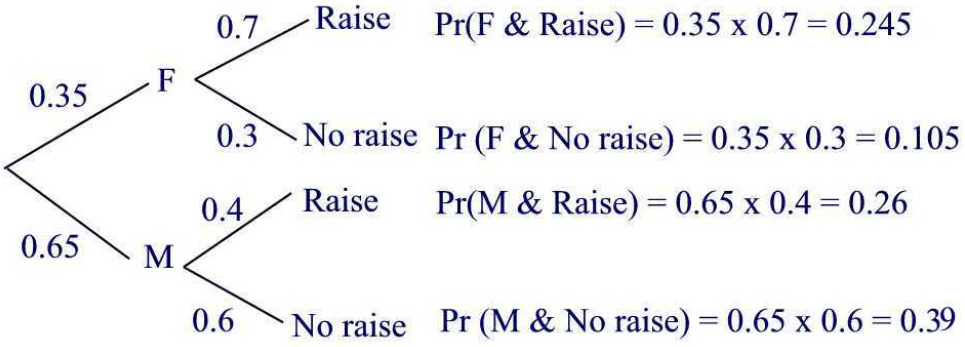


Figure 8.3

The first and third branches correspond to favoring the tuition raise. We add their probabilities.

$$P(\text{tuition raise}) = 0.245 + 0.26 = 0.505 \blacksquare$$

Example 8.6
A regular insurance claimant is trying to hide 3 fraudulent claims among 7 genuine claims. The claimant knows that the insurance company processes claims in batches of 5 or in batches of 10. For batches of 5, the insurance company will investigate one claim at random to check for fraud; for batches of 10, two of the claims are randomly selected for investigation. The claimant has three possible strategies:
(a) submit all 10 claims in a single batch,
(b) submit two batches of 5, one containing 2 fraudulent claims and the other containing 1,
(c) submit two batches of 5, one containing 3 fraudulent claims and the other containing 0.
What is the probability that all three fraudulent claims will go undetected in each case? What is the best strategy?

Solution.
(a) P(fraud not detected) $= \frac{7}{10} \cdot \frac{6}{9} = \frac{7}{15}$
(b) P(fraud not detected) $= \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}$
(c) P(fraud not detected) $= \frac{2}{5} \cdot 1 = \frac{2}{5}$
Claimant's best strategy is to split fraudulent claims between two batches of 5 $\blacksquare$

Practice Problems

Problem 8.1
A box contains three red balls and two blue balls. Two balls are to be drawn without replacement. Use a tree diagram to represent the various outcomes that can occur. What is the probability of each outcome?

Problem 8.2
Repeat the previous exercise but this time replace the first ball before drawing the second.

Problem 8.3
An urn contains three red marbles and two green marbles. An experiment consists of drawing one marble at a time without replacement, until a red one is obtained. Find the probability of the following events.

A : Only one draw is needed.
B : Exactly two draws are needed.
C : Exactly three draws are needed.

Problem 8.4
Consider a jar with three black marbles and one red marble. For the experiment of drawing two marbles with replacement, what is the probability of drawing a black marble and then a red marble in that order? Assume that the balls are equally likely to be drawn.

Problem 8.5
An urn contains two black balls and one red ball. Two balls are drawn with replacement. What is the probability that both balls are black? Assume that the balls are equally likely to be drawn.

Problem 8.6
An urn contains four balls: one red, one green, one yellow, and one white. Two balls are drawn without replacement from the urn. What is the probability of getting a red ball and a white ball? Assume that the balls are equally likely to be drawn

Problem 8.7
An urn contains 3 white balls and 2 red balls. Two balls are to be drawn one at a time and without replacement. Draw a tree diagram for this experiment and find the probability that the two drawn balls are of different colors. Assume that the balls are equally likely to be drawn

Problem 8.8
Repeat the previous problem but with each drawn ball to be put back into the urn.

Problem 8.9
An urn contains 16 black balls and 3 purple balls. Two balls are to be drawn one at a time without replacement. What is the probability of drawing out a black on the first draw and a purple on the second?

Problem 8.10
A board of trustees of a university consists of 8 men and 7 women. A committee of 3 must be selected at random and without replacement. The role of the committee is to select a new president for the university. Calculate the probability that the number of men selected exceeds the number of women selected.

Problem 8.11 ‡
A store has 80 modems in its inventory, 30 coming from Source A and the remainder from Source B. Of the modems from Source A, 20% are defective. Of the modems from Source B, 8% are defective.
Calculate the probability that exactly two out of a random sample of five modems from the store's inventory are defective.

Problem 8.12 ‡
From 27 pieces of luggage, an airline luggage handler damages a random sample of four. The probability that exactly one of the damaged pieces of luggage is insured is twice the probability that none of the damaged pieces are insured. Calculate the probability that exactly two of the four damaged pieces are insured.

Problem 8.13
Melanie is going to play a tennis match and a squash match. The probability that she will win the tennis match is $\frac{7}{10}$ and that of winning the squash mash

is $\frac{3}{5}$.
(a) Construct the probability tree diagram.
(b) What is the probability that Melanie will win both matches?

Problem 8.14
On a shelf, there are 4 tuna sandwiches, 5 cheese sandwiches, and 2 peanut butter sandwiches. Jeanine takes two sandwiches at random. What is the probability that she takes two different types of sandwiches?

Problem 8.15
A jar contains 7 balls. Two of the balls are labeled 1, three are labeled 2, and two are labeled 3.
(a) Pick two balls without replacement. What is the probability that both balls have the number 1 on them?
(b) Calculate the probability that the number on the second ball is greater than the number on the first tile.

Problem 8.16
A tin has 12 chocolate bars, 5 strawberry bars and 3 vanilla bars. You pick, without replacement, two bars from the tin. What is the probability that the two bars are of different flavors?

Problem 8.17
A box has five red marbles, three blue marbles and two green marbles. You pick successively and without replacement two marbles from the box. What is the probability that the two marbles have the same color?

Problem 8.18
A race is run with 10 competitors. What is the probability of selecting 3 finishers of the race in the correct order?

Problem 8.19
A lottery consists of selecting 6 numbers from 1 - 53. What is the probability of getting exactly 4 winning numbers correct with one ticket?

Problem 8.20
A company consists of 10 male and 12 female. What is the probability of a having a committee of 2 male and 2 female?

Problem 8.21 ‡
An insurance agent meets twelve potential customers independently, each of whom is equally likely to purchase an insurance product. Six are interested only in auto insurance, four are interested only in homeowners insurance, and two are interested only in life insurance.
The agent makes six sales. Calculate the probability that two are for auto insurance, two are for homeowners insurance, and two are for life insurance.

Problem 8.22 ‡
Six claims are to be randomly selected from a group of thirteen different claims, which includes two workers compensation claims, four homeowners claims and seven auto claims.
Calculate the probability that the six claims selected will include one workers compensation claim, two homeowners claims and three auto claims.

Problem 8.23 ‡
A drawer contains four pairs of socks, with each pair a different color. One sock at a time is randomly drawn from the drawer until a matching pair is obtained. Calculate the probability that the maximum number of draws is required.

Conditional Probability and Independence

In this chapter we introduce the concept of conditional probability. So far, the notation $P(A)$ stands for the probability of A regardless of the occurrence of any other events. If the occurrence of an event B influences the probability of A then this new probability is called conditional probability.

9 Conditional Probabilities

We desire to know the probability of an event A conditional on the knowledge that another event B has occurred. The information the event B has occurred causes us to update the probabilities of other events in the sample space. To illustrate, suppose you cast two dice; one red, and one green. The Sample space is given in the Table 9.1.

G\ R	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Table 9.1

Let A be the event of getting two ones. That is, $A = \{(1,1)\}$. Then $P(A) = 1/36$. However, if, after casting the dice, you ascertain that the green die shows a one (but know nothing about the red die). That is, the event $B = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$. Then there is 1/6 chance that both of

them will be one. In other words, the probability of getting two ones changes if you have partial information, and we refer to this (altered) probability as **conditional probability**.

If the occurrence of the event A depends on the occurrence of B then the conditional probability will be denoted by $P(A|B)$, read as the *probability of A given B*. Conditioning restricts the sample space to those outcomes which are in the set being conditioned on (in this case B). In this case,

$$P(A|B) = \frac{\text{number of outcomes corresponding to event A and B}}{\text{number of outcomes of B}}.$$

Thus,

$$P(A|B) = \frac{\#(A \cap B)}{\#(B)} = \frac{\frac{\#(A \cap B)}{\#(S)}}{\frac{\#(B)}{\#(S)}} = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

Example 9.1

Let M denote the event "student is male" and let H denote the event "student is hispanic". In a class of 100 students suppose 60 are hispanic, and suppose that 10 of the hispanic students are males. Find the probability that a randomly chosen hispanic student is a male, that is, find $P(M|H)$.

Solution.

Since 10 out of 100 students are both hispanic and male, $P(M \cap H) = \frac{10}{100} = 0.1$. Also, 60 out of the 100 students are hispanic, so $P(H) = \frac{60}{100} = 0.6$. Hence, $P(M|H) = \frac{0.1}{0.6} = \frac{1}{6}$ ■

Using the fact that $P(A \cap B) = P(B \cap A)$ and the formulas

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

we can write

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A). \tag{9.1}$$

From this last equation, we can write

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

This formula is known as **Bayes' formula** which will be the topic of the next section.

Example 9.2
The probability of an applicant to be admitted to a certain college is 0.8. The probability for a student in the college to live on campus is 0.6. What is the probability that an applicant will be admitted to the college and will be assigned a dormitory housing?

Solution.
The probability of the applicant being admitted and receiving dormitory housing is defined by

$$\begin{aligned} \text{P(Accepted and Housing)} &= \text{P(Housing|Accepted)P(Accepted)} \\ &= (0.6)(0.8) = 0.48 \ \blacksquare \end{aligned}$$

Equation (9.1) can be generalized to any finite number of events.

Theorem 9.1
Consider n events $A_1, A_2, \cdots, A_n$. Then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

Proof.
The proof is by induction on $n \geq 2$. By Equation (9.1), the relation holds for $n = 2$. Suppose that the relation is true for $2, 3, \cdots, n$. We wish to establish

$$P(A_1 \cap A_2 \cap \cdots \cap A_{n+1}) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n).$$

We have

$$\begin{aligned} P(A_1 \cap A_2 \cap \cdots \cap A_{n+1}) =& P((A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}) \\ =& P(A_{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n)P(A_1 \cap A_2 \cap \cdots \cap A_n) \\ =& P(A_{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n)P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \\ & \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \\ =& P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \\ & \times P(A_{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n) \ \blacksquare \end{aligned}$$

Example 9.3
Suppose 5 cards are drawn from a deck of 52 playing cards. What is the probability that all cards are the same suit, i.e., a flush?

Solution.
We must find

P(a flush) = P(5 spades) + P(5 hearts) + P(5 diamonds) + P(5 clubs).

Now, the probability of getting 5 spades is found as follows:

$$\begin{aligned} \text{P(5 spades)} &= \text{P(1st card is a spade)P(2nd card is a spade}|\text{1st card is a spade)} \\ &\times \cdots \times \text{P(5th card is a spade}|\text{1st,2nd,3rd,4th cards are spades)} \\ &= \tfrac{13}{52} \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{10}{49} \times \tfrac{9}{48} \end{aligned}$$

Alternatively,

$$\text{P(5 spades)} = \tfrac{{}_{13}C_5}{{}_{52}C_5}.$$

Since the above calculation is the same for any of the four suits,

$$\text{P(a flush)} = 4 \times \tfrac{13}{52} \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{10}{49} \times \tfrac{9}{48} = 0.0019807923 \ \blacksquare$$

We end this section by showing that $P(\cdot|A)$ satisfies the properties of ordinary probabilities.

Theorem 9.2
The function $B \to P(B|A)$ defines a probability measure.

Proof.
1. Since $A \cap B \subseteq A,\ 0 \leq P(A \cap B) \leq P(A)$, so that by dividing through by $P(A)$ we obtain $0 \leq P(B|A) \leq 1$.
2. $P(S|A) = \frac{P(S \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$.
3. Suppose that $B_1, B_2, \cdots$, are mutually exclusive events. Then $B_1 \cap A, B_2 \cap A, \cdots$, are mutually exclusive. Thus,

$$\begin{aligned} P(\bigcup_{n=1}^{\infty} B_n|A) &= \frac{P\left(\bigcup_{n=1}^{\infty}(B_n \cap A)\right)}{P(A)} \\ &= \frac{\sum_{n=1}^{\infty} P(B_n \cap A)}{P(A)} \\ &= \sum_{n=1}^{\infty} \frac{P(B_n \cap A)}{P(A)} = \sum_{n=1}^{\infty} P(B_n|A) \ \blacksquare \end{aligned}$$

From the above thereom, every theorem we have proved for unconditional probability function holds for a conditional probability function. For example, we have

$$P(B^c|A) = 1 - P(B|A).$$

Prior and Posterior Probabilities

The unconditional probability $P(A)$ is the probability of the event A prior to introducing new events that might affect A. This is also known as the **prior probability** of A. When the occurrence of an event B will affect the event A then the conditional probability $P(A|B)$ is known as the **posterior probability** of A.

Practice Problems

Problem 9.1 ‡
A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.
Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

Problem 9.2 ‡
An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

Problem 9.3 ‡
An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is $\frac{1}{3}$.
What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

Problem 9.4
You are given $P(A) = \frac{2}{5},\ P(A \cup B) = \frac{3}{5},\ P(B|A) = \frac{1}{4},\ P(C|B) = \frac{1}{3}$, and $P(C|A \cap B) = \frac{1}{2}$. Find $P(A|B \cap C)$.

Problem 9.5
A pollster surveyed 100 people about watching the TV show "The big bang theory". The results of the poll are shown in the table.

	Yes	No	Total
Male	19	41	60
Female	12	28	40
Total	31	69	100

(a) What is the probability of a randomly selected individual is a male and watching the show?
(b) What is the probability of a randomly selected individual is a male?
(c) What is the probability of a randomly selected individual watches the show?
(d) What is the probability of a randomly selected individual watches the show, given that the individual is a male?
(e) What is the probability that a randomly selected individual watching the show is a male?

Problem 9.6
An urn contains 22 marbles: 10 red, 5 green, and 7 orange. You pick two at random without replacement. What is the probability that the first is red and the second is orange?

Problem 9.7
You roll two fair dice. Find the (conditional) probability that the sum of the two faces is 6 given that the two dice are showing different faces.

Problem 9.8
A machine produces small cans that are used for baked beans. The probability that the can is in perfect shape is 0.9. The probability of the can having an unnoticeable dent is 0.02. The probability that the can is obviously dented is 0.08. Produced cans get passed through an automatic inspection machine, which is able to detect obviously dented cans and discard them. What is the probability that a can that gets shipped for use will be of perfect shape?

Problem 9.9
An urn contains 225 white marbles and 15 black marbles. If we randomly pick (without replacement) two marbles in succession from the urn, what is the probability that they will both be black?

Problem 9.10
Find the probabilities of randomly drawing two kings in succession from an ordinary deck of 52 playing cards if we sample
(a) without replacement
(b) with replacement

Problem 9.11
A box of television tubes contains 20 tubes, of which five are defective. If three of the tubes are selected at random and removed from the box in succession without replacement, what is the probability that all three tubes are defective?

Problem 9.12
A study of texting and driving has found the following: Given that an accident is fatal, the probability that it is caused by a texting driver is 40%. Also, 1% of all auto accidents are fatal, and drivers who text while driving are responsible for 20% of all accidents. Find the probability that an accident is caused by a driver who does not text given that the accident is non-fatal.

Problem 9.13
A TV manufacturer buys TV tubes from three sources. Source A supplies 50% of all tubes and has a 1% defective rate. Source B supplies 30% of all tubes and has a 2% defective rate. Source C supplies the remaining 20% of tubes and has a 5% defective rate.
(a) What is the probability that a randomly selected purchased tube is defective?
(b) Given that a purchased tube is defective, what is the probability it came from Source A? From Source B? From Source C?

Problem 9.14
In a certain town in the United States, 40% of the population are liberals and 60% are conservatives. The city council has proposed selling alcohol illegal in the town. It is known that 75% of conservatives and 30% of liberals support this measure.
(a) What is the probability that a randomly selected resident from the town will support the measure?
(b) If a randomly selected person does support the measure, what is the probability the person is a liberal?
(c) If a randomly selected person does not support the measure, what is the probability that he or she is a liberal?

Problem 9.15
Let A and B be arbitrary events of a sample space S. Show that
(a) $A = (A \cap B) \cup (A \cap B^c)$ and $(A \cap B) \cap (A \cap B^c) = \emptyset$.
(b) $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$.

Problem 9.16
Prove:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Problem 9.17
Show that $P(A|B) > P(A)$ if and only if $P(A^c|B) < P(A^c)$. We assume that $0 < P(A) < 1$ and $0 < P(B) < 1$

Problem 9.18
Two events A and B are said to be **independent** if and only if $P(A|B) = P(A)$. Prove that A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

Problem 9.19
If A and B are independent then so are A and B^c.

Problem 9.20
Show that if A and B are independent so are A^c and B^c.

Problem 9.21 ‡
In a shipment of 20 packages, 7 packages are damaged. The packages are randomly inspected, one at a time, without replacement, until the fourth damaged package is discovered. Calculate the probability that exactly 12 packages are inspected.

Problem 9.22 ‡
Bowl I contains eight red balls and six blue balls. Bowl II is empty. Four balls are selected at random, without replacement, and transferred from bowl I to bowl II. One ball is then selected at random from bowl II.
Calculate the conditional probability that two red balls and two blue balls were transferred from bowl I to bowl II, given that the ball selected from bowl II is blue.

Problem 9.23 ‡
A machine has two parts labeled A and B. The probability that part A works for one year is 0.8 and the probability that part B works for one year is 0.6. The probability that at least one part works for one year is 0.9.
Calculate the probability that part B works for one year, given that part A works for one year.

10 Posterior Probabilities: Bayes' Formula

It is often the case that we know the probabilities of certain events conditional on other events, but what we would like to know is the "reverse". That is, given $P(A|B)$ we would like to find $P(B|A)$.
Bayes' formula is a simple mathematical formula used for calculating $P(B|A)$ given $P(A|B)$. We derive this formula as follows. Let A and B be two events. Then

$$A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Since the events $A \cap B$ and $A \cap B^c$ are mutually exclusive, we can write

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c). \end{aligned} \tag{10.1}$$

Example 10.1
The completion of a highway construction may be delayed because of a projected storm. The probabilities are 0.60 that there will be a storm, 0.85 that the construction job will be completed on time if there is no storm, and 0.35 that the construction will be completed on time if there is a storm. What is the probability that the construction job will be completed on time?

Solution.
Let A be the event that the construction job will be completed on time and B is the event that there will be a storm. We are given $P(B) = 0.60$, $P(A|B^c) = 0.85$, and $P(A|B) = 0.35$. From Equation (10.1) we find

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c) = (0.60)(0.35) + (0.4)(0.85) = 0.55 \blacksquare$$

From Equation (10.1) we can get Bayes' formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A \cap B) + P(A \cap B^c)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \tag{10.2}$$

Example 10.2
A small manufacturing company uses two machines A and B to make shirts. Observation shows that machine A produces 10% of the total production of shirts while machine B produces 90% of the total production of shirts. Assuming that 1% of all the shirts produced by A are defective while 5% of

all the shirts produced by B are defective, find the probability that a shirt taken at random from a day's production was made by machine A, given that it is defective.

Solution.
We are given $P(A) = 0.1,\ P(B) = 0.9,\ P(D|A) = 0.01$, and $P(D|B) = 0.05$. We want to find $P(A|D)$. Using Bayes' formula we find

$$\begin{aligned} P(A|D) =& \frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B)} \\ =& \frac{(0.01)(0.1)}{(0.01)(0.1) + (0.05)(0.9)} \approx 0.0217 \ \blacksquare \end{aligned}$$

Example 10.3
A credit card company offers two types of cards: a basic card (B) and a gold card (G). Over the past year, 40% of the cards issued have been of the basic type. Of those getting the basic card, 30% enrolled in an identity theft plan, whereas 50% of all gold cards holders do so. If you learn that a randomly selected cardholder has an identity theft plan, how likely is it that he/she has a basic card?

Solution.
Let I denote the identity theft plan. We are given $P(B) = 0.4, P(G) = 0.6,\ P(I|B) = 0.3$, and $P(I|G) = 0.5$. By Bayes' formula we have

$$\begin{aligned} P(B|I) =& \frac{P(B \cap I)}{P(I)} = \frac{P(I|B)P(B)}{P(I|B)P(B) + P(I|G)P(G)} \\ =& \frac{(0.3)(0.4)}{(0.3)(0.4) + (0.5)(0.6)} = 0.286 \ \blacksquare \end{aligned}$$

Formula (10.2) is a special case of the more general result:

Theorem 10.1 *(Bayes' formula)*
Suppose that the sample space S is the union of mutually exclusive events $H_1, H_2, \cdots, H_n$ with $P(H_i) > 0$ for each i. Then for any event A and $1 \leq i \leq n$ we have

$$P(H_i|A) = \frac{P(A|H_i)P(H_i)}{P(A)}$$

where

$$P(A) = P(H_1)P(A|H_1) + P(H_2)P(A|H_2) + \cdots + P(H_n)P(A|H_n). \quad (10.3)$$

Equation (10.3) is known as the **Law of Total Probability**.

Proof.
First note that

$$\begin{aligned} P(A) =& P(A \cap S) = P(A \cap (\bigcup_{i=1}^{n} H_i)) \\ =& P(\bigcup_{i=1}^{n} (A \cap H_i)) \\ =& \sum_{i=1}^{n} P(A \cap H_i) = \sum_{i=1}^{n} P(A|H_i)P(H_i) \end{aligned}$$

Hence,

$$P(H_i|A) = \frac{P(A|H_i)P(H_i)}{P(A)} = \frac{P(A|H_i)P(H_i)}{\sum_{i=1}^{n} P(A|H_i)P(H_i)} \blacksquare$$

Example 10.4
A survey about a measure to legalize medical marijuana is taken in three states: Kentucky, Maine and Arkansas. In Kentucky, 50% of voters support the measure, in Maine, 60% of the voters support the measure, and in Arkansas, 35% of the voters support the measure. Of the total population of the three states, 40% live in Kentucky , 25% live in Maine, and 35% live in Arkansas. Given that a voter supports the measure, what is the probability that he/she lives in Maine?

Solution.
Let L_I denote the event that a voter lives in state I, where I = K (Kentucky), M (Maine), or A (Arkansas). Let S denote the event that a voter supports the measure. We want to find $P(L_M|S)$. By Bayes' formula we have

$$\begin{aligned} P(L_M|S) &= \frac{P(S|L_M)P(L_M)}{P(S|L_K)P(L_K)+P(S|L_M)P(L_M)+P(S|L_A)P(L_A)} \\ &= \frac{(0.6)(0.25)}{(0.5)(0.4)+(0.6)(0.25)+(0.35)(0.35)} \approx 0.3175 \blacksquare \end{aligned}$$

Example 10.5
Passengers in Little Rock Airport rent cars from three rental companies: 60% from Avis, 30% from Enterprise, and 10% from National. Past statistics show that 9% of the cars from Avis, 20% of the cars from Enterprise , and 6% of the cars from National need oil change. If a rental car delivered to a passenger needs an oil change, what is the probability that it came from Enterprise?

Solution.
Define the events

$$\begin{aligned} A &= \text{car comes from Avis} \\ E &= \text{car comes from Enterprise} \\ N &= \text{car comes from National} \\ O &= \text{car needs oil change} \end{aligned}$$

Then

$$\begin{array}{ccc} P(A) = 0.6 & P(E) = 0.3 & P(N) = 0.1 \\ P(O|A) = 0.09 & P(O|E) = 0.2 & P(O|N) = 0.06 \end{array}$$

From Bayes' theorem we have

$$\begin{aligned} P(E|O) &= \frac{P(O|E)P(E)}{P(O|A)P(A) + P(O|E)P(E) + P(O|N)P(N)} \\ &= \frac{0.2 \times 0.3}{0.09 \times 0.6 + 0.2 \times 0.3 + 0.06 \times 0.1} = 0.5 \ \blacksquare \end{aligned}$$

Example 10.6
A toy factory produces its toys with three machines A, B, and C. From the total production, 50% are produced by machine A, 30% by machine B, and 20% by machine C. Past statistics show that 4% of the toys produced by machine A are defective, 2% produced by machine B are defective, and 4% of the toys produced by machine C are defective.
(a) What is the probability that a randomly selected toy is defective?
(b) If a randomly selected toy was found to be defective, what is the probability that this toy was produced by machine A?

Solution.
Let D be the event that the selected product is defective. Then, $P(A) =$

$0.5, P(B) = 0.3, P(C) = 0.2, P(D|A) = 0.04,\ P(D|B) = 0.02, P(D|C) = 0.04$. We have

$$\begin{aligned} P(D) =& P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ =& (0.04)(0.50) + (0.02)(0.30) + (0.04)(0.20) = 0.034 \end{aligned}$$

(b) By Bayes' theorem, we find

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.04)(0.50)}{0.034} \approx 0.5882 \ \blacksquare$$

Example 10.7
A group of traffic violators consists of 45 men and 15 women. The men have probability 1/2 for being ticketed for crossing a red light while the women have probability 1/3 for the same offense.
(a) Suppose you choose at random a person from the group. What is the probability that the person will be ticketed for crossing a red light?
(b) Determine the conditional probability that you chose a woman given that the person you chose was being ticketed for crossing the red light.

Solution.
Let

$$\begin{aligned} W =& \{\text{the one selected is a woman}\} \\ M =& \{\text{the one selected is a man}\} \\ T =& \{\text{the one selected is ticketed for crossing a red light}\} \end{aligned}$$

(a) We are given the following information: $P(W) = \frac{15}{60} = \frac{1}{4}, P(M) = \frac{3}{4},\ P(T|W) = \frac{1}{3}$, and $P(T|M) = \frac{1}{2}$. We have,

$$P(T) = P(T|W)P(W) + P(T|M)P(M) = \frac{11}{24}.$$

(b) Using Bayes' theorem we find

$$P(W|T) = \frac{P(T|W)P(W)}{P(T)} = \frac{(1/3)(1/4)}{(11/24)} = \frac{2}{11} \ \blacksquare$$

Practice Problems

Problem 10.1
An insurance company believes that auto drivers can be divided into two categories: those who are a high risk for accidents and those who are low risk. Past statistics show that the probability for a high risk driver to have an accident within a one-year period is 0.4, whereas this probability is 0.2 for a low risk driver.
(a) If we assume that 30% of the population is high risk, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?
(b) Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is high risk?

Problem 10.2 ‡
An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16 - 20	0.06	0.08
21 - 30	0.03	0.15
31 - 65	0.02	0.49
66 - 99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16-20.

Problem 10.3 ‡
An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year.
A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

Problem 10.4 ‡
Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) the rest of the emergency room patients were stable;
(iv) 40% of the critical patients died;
(v) 10% of the serious patients died; and
(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

Problem 10.5 ‡
A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers.
Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers.
A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

Problem 10.6 ‡
An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

Problem 10.7 ‡
A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease.
Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

Problem 10.8 ‡
The probability that a randomly chosen male has a circulation problem is 0.25 . Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem.
What is the conditional probability that a male has a circulation problem, given that he is a smoker?

Problem 10.9 ‡
A study of automobile accidents produced the following data:

Model year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

Problem 10.10
A study was conducted about the excessive amounts of pollutants emitted by cars in a certain town. The study found that 25% of all cars emit excessive amounts of pollutants. The probability for a car emitting excessive amounts of pollutants to fail the town's vehicular emission test is found to be 0.99. Cars who do not emit excessive amounts of pollutants have a probability of 0.17 to fail to emission test. A car is selected at random. What is the probability that the car emits excessive amounts of pollutants given that it failed the emission test?

Problem 10.11
A medical agency is conducting a study about injuries resulted from activities for a group of people. For this group, 50% were skiing, 30% were hiking, and 20% were playing soccer. The (conditional) probability of a person getting injured from skiing is 30% , it is 10% from hiking, and 20% from playing soccer.
(a) What is the probability for a randomly selected person in the group for getting injured?
(b) Given that a person is injured, what is the probability that his injuries are due to skiing?

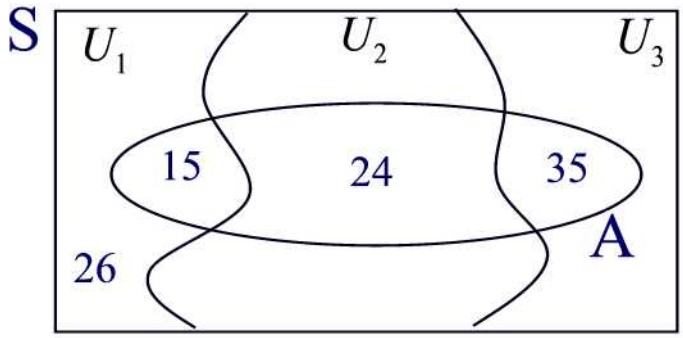
Problem 10.12
A written driving test is graded either pass or fail. A randomly chosen person from a driving class has a 40% chance of knowing the material well. If the person knows the material well, the probability for this person to pass the written test is 0.8. For a person not knowing the material well, the probability is 0.4 for passing the test.
(a) What is the probability of a randomly chosen person from the class for passing the test?
(b) Given that a person in the class passes the test, what is the probability that this person knows the material well?

Problem 10.13 ‡
Ten percent of a company's life insurance policyholders are smokers. The rest are nonsmokers. For each nonsmoker, the probability of dying during the year is 0.01. For each smoker, the probability of dying during the year is 0.05.
Given that a policyholder has died, what is the probability that the policyholder was a smoker?

Problem 10.14
A prerequisite for students to take a probability class is to pass calculus. A study of correlation of grades for students taking calculus and probability was conducted. The study shows that 25% of all calculus students get an A, and that students who had an A in calculus are 50% more likely to get an A in probability as those who had a lower grade in calculus. If a student who received an A in probability is chosen at random, what is the probability that he/she also received an A in calculus?

Problem 10.15
A group of people consists of 70 men and 70 women. Seven men and ten women are found to be color-blind.
(a) What is the probability that a randomly selected person is color-blind?
(b) If the randomly selected person is color-blind, what is the probability that the person is a man?

Problem 10.16
Calculate $P(U_1|A)$.



Problem 10.17
The probability that a person with certain symptoms has prostate cancer is 0.8. A PSA test used to confirm this diagnosis gives positive results for 90% of those who have the disease, and 5% of those who do not have the disease. What is the probability that a person who reacts positively to the test actually has the disease ?

Problem 10.18
At a dairy farm, customers are handed empty one gallon containers that can be used to buy milk. 40% of the customers buy whole milk, 35% buy 2% fat free milk, and 25% buy a 1% fat free. Of those customers buying whole milk, only 30% fill the whole gallon. Of those customers buying 2% fat free, 60% fill their gallons, while of those buying 1% fat free milk, 50% fill their gallons. If the next customer fills his/her gallon, what is the probability that the milk bought is whole milk?

Problem 10.19
Seventy percent of the lost luggage at a certain airport are subsequently

recovered. Of the recovered luggage, 60% are of a premium brand FANCY, whereas 90% of the non-recovered luggage are not of the brand FANCY. Suppose that a luggage is lost. If it is a brand FANCY, what is the probability that it will be recovered?

Problem 10.20
A bolt factory uses three machines to manufacture bolts: A, B, and C. Machine A produces 25% of the bolt, machine B produces 35% and the rest of the bolts are produced by machine C. Studies show that of the bolts produced by machine A, 5% are defective, whereas 4% from machine B and 2% from machine C. A randomly chosen bolt was found to be defective. What is the probability that it was produced by machine A.

Problem 10.21 ‡
An urn contains four fair dice. Two have faces numbered 1, 2, 3, 4, 5, and 6; one has faces numbered 2, 2, 4, 4, 6, and 6; and one has all six faces numbered 6. One of the dice is randomly selected from the urn and rolled. The same die is rolled a second time. Calculate the probability that a 6 is rolled both times.

Problem 10.22 ‡
In a group of health insurance policyholders, 20% have high blood pressure and 30% have high cholesterol. Of the policyholders with high blood pressure, 25% have high cholesterol. A policyholder is randomly selected from the group. Calculate the probability that a policyholder has high blood pressure, given that the policyholder has high cholesterol.

Problem 10.23 ‡
An insurance company insures red and green cars. An actuary compiles the following data:

Color of car	Red	Green
Number insured	300	700
Probability an accident occurs	0.10	0.05
Probability that the claim exceeds the deductible, given an accident occurs from this group	0.90	0.80

The actuary randomly picks a claim from all claims that exceed the deductible. Calculate the probability that the claim is on a red car.

Problem 10.24 ‡
A student takes an examination consisting of 20 true-false questions. The student knows the answer to N of the questions, which are answered correctly, and guesses the answers to the rest. The conditional probability that the student knows the answer to a question, given that the student answered it correctly, is 0.824. Calculate N.

Problem 10.25 ‡
Two fair dice are tossed. One die is red and one die is green. Calculate the probability that the sum of the numbers on the two dice is an odd number given that the number that shows on the red die is larger than the number that shows on the green die.

Problem 10.26 ‡
The following information is given about a group of high-risk borrowers.
i) Of all these borrowers, 30% defaulted on at least one student loan.
ii) Of the borrowers who defaulted on at least one car loan, 40% defaulted on at least one student loan.
iii) Of the borrowers who did not default on any student loans, 28% defaulted on at least one car loan.
A statistician randomly selects a borrower from this group and observes that the selected borrower defaulted on at least one student loan.
Calculate the probability that the selected borrower defaulted on at least one car loan.

Problem 10.27 ‡
An insurance company categorizes its policyholders into three mutually exclusive groups: high-risk, medium-risk, and low-risk. An internal study of the company showed that 45% of the policyholders are low-risk and 35% are medium-risk. The probability of death over the next year, given that a policyholder is high-risk is two times the probability of death of a medium-risk policyholder. The probability of death over the next year, given that a policyholder is medium-risk is three times the probability of death of a low-risk policyholder. The probability of death of a randomly selected policyholder, over the next year, is 0.009.
Calculate the probability of death of a policyholder over the next year, given that the policyholder is high-risk.

Problem 10.28 ‡
At a mortgage company, 60% of calls are answered by an attendant. The remaining 40% of callers leave their phone numbers. Of these 40%, 75% receive a return phone call the same day. The remaining 25% receive a return call the next day.
Of those who initially spoke to an attendant, 80% will apply for a mortgage. Of those who received a return call the same day, 60% will apply. Of those who received a return call the next day, 40% will apply.
Calculate the probability that a person initially spoke to an attendant, given that he or she applied for a mortgage.

Problem 10.29 ‡
An insurance company studies back injury claims from a manufacturing company. The insurance company finds that 40% of workers do no lifting on the job, 50% do moderate lifting and 10% do heavy lifting. During a given year, the probability of filing a claim is 0.05 for a worker who does no lifting, 0.08 for a worker who does moderate lifting and 0.20 for a worker who does heavy lifting. A worker is chosen randomly from among those who have filed a back injury claim.
Calculate the probability that the worker's job involves moderate or heavy lifting.

11 Independent Events

Intuitively, when the occurrence of an event B has no influence on the probability of occurrence of an event A then we say that the two events are independent. For example, in the experiment of tossing two coins, the first toss has no effect on the second toss. In terms of conditional probability, two events A and B are said to be **independent** if and only if

$$P(A|B) = P(A).$$

We next introduce the two most basic theorems regarding independence.

Theorem 11.1
A and B are independent events if and only if $P(A \cap B) = P(A)P(B)$.

Proof.
A and B are independent if and only if $P(A|B) = P(A)$ and this is equivalent to

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B) \blacksquare$$

Example 11.1
Show that $P(A|B) > P(A)$ if and only if $P(A^c|B) < P(A^c)$. We assume that $0 < P(A) < 1$ and $0 < P(B) < 1$

Solution.
We have

$$\begin{aligned}
P(A|B) > P(A) &\Leftrightarrow \frac{P(A \cap B)}{P(B)} > P(A) \\
&\Leftrightarrow P(A \cap B) > P(A)P(B) \\
&\Leftrightarrow P(B) - P(A \cap B) < P(B) - P(A)P(B) \\
&\Leftrightarrow P(A^c \cap B) < P(B)(1 - P(A)) \\
&\Leftrightarrow P(A^c \cap B) < P(B)P(A^c) \\
&\Leftrightarrow \frac{P(A^c \cap B)}{P(B)} < P(A^c) \\
&\Leftrightarrow P(A^c|B) < P(A^c) \blacksquare
\end{aligned}$$

Example 11.2
A coal exploration company is set to look for coal mines in two states Virginia and New Mexico. Let A be the event that a coal mine is found in Virginia and B the event that a coal mine is found in New Mexico. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$. What is the probability that at least one coal mine is found in one of the states?

Solution.
The probability that at least one coal mine is found in one of the two states is $P(A \cup B)$. Thus,

$$\begin{aligned} P(A \cup B) =& P(A) + P(B) - P(A \cap B) \\ =& P(A) + P(B) - P(A)P(B) \\ =& 0.4 + 0.7 - 0.4(0.7) = 0.82 \end{aligned}$$ ■

Example 11.3
Let A and B be two independent events such that $P(B|A \cup B) = \frac{2}{3}$ and $P(A|B) = \frac{1}{2}$. What is $P(B)$?

Solution.
First, note that by indepedence we have

$$\frac{1}{2} = P(A|B) = P(A).$$

Next,

$$\begin{aligned} P(B|A \cup B) =& \frac{P(B)}{P(A \cup B)} \\ =& \frac{P(B)}{P(A) + P(B) - P(A \cap B)} \\ =& \frac{P(B)}{P(A) + P(B) - P(A)P(B)}. \end{aligned}$$

Thus,

$$\frac{2}{3} = \frac{P(B)}{\frac{1}{2} + \frac{P(B)}{2}}$$

Solving this equation for $P(B)$ we find $P(B) = \frac{1}{2}$ ■

Theorem 11.2
If A and B are independent then so are A and B^c.

Proof.
First note that A can be written as the union of two mutually exclusive events: $A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$. Thus, $P(A) = P(A \cap B) + P(A \cap B^c)$. It follows that

$$\begin{aligned} P(A \cap B^c) =& P(A) - P(A \cap B) \\ =& P(A) - P(A)P(B) \\ =& P(A)(1 - P(B)) = P(A)P(B^c) \end{aligned}$$ ■

Example 11.4
Show that if A and B are independent so are A^c and B^c.

Solution.
Using De Morgan's formula we have

$$\begin{aligned} P(A^c \cap B^c) =& 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\ =& [1 - P(A)] - P(B) + P(A)P(B) \\ =& P(A^c) - P(B)[1 - P(A)] = P(A^c) - P(B)P(A^c) \\ =& P(A^c)[1 - P(B)] = P(A^c)P(B^c) \end{aligned}$$ ■

When the outcome of one event affects the outcome of a second event, the events are said to be **dependent.** The following is an example of events that are dependent.

Example 11.5
Draw two cards from a deck without replacement, one at a time. Let A = "The first card is a spade," and B = "The second card is a spade." Show that A and B are dependent.

Solution.
We have $P(A) = \frac{13}{52} = \frac{1}{4}$. Since the drawing is without replacement, the event B is conditioned on A and therefore A affects the probability of B, where $P(B) = \frac{3}{51}$. Hence, the events A and B are dependent ■

The definition of independence for a finite number of events is defined as follows: Events $A_1, A_2, \cdots, A_n$ are said to be **mutually independent** or simply **independent** if for any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ we have

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

In particular, three events A, B, C are independent if and only if

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap B \cap C) &= P(A)P(B)P(C) \end{aligned}$$

Example 11.6
Consider the experiment of tossing a coin n times. Let A_i = "the i$^{\text{th}}$ coin shows Heads". Show that $A_1, A_2, \cdots, A_n$ are independent.

Solution.
For any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ we have $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \frac{1}{2^k}$. But $P(A_i) = \frac{1}{2}$. Thus, $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$ ■

Example 11.7
In a clinic laboratory, the probability that a blood sample shows cancerous cells is 0.05. Four blood samples are tested, and the samples are independent.
(a) What is the probability that none shows cancerous cells?
(b) What is the probability that exactly one sample shows cancerous cells?
(c) What is the probability that at least one sample shows cancerous cells?

Solution.
Let H_i denote the event that the i^{th} sample contains cancerous cells for $i = 1, 2, 3, 4$.
(a) The event that none contains cancerous cells is equivalent to $H_1^c \cap H_2^c \cap H_3^c \cap H_4^c$. So, by independence, the desired probability is

$$\begin{aligned} P(H_1^c \cap H_2^c \cap H_3^c \cap H_4^c) &= P(H_1^c)P(H_2^c)P(H_3^c)P(H_4^c) \\ &= (1 - 0.05)^4 = 0.8145. \end{aligned}$$

(b) Let

$$\begin{aligned} A_1 =& H_1 \cap H_2^c \cap H_3^c \cap H_4^c \\ A_2 =& H_1^c \cap H_2 \cap H_3^c \cap H_4^c \\ A_3 =& H_1^c \cap H_2^c \cap H_3 \cap H_4^c \\ A_4 =& H_1^c \cap H_2^c \cap H_3^c \cap H_4 \end{aligned}$$

Then, the requested probability is the probability of the union $A_1 \cup A_2 \cup A_3 \cup A_4$ and these events are mutually exclusive. Also, by independence, $P(A_i) = (0.95)^3(0.05) = 0.0429, i = 1, 2, 3, 4$. Therefore, the answer is $4(0.0429) = 0.1716$.

(c) Let B be the event that no sample contains cancerous cells. The event that at least one sample contains cancerous cells is the complement of B, i.e. B^c. By part (a), it is known that $P(B) = 0.8145$. So, the requested probability is

$$P(B^c) = 1 - P(B) = 1 - 0.8145 = 0.1855 \blacksquare$$

Example 11.8
Find the probability of getting four sixes and then another number in five random rolls of a balanced die.

Solution.
Because the events are independent, the probability in question is

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{7776} \blacksquare$$

A collection of events $A_1, A_2, \cdots, A_n$ are said to be **pairwise independent** if and only if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$ where $1 \leq i, j \leq n$. Pairwise independence does not imply mutual independence as the following example shows.

Example 11.9
Consider the experiment of flipping two fair coins. Consider the three events: $A =$ the first coin shows heads; $B =$ the second coin shows heads, and $C =$ the two coins show the same result. Show that these events are pairwise independent, but not independent.

Solution.
Note that $A = \{(H,H),(H,T)\},\ B = \{(H,H),(T,H)\},\ C = \{(H,H),(T,T)\}$.
We have

$$\begin{aligned} P(A \cap B) =& P(\{(H,H)\}) = \frac{1}{4} = \frac{2}{4} \cdot \frac{2}{4} = P(A)P(B) \\ P(A \cap C) =& P(\{(H,H)\}) = \frac{1}{4} = \frac{2}{4} \cdot \frac{2}{4} = P(A)P(C) \\ P(B \cap C) =& P(\{(H,H)\}) = \frac{1}{4} = \frac{2}{4} \cdot \frac{2}{4} = P(B)P(C) \end{aligned}$$

Hence, the events A, B, and C are pairwise independent. On the other hand

$$P(A \cap B \cap C) = P(\{(H,H)\}) = \frac{1}{4} \neq \frac{2}{4} \cdot \frac{2}{4} \cdot \frac{2}{4} = P(A)P(B)P(C)$$

so that A, B, and C are not independent ■

Practice Problems

Problem 11.1
Determine whether the events are independent or dependent.
(a) Selecting a marble from an urn and then choosing a second marble from the same urn without replacing the first marble.
(b) Rolling a die and spinning a spinner.

Problem 11.2
Amin and Nadia are allowed to have one topping on their ice cream. The choices of toppings are Butterfingers, M and M, chocolate chips, Gummy Bears, Kit Kat, Peanut Butter, and chocolate syrup. If they choose at random, what is the probability that they both choose Kit Kat as a topping?

Problem 11.3
You randomly select two cards from a standard 52-card deck. What is the probability that the first card is not a face card (a king, queen, jack, or an ace) and the second card is a face card if
(a) you replace the first card before selecting the second, and
(b) you do not replace the first card?

Problem 11.4
Marlon, John, and Steve are given the choice for only one topping on their personal size pizza. There are 10 toppings to choose from. What is the probability that each of them orders a different topping?

Problem 11.5 ‡
One urn contains 4 red balls and 6 blue balls. A second urn contains 16 red balls and x blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44 .
Calculate x.

Problem 11.6 ‡
An actuary studying the insurance preferences of automobile owners makes the following conclusions:
(i) An automobile owner is twice as likely to purchase a collision coverage as opposed to a disability coverage.
(ii) The event that an automobile owner purchases a collision coverage is independent of the event that he or she purchases a disability coverage.

(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.
What is the probability that an automobile owner purchases neither collision nor disability coverage?

Problem 11.7 ‡
An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims.
Calculate the probability that a claim submitted to the insurance company includes operating room charges.

Problem 11.8
Let $S = \{1, 2, 3, 4\}$ with each outcome having equal probability $\frac{1}{4}$ and define the events $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{1, 4\}$. Show that the three events are pairwise independent but not independent.

Problem 11.9
Assume A and B are independent events with $P(A) = 0.2$ and $P(B) = 0.3$. Let C be the event that neither A nor B occurs, let D be the event that exactly one of A or B occurs. Find $P(C)$ and $P(D)$.

Problem 11.10
Suppose A, B, and C are mutually independent events with probabilities $P(A) = 0.5$, $P(B) = 0.8$, and $P(C) = 0.3$. Find the probability that at least one of these events occurs.

Problem 11.11
Suppose A, B, and C are mutually independent events with probabilities $P(A) = 0.5, P(B) = 0.8$, and $P(C) = 0.3$. Find the probability that exactly two of the events A, B, C occur.

Problem 11.12
If events A, B, and C are independent, show that
(a) A and $B \cap C$ are independent
(b) A and $B \cup C$ are independent

Problem 11.13
Suppose you flip a nickel, a dime and a quarter. Each coin is fair, and the flips of the different coins are independent. Let A be the event "the total value of the coins that came up heads is at least 15 cents". Let B be the event "the quarter came up heads". Let C be the event "the total value of the coins that came up heads is divisible by 10 cents".
(a) Write down the sample space, and list the events A, B, and C.
(b) Find $P(A), P(B)$ and $P(C)$.
(c) Compute $P(B|A)$.
(d) Are B and C independent? Explain.

Problem 11.14 ‡
Workplace accidents are categorized into three groups: minor, moderate and severe. The probability that a given accident is minor is 0.5, that it is moderate is 0.4, and that it is severe is 0.1. Two accidents occur independently in one month.
Calculate the probability that neither accident is severe and at most one is moderate.

Problem 11.15
Among undergraduate students living on a college campus, 20% have an automobile. Among undergraduate students living off campus, 60% have an automobile. Among undergraduate students, 30% live on campus. Give the probabilities of the following events when a student is selected at random:
(a) Student lives off campus
(b) Student lives on campus and has an automobile
(c) Student lives on campus and does not have an automobile
(d) Student lives on campus or has an automobile
(e) Student lives on campus given that he/she does not have an automobile.

Problem 11.16
Let A, B, and C be three mutually independent events such that $P(A) = 0.1$, $P(B) = 0.3$ and $P(C) = 0.4$. Find $P(A \cup B^c \cup C^c)$.

Problem 11.17
We toss a coin twice. Let A be the event that a head occurs on the first toss. Let B be the event that two heads turn up. Are A and B independent?

Problem 11.18
Let A be an event in a sample space S. Show that A and S are independent.

Problem 11.19
Throw a dice twice. Let A be the event the sum of the points is 7. Let B be the event the first throw came up 3. Let C be the event the second throw came up 4. Show that A, B, and C are pairwise independent but not mutually independent.

Problem 11.20
Throw a dice twice. Let A be the event the first throw came up 1, 2, or 3. Let B be the event that the first throw cam up 3,4, or 5. Let C be the event that the sum of the two throws is 9. Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$ but A, B, and C are not pairwise independent.

Problem 11.21 ‡
In a certain game of chance, a square board with area 1 is colored with sectors of either red or blue. A player, who cannot see the board, must specify a point on the board by giving an $x-$coordinate and a $y-$coordinate. The player wins the game if the specified point is in a blue sector. The game can be arranged with any number of red sectors, and the red sectors are designed so that

$$R_i = \left(\frac{9}{20}\right)^i$$

where R_i is the area of the i^{th} red sector.
Calculate the minimum number of red sectors that makes the chance of a player winning less than 20%.

Problem 11.22 ‡
Two fair dice, one red and one blue, are rolled.
Let I be the event that the number rolled on the red die is odd.
Let J be the event that the number rolled on the blue die is odd.
Let H be the event that the sum of the numbers rolled on the two dice is odd.

Determine which of the following is true.

(A) I, J, and H are not mutually independent, but each pair is independent.
(B) I, J, and H are mutually independent.
(C) Exactly one pair of the three events is independent.
(D) Exactly two of the three pairs are independent.
(E) No pair of the three events is independent.

Problem 11.23 ‡
A company sells two types of life insurance policies (P and Q) and one type of health insurance policy. A survey of potential customers revealed the following:
i) No survey participant wanted to purchase both life policies.
ii) Twice as many survey participants wanted to purchase life policy P as life policy Q.
iii) 45% of survey participants wanted to purchase the health policy.
iv) 18% of survey participants wanted to purchase only the health policy.
v) The event that a survey participant wanted to purchase the health policy was independent of the event that a survey participant wanted to purchase a life policy.
Calculate the probability that a randomly selected survey participant wanted to purchase exactly one policy.

Problem 11.24 ‡
Let A, B, and C be events such that $P(A) = 0.2,\ P(B) = 0.1$, and $P(C) = 0.3$. The events A and B are independent, the events B and C are independent, and the events A and C are mutually exclusive. Calculate $P(A \cup B \cup C)$.

Problem 11.25 ‡
Events E and F are independent with $P(E) = 0.84$ and $P(F) = 0.65$. Calculate the probability that exactly one of the two events occurs.

12 Odds and Conditional Probability

What's the difference between probabilities and odds? To answer this question, let's consider a game that involves rolling a die. If one gets the face 1 then he wins the game, otherwise he loses. The probability of winning is $\frac{1}{6}$ whereas the probability of losing is $\frac{5}{6}$. The odds of winning is 1:5(read 1 to 5). This expression means that the probability of losing is five times the probability of winning. Thus, probabilities describe the frequency of a favorable result in relation to all possible outcomes whereas the **odds in favor** of an event compare the favorable outcomes to the unfavorable outcomes. More formally,

$$\text{odds in favor} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}}.$$

If E is the event of all favorable outcomes then its complementary, E^c, is the event of unfavorable outcomes. Hence,

$$\text{odds in favor} = \frac{n(E)}{n(E^c)}$$

Also, we define the **odds against** an event as

$$\text{odds against} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{\#(E^c)}{\#(E)}.$$

Any probability can be converted to odds, and any odds can be converted to a probability.

Converting Odds to Probability
Suppose that the odds in favor for an event E is a:b. Thus, $\#(E) = ak$ and $\#(E^c) = bk$ where k is a positive integer. Since $S = E \cup E^c$ and $E \cap E^c = \emptyset$, by Theorem 2.3(b) we have $\#(S) = \#(E) + \#(E^c)$. Therefore,

$$P(E) = \frac{\#(E)}{\#(S)} = \frac{\#(E)}{\#(E)+\#(E^c)} = \frac{ak}{ak+bk} = \frac{a}{a+b}$$

and

$$P(E^c) = \frac{\#(E^c)}{\#(S)} = \frac{\#(E^c)}{\#(E)+\#(E^c)} = \frac{bk}{ak+bk} = \frac{b}{a+b}.$$

Example 12.1
If the odds in favor of an event E is 5:4, compute $P(E)$ and $P(E^c)$.

Solution.
We have

$$P(E) = \frac{5}{5+4} = \frac{5}{9} \quad \text{and} \quad P(E^c) = \frac{4}{5+4} = \frac{4}{9} \blacksquare$$

Converting Probability to Odds
Given $P(E)$, we want to find the odds in favor of E and the odds against E. The odds in favor of E are

$$\begin{aligned} \frac{\#(E)}{\#(E^c)} &= \frac{\#(E)}{\#(S)} \cdot \frac{\#(S)}{\#(E^c)} \\ &= \frac{P(E)}{P(E^c)} \\ &= \frac{P(E)}{1 - P(E)} \end{aligned}$$

and the odds against E are

$$\frac{\#(E^c)}{\#(E)} = \frac{1 - P(E)}{P(E)}$$

Example 12.2
For each of the following, find the odds in favor of the event's occurring:
(a) Rolling a number less than 5 on a die.
(b) Tossing heads on a fair coin.
(c) Drawing an ace from an ordinary 52-card deck.

Solution.
(a) The probability of rolling a number less than 5 is $\frac{4}{6}$ and that of rolling 5 or 6 is $\frac{2}{6}$. Thus, the odds in favor of rolling a number less than 5 is $\frac{4}{6} \div \frac{2}{6} = \frac{2}{1}$ or 2:1
(b) Since $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$, the odds in favor of getting heads is $\left(\frac{1}{2}\right) \div \left(\frac{1}{2}\right)$ or 1:1
(c) We have P(ace) $=\frac{4}{52}$ and P(not an ace) $=\frac{48}{52}$ so that the odds in favor of drawing an ace is $\left(\frac{4}{52}\right) \div \left(\frac{48}{52}\right) = \frac{1}{12}$ or 1:12 ■

Remark 12.1
A probability such as $P(E) = \frac{5}{6}$ is just a ratio. The exact number of favorable outcomes and the exact total of all outcomes are not necessarily known.

Practice Problems

Problem 12.1
If the probability of a boy being born is $\frac{1}{2}$, and a family plans to have four children, what are the odds against having all boys?

Problem 12.2
If the odds against Nadia's winning first prize in a chess tournament are 3:5, what is the probability that she will win first prize?

Problem 12.3
What are the odds in favor of getting at least two heads if a fair coin is tossed three times?

Problem 12.4
If the probability of snow for the day is 60%, what are the odds against snowing?

Problem 12.5
On a tote board at a race track, the odds for Smarty Harper are listed as 26:1. Tote boards list the odds that the horse will lose the race. If this is the case, what is the probability of Smarty Harper's winning the race?

Problem 12.6
If a die is tossed, what are the odds in favor of the following events?
(a) Getting a 4
(b) Getting a prime
(c) Getting a number greater than 0
(d) Getting a number greater than 6.

Problem 12.7
Find the odds against E if $P(E) = \frac{3}{4}$.

Problem 12.8
Find $P(E)$ in each case.
(a) The odds in favor of E are 3:4
(b) The odds against E are 7:3

Problem 12.9
A jar contains 5 red balls, 3 green balls, and 7 yellow balls. Without replacement, three balls are chosen randomly. Find the odds against choosing 3 balls, each of a different color.

Problem 12.10
A game between two teams A and B is scheduled next week. You are told that the odds against Team A is 300:1 and that against Team B is 100:1. True or false: The chances that Team A not winning is three times greater than Team B.

Discrete Random Variables

This chapter is one of two chapters dealing with random variables. After introducing the notion of a random variable, we discuss discrete random variables. Continuous random variables are left to the next chapter.

13 Random Variables

By definition, a **random variable** X is a function with domain the sample space and range a subset of the real numbers. For example, in rolling two dice X might represent the sum of the points on the two dice. Similarly, in taking samples of college students X might represent the number of hours per week a student studies, a student's GPA, or a student's height.
The notation $X(s) = x$ means that x is the value associated with the outcome s by the random variable X.
There are three types of random variables: discrete random variables, continuous random variables, and mixed random variables.
A **discrete** random variable is a random variable whose range is either finite or countably infinite[5]. A **continuous** random variable is a random variable whose range is an interval in $\mathbb{R}$. A **mixed** random variable is partially discrete and partially continuous.
In this chapter, we will just consider discrete random variables.

Example 13.1
State whether the random variables are discrete, continuous or mixed.
(a) A coin is tossed ten times. The random variable X is the number of tails that are noted.
(b) A light bulb is burned until it burns out. The random variable Y is its

[5]See page 5.

lifetime in hours.
(c) $Z : (0, 1) \to \mathbb{R}$ where

$$Z(s) = \begin{cases} 1 - s, & 0 < s < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leq s < 1. \end{cases}$$

Solution.
(a) X can only take the values 0, 1, ..., 10, so X is a discrete random variable.
(b) Y can take any positive real value, so Y is a continuous random variable.
(c) Z is a mixed random variable since Z is continuous in the interval $(0, \frac{1}{2})$ and discrete on the interval $[\frac{1}{2}, 1)$ ■

Example 13.2
The sample space of the experiment of tossing a coin 3 times is given by

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Let X = # of Heads in 3 tosses. Find the range of X.

Solution.
We have

$$\begin{array}{cccccccccccc} X(HHH) & = & 3 & X(HHT) & = & 2 & X(HTH) & = & 2 & X(HTT) & = & 1 \\ X(THH) & = & 2 & X(THT) & = & 1 & X(TTH) & = & 1 & X(TTT) & = & 0 \end{array}$$

Thus, the range of X consists of $\{0, 1, 2, 3\}$ so that X is a discrete random variable ■

We use upper-case letters X, Y, Z, etc. to represent random variables. We use small letters x, y, z, etc to represent possible values that the corresponding random variables X, Y, Z, etc. can take. The statement $X = x$ defines an event consisting of all outcomes with X-measurement equal to x which is the set $\{s \in S : X(s) = x\}$. For instance, considering the random variable of the previous example, the statement "X = 2" is the event $\{HHT, HTH, THH\}$. Because the value of a random variable is determined by the outcomes of the experiment, we may assign probabilities to the possible values of the random variable. For example, $P(X = 2) = \frac{3}{8}$.

Example 13.3
Consider the experiment consisting of 2 rolls of a fair 4-sided die. Let X be a random variable, equal to the maximum of the 2 rolls. Complete the following table

x	1	2	3	4
$P(X=x)$				

Solution.
The sample space of this experiment is

$$S = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),$$
$$(3,4),(4,1),(4,2),(4,3),(4,4)\}.$$

Thus,

x	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

■

Example 13.4
A class consisting of five male students and five female students has taken the GRE examination. All ten students got different scores on the test. The students are ranked according to their scores on the test. Assume that all possible rankings are equally likely. Let X denote the highest ranking achieved by a male student. Find $P(X=i), i=1,2,\cdots,10$.

Solution.
The sample space of this experiment consists of all possible rankings of the 10 students. The number of outcomes of this experiment is 10! Now, 6 is the lowest possible rank attainable by the highest-scoring male. Thus, we must have $P(X=7)=P(X=8)=P(X=9)=P(X=10)=0$.
For $X=1$ (male is highest-ranking scorer), we have 5 possible choices out of 10 for the top spot that satisfy this requirement and 9! for the remaining spots. Hence,

$$P(X=1)=\frac{5\cdot 9!}{10!}=\frac{1}{2}.$$

For $X=2$ (male is 2nd-highest scorer), we have 5 possible choices for the top female, then 5 possible choices for the male who ranked 2nd overall, and then any arrangement of the remaining 8 individuals is acceptable (out of 10! possible arrangements of 10 individuals); hence,

$$P(X=2)=\frac{5\cdot 5\cdot 8!}{10!}=\frac{5}{18}.$$

For $X=3$ (male is 3rd-highest scorer), acceptable configurations yield (5)(4)=20 possible choices for the top 2 females, 5 possible choices for the

male who ranked 3rd overall, and 7! different arrangement of the remaining 7 individuals (out of a total of 10! possible arrangements of 10 individuals); hence,

$$P(X = 3) = \frac{5 \cdot 4 \cdot 5 \cdot 7!}{10!} = \frac{5}{36}.$$

Similarly, we have

$$\begin{aligned} P(X = 4) &= \frac{5 \cdot 4 \cdot 3 \cdot 5 \cdot 6!}{10!} = \frac{5}{84} \\ P(X = 5) &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 5!}{10!} = \frac{5}{252} \\ P(X = 6) &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4!}{10!} = \frac{1}{252} \end{aligned}$$

■

Practice Problems

Problem 13.1
Determine whether the random variable is discrete, continuous or mixed.
(a) X is a randomly selected number in the interval $(0, 1)$.
(b) Y is the number of heart beats per minute.
(c) Z is the number of calls at a switchboard in a day.
(d) $U : (0, 1) \to \mathbb{R}$ defined by $U(s) = 2s - 1$.
(e) $V : (0, 1) \to \mathbb{R}$ defined by $V(s) = 2s - 1$ for $0 < s < \frac{1}{2}$ and $V(s) = 1$ for $\frac{1}{2} \leq s < 1$.

Problem 13.2
Two apples are selected at random and removed in succession and without replacement from a bag containing five golden apples and three red apples. List the elements of the sample space, the corresponding probabilities, and the corresponding values of the random variable X, where X is the number of golden apples selected.

Problem 13.3
Suppose that two fair dice are rolled so that the sample space is $S = \{(i, j) : 1 \leq i, j \leq 6\}$. Let X be the random variable $X(i, j) = i + j$. Find $P(X = 6)$.

Problem 13.4
Let X be a random variable with probability distribution table given below

x	0	10	20	50	100
$P(X = x)$	0.4	0.3	0.15	0.1	0.05

Find $P(X < 50)$.

Problem 13.5
You toss a coin repeatedly until you get heads. Let X be the random variable representing the number of times the coin flips until the first head appears. Find $P(X = n)$ where n is a positive integer.

Problem 13.6
A couple is expecting the arrival of a new boy. They are deciding on a name from the list $S = \{$ Steve, Stanley, Joseph, Elija $\}$. Let $X(\omega) =$ first letter in name. Find $P(X = S)$.

Problem 13.7 ‡
The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}, \quad n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

Problem 13.8
Let X be a discrete random variable with the following probability table

x	1	5	10	50	100
$P(X = x)$	0.02	0.41	0.21	0.08	0.28

Compute $P(X > 4 | X \leq 50)$.

Problem 13.9
Shooting is one of the sports listed in the Olympic games. A contestant shoots three times, independently. The probability of hiting the target in the first try is 0.7, in the second try 0.5, and in the third try 0.4. Let X be the discrete random variable representing the number of successful shots among these three.
(a) Find a formula for the piecewise defined function $X : \Omega \to \mathbb{R}$, where Ω denotes the sample space.
(b) Find the event corresponding to $X = 0$. What is the probability that he misses all three shots; i.e., $P(X = 0)$?
(c) What is the probability that he succeeds exactly once among these three shots; i.e $P(X = 1)$?
(d) What is the probability that he succeeds exactly twice among these three shots; i.e $P(X = 2)$?
(e) What is the probability that he makes all three shots; i.e $P(X = 3)$?

Problem 13.10
Let X be a discrete random variable with range $\{0, 1, 2, 3, \cdots\}$. Suppose that

$$P(X = 0) = P(X = 1), \quad P(X = k + 1) = \frac{1}{k} P(X = k), \quad k = 1, 2, 3, \cdots$$

Find $P(0)$.

Problem 13.11 ‡
Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:
(i) $p_n \geq p_{n+1}$ for $0 \leq n \leq 4$.
(ii) The difference between p_n and p_{n+1} is the same for $0 \leq n \leq 4$.
(iii) Exactly 40% of policyholders file fewer than two claims during a given year.
Calculate the probability that a random policyholder will file more than three claims during a given year.

Problem 13.12
A committee of 4 is to be selected from a group consisting of 5 men and 5 women. Let X be the random variable that represents the number of women in the committee. Complete the following table

x	0	1	2	3	4
$p(x)$					

Problem 13.13
Consider the experiment of rolling a fair die twice. Let $X(i, j) = \max\{i, j\}$. Find a formula of $P(X = x)$ in terms of x.

Problem 13.14
Let X be a discrete random variable such that $P(X = n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$, where $n = 0, 1, 2, \cdots$. Find a formula for the function $F(n) = P(X \leq n)$.

Problem 13.15
Toss a pair of fair dice. Let X denote the sum of the dots on the two faces. Find $P(X = x)$ for all the possible values of x.

Problem 13.16
A box of six apples has one rotten apple. Randomly draw one apple from the box, without replacement, until the rotten apple is found. Let X denote the number of apples drawn until the rotten apple is found. Find $P(X = x)$.

Problem 13.17
In the experiment of rolling two dice, let X be the random variable representing the number of even numbers that appear. Find $X(s)$, where $s =$ (dice1,dice2).

Problem 13.18
In the experiment of rolling two dice, let X be the random variable representing the number of even numbers that appear. Find $P(X = x)$.

Problem 13.19
A box contains 100 computer mice of which 95 are defective. One mouse is taken from the box at a time (without replacement) until a non-defective mouse is found. Let Xbe the number of mouses you have to take out in order to find one that is not defective. Find $P(X = n)$.

Problem 13.20
A game consists of randomly selecting two balls without replacement from an urn containing 3 red balls and 4 blue balls. If the two selected balls are of the same color then you win \$2. If they are of different colors then you lose \$1. Let X denote your gain/lost. Find $P(X = x)$.

Problem 13.21 ‡
Two fair dice are rolled. Let X be the absolute value of the difference between the two numbers on the dice. Calculate the probability that $X < 3$.

Problem 13.22 ‡
On a block of ten houses, k are not insured. A tornado randomly damages three houses on the block. The probability that none of the damaged houses are insured is 1/120. Calculate the probability that at most one of the damaged houses is insured.

Problem 13.23 ‡
Four letters to different insureds are prepared along with accompanying envelopes. The letters are put into the envelopes randomly. Calculate the probability that at least one letter ends up in its accompanying envelope.

14 Probability Mass Function and Cumulative Distribution Function

For a discrete random variable X, we define the **probability distribution** or the **probability mass function**(abbreviated pmf) by the equation

$$p(x) = P(X = x).$$

That is, a probability mass function gives the probability that a discrete random variable is exactly equal to some value.
The pmf can be an equation, a table, or a graph that shows how probability is assigned to possible values of the random variable.

Example 14.1
Suppose a variable X can take the values 1, 2, 3, or 4. The probabilities associated with each outcome are described by the following table:

x	1	2	3	4
$p(x)$	0.1	0.3	0.4	0.2

and $p(x) = 0$ for $x \neq 1, 2, 3, 4$. Draw the probability histogram.

Solution.
The probability histogram is shown in Figure 14.1 ■

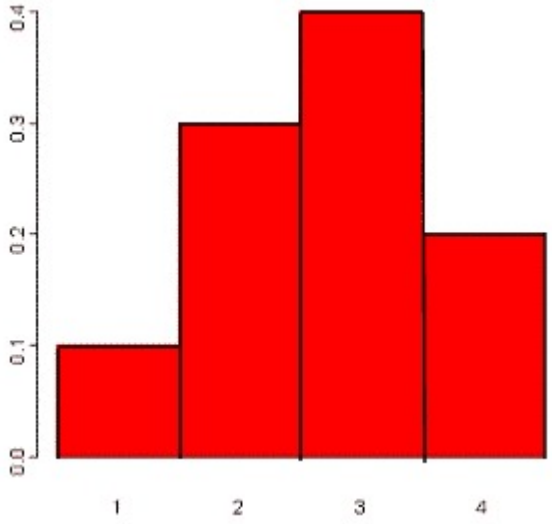


Figure 14.1

Example 14.2
A committee of 4 is to be selected from a group consisting of 5 men and 5 women. Let X be the random variable that represents the number of women in the committee. Create the probability mass distribution.

Solution.
For $x = 0, 1, 2, 3, 4$ we have

$$p(x) = \frac{\begin{pmatrix} 5 \\ x \end{pmatrix}\begin{pmatrix} 5 \\ 4-x \end{pmatrix}}{\begin{pmatrix} 10 \\ 4 \end{pmatrix}}.$$

The probability mass function can be described by the table

x	0	1	2	3	4
$p(x)$	$\frac{5}{210}$	$\frac{50}{210}$	$\frac{100}{210}$	$\frac{50}{210}$	$\frac{5}{210}$

and 0 otherwise ■

Example 14.3
Consider the experiment of rolling a fair die twice. Let $X(i, j) = \max\{i, j\}$. Find the equation of $p(x)$.

Solution.
The pmf of X is

$$\begin{aligned} p(x) &= \begin{cases} \frac{2x-1}{36} & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{2x-1}{36} I_{\{1,2,3,4,5,6\}}(x) \end{aligned}$$

where

$$I_{\{1,2,3,4,5,6\}}(x) = \begin{cases} 1 & \text{if } x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} ■$$

In general, we define the **indicator function** of a set A to be the function

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Note that if the range of a random variable X is $\text{Im}(X) = \{x_1, x_2, \cdots\}$ then

$$p(x) \geq 0, \ x \in \text{Im}(X)$$

and

$$\sum_{x \in \text{Im}(X)} p(x) = 1.$$

All random variables (discrete, continuous or mixed) have a **distribution function** or a **cumulative distribution function**, abbreviated cdf. It is a function giving the probability that the random variable X is less than or equal to x, for every value x. For a discrete random variable, the cumulative distribution function is found by summing up the probabilities. That is,

$$F(a) = P(X \leq a) = \sum_{x \leq a} p(x).$$

Example 14.4
Given the following pmf

$$p(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise.} \end{cases}$$

Find a formula for $F(x)$ and sketch its graph.

Solution.
A formula for $F(x)$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{otherwise.} \end{cases}$$

Its graph is given in Figure 14.2 ■

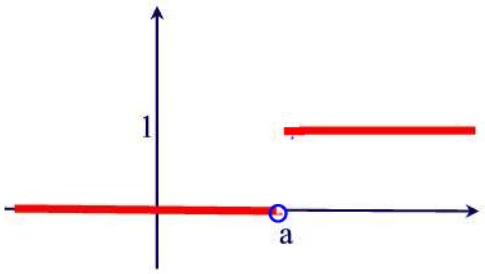


Figure 14.2

For discrete random variables the cumulative distribution function will always be a step function with jumps at each value of x that has probability greater than 0. Note that the value of $F(x)$ is assigned to the top of the jump.

Example 14.5
Consider the following probability mass distribution

x	1	2	3	4
$p(x)$	0.25	0.5	0.125	0.125

and 0 otherwise. Find a formula for $F(x)$ and sketch its graph.

Solution.
The cdf is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.25 & 1 \leq x < 2 \\ 0.75 & 2 \leq x < 3 \\ 0.875 & 3 \leq x < 4 \\ 1 & 4 \leq x. \end{cases}$$

Its graph is given in Figure 14.3 ■

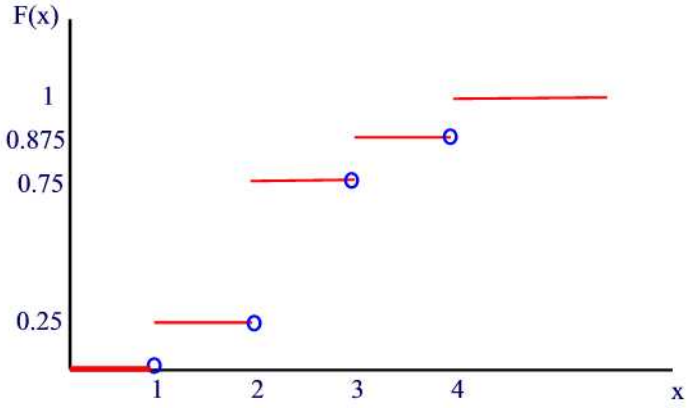


Figure 14.3

Note that the size of the step at any of the values 1,2,3,4 is equal to the probability that X assumes that particular value. That is, we have

Theorem 14.1
If the range of a discrete random variable X consists of the values $x_1 < x_2 < \cdots < x_n$ then $p(x_1) = F(x_1)$ and

$$p(x_i) = F(x_i) - F(x_{i-1}), \quad i = 2, 3, \cdots, n.$$

Proof.
Because $F(x) = 0$ for $x < x_1$, then $F(x_1) = P(X \leq x_1) = P(X < x_1) + P(X = x_1) = P(X = x_1) = p(x_1)$. Now, for $i = 2, 3, \cdots, n$, let $A = \{s \in S : X(s) > x_{i-1}\}$ and $B = \{s \in S : X(s) \leq x_i\}$. Thus, $A \cup B = S$. We have

$$\begin{aligned} P(x_{i-1} < X \leq x_i) =& P(A \cap B) \\ =& P(A) + P(B) - P(A \cup B) \\ =& 1 - F(x_{i-1}) + F(x_i) - 1 \\ =& F(x_i) - F(x_{i-1}) \ \blacksquare \end{aligned}$$

Example 14.6
The cumulative distribution function of X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16} & 0 \leq x < 1 \\ \frac{5}{16} & 1 \leq x < 2 \\ \frac{11}{16} & 2 \leq x < 3 \\ \frac{15}{16} & 3 \leq x < 4. \\ 1 & x \geq 4. \end{cases}$$

Find the pmf of X.

Solution.
Making use of the previous theorem, we get $p(0) = \frac{1}{16}, p(1) = \frac{1}{4}, p(2) = \frac{3}{8}, p(3) = \frac{1}{4}$, and $p(4) = \frac{1}{16}$, and $p(x) = 0$ for $x \geq 5$ $\blacksquare$

Practice Problems

Problem 14.1
Consider the experiment of tossing a fair coin three times. Let X denote the random variable representing the total number of heads.
(a) Describe the probability mass function by a table.
(b) Describe the probability mass function by a histogram.

Problem 14.2
In the previous problem, describe the cumulative distribution function by a formula and by a graph.

Problem 14.3
Toss a pair of fair dice. Let X denote the sum of the dots on the two faces. Find the probability mass function.

Problem 14.4
A box of six apples has one rotten apple. Randomly draw one apple from the box, without replacement, until the rotten apple is found. Let X denote the number of apples drawn until the rotten apple is found. Find the probability mass function of X and draw its histogram.

Problem 14.5
In the experiment of rolling two dice, let X be the random variable representing the number of even numbers that appear. Find the probability mass function of X.

Problem 14.6
Let X be a random variable with pmf

$$p(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n, \quad n = 0, 1, 2, \cdots$$

and 0 otherwise. Find a formula for $F(n)$.

Problem 14.7
A box contains 100 computer mice of which 95 are defective.
(a) One mouse is taken from the box at a time (without replacement) until a non-defective mouse is found. Let X be the number of mouses you have

to take out in order to find one that is not defective. Find the probability distribution of X.
(b) Exactly 10 mouses were taken from the box and then each of the 10 mouses is tested. Let Y denote the number of non-defective mouses among the 10 that were taken out. Find the probability distribution of Y.

Problem 14.8
Let X be a discrete random variable with cdf given by

$$F(x) = \begin{cases} 0 & x < -4 \\ \frac{3}{10} & -4 \leq x < 1 \\ \frac{7}{10} & 1 \leq x < 4 \\ 1 & x \geq 4. \end{cases}$$

Find a formula of $p(x)$.

Problem 14.9
A game consists of randomly selecting two balls without replacement from an urn containing 3 red balls and 4 blue balls. If the two selected balls are of the same color then you win \$2. If they are of different colors then you lose \$1. Let X denote your gain/lost. Find the probability mass function of X.

Problem 14.10
An unfair coin is tossed three times. The probability of tails on any particular toss is known to be $\frac{2}{3}$. Let X denote the number of heads.
(a) Find the probability mass function.
(b) Graph the cumulative distribution function for X.

Problem 14.11
A lottery game consists of matching three numbers drawn (without replacement) from a set of 15 numbers. Let X denote the random variable representing the numbers on your tickets that match the winning numbers. Find the cumulative distribution of X.

Problem 14.12
The cumulative distribution function of X is given by

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{36} & 2 \leq x < 3 \\ \frac{3}{36} & 3 \leq x < 4 \\ \frac{6}{36} & 4 \leq x < 5 \\ \frac{10}{36} & 5 \leq x < 6 \\ \frac{15}{36} & 6 \leq x < 7 \\ \frac{21}{36} & 7 \leq x < 8 \\ \frac{26}{36} & 8 \leq x < 9 \\ \frac{30}{36} & 9 \leq x < 10 \\ \frac{33}{36} & 10 \leq x < 11 \\ \frac{35}{36} & 11 \leq x < 12 \\ 1 & x \geq 12. \end{cases}$$

Find the probability distribution of X.

Problem 14.13
An urn contains 30 marbles of which 8 are black, 12 are red, and 10 are blue. Randomly, select four marbles without replacement. Let X be the number black marbles in the sample of four.
(a) What is the probability that no black marble was selected?
(b) What is the probability that exactly one black marble was selected?
(c) Find a formula for $p(x)$.

Problem 14.14
The distribution function of a discrete random variable X is given by

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \leq x < 0 \\ 0.5 & 0 \leq x < 2.2 \\ 0.6 & 2.2 \leq x < 3 \\ 0.6 + q & 3 \leq x < 4 \\ 1 & x \geq 4. \end{cases}$$

Suppose that $P(X > 3) = 0.1$.
(a) Determine the value of q?
(b) Compute $P(X^2 > 2)$.
(c) Find $p(0),\ p(1)$ and $p(P(X \leq 0))$.
(d) Find the formula of the probability mass function $p(x)$.

Problem 14.15
An urn contains 10 marbles in which 3 are black. Four of the marbles are selected at random (without repalcement) and are tested for the black color. Define the random variable X to be the number of the selected marbles that are not black.
(a) Find the probability mass function of X.
(b) What is the cumulative distribution function of X?

Problem 14.16
Suppose that X is a discrete random variable with probability mass function

$$p(x) = cx^2, \quad x = 1, 2, 3, 4$$

and 0 otherwise. Find the value of c.

Problem 14.17
A discrete random variable X has the following probability mass function defined in tabular form

x	-1	1	2
$p(x)$	2c	3c	4c

and 0 otherwise.
(a) Find the value of c.
(b) Compute $p(-1), p(1)$, and $p(2)$.

Problem 14.18
Let X be a discrete random variable with range $\{1, 2, 3, 4, 5, 6\}$. Suppose that $p(x) = kx$ for some positive constant k and 0 otherwise.
(a) Determine the value of k.
(b) Find $P(X = x)$ for x even.

Problem 14.19
An insurance company sells a one-year automobile policy with a deductible of 2 . The probability that the insured will incur a loss is 0.05 . If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \cdots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur.
(a) Find the value of K.
(b) Find a formula for $p(n) = P(N = n)$.

Problem 14.20
A box contains 7 marbles of which 3 are red and 4 are blue. Randomly select two marbles without replacement. If the marbles are of the same color then you win \$2, otherwise you lose \$1. Let X be the random variable representing your net winnings.
Find the probability mass function of X.

Problem 14.21 ‡
Four distinct integers are chosen randomly and without replacement from the first twelve positive integers. Let X be the random variable representing the second largest of the four selected integers, and let $p(x)$ be the probability mass function of X. Determine $p(x)$, for integer values of x, where $p(x) > 0$.

Problem 14.22 ‡
A health insurance policy covers visits to a doctors office. Each visit costs 100. The annual deductible on the policy is 350. For a policy, the number of visits per year has the following probability distribution:

Number of Visits	0	1	2	3	4	5	6
Probability	0.6	0.15	0.10	0.08	0.04	0.02	0.01

A policy is selected at random from those where costs exceed the deductible. Calculate the probability that this policyholder had exactly five office visits.

Problem 14.23 ‡
Four distinct integers are chosen randomly and without replacement from the first twelve positive integers. Let X be the random variable representing the second smallest of the four selected integers, and let $p(x)$ be the probability mass function of X. Determine $p(x)$, for $x = 2, 3, 4, \cdots$

15 Expected Value of a Discrete Random Variable

A cube has three red faces, two green faces, and one blue face. A game consists of rolling the cube twice. You pay \$2 to play. If both faces are the same color, you are paid \$5(that is you win \$3). If not, you lose the \$2 it costs to play. Will you win money in the long run? Let W denote the event that you win. Then $W = \{RR, GG, BB\}$ and

$$P(W) = P(RR) + P(GG) + P(BB) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6} = \frac{7}{18} \approx 39\%.$$

Thus, $P(L) = \frac{11}{18} = 61\%$. Hence, if you play the game 18 times you expect to win 7 times and lose 11 times on average. So your winnings in dollars will be $3 \times 7 - 2 \times 11 = -1$. That is, you can expect to lose \$1 if you play the game 18 times. On the average, you will lose \$ $\frac{1}{18}$ per game (about 6 cents). This can be found also using the equation

$$3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}.$$

If we let X denote the winnings of this game then the range of X consists of the two numbers 3 and -2 which occur with respective probability 0.39 and 0.61. Thus, we can write

$$E(X) = 3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}.$$

We call this number the expected value of X. More formally, let the range of a discrete random variable X be a sequence of numbers $x_1, x_2, \cdots, x_k$, and let $p(x)$ be the corresponding probability mass function. Then the **expected value** of X is

$$E(X) = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_k p(x_k).$$

The following is a justification of the above formula. Suppose that X has k possible values $x_1, x_2, \cdots, x_k$ and that

$$p_i = P(X = x_i) = p(x_i), i = 1, 2, \cdots, k$$

and 0 otherwise. Suppose that in n repetitions of the experiment, the number of times that X takes the value x_i is n_i. Then the sum of the values of X over the n repetitions is

$$n_1x_1 + n_2x_2 + \cdots + n_kx_k$$

and the average value of X is

$$\frac{n_1x_1 + n_2x_2 + \cdots + n_kx_k}{n} = \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2 + \cdots + \frac{n_k}{n}x_k.$$

But $p(x_i) = P(X = x_i) = \lim_{n\to\infty} \frac{n_i}{n}$. Thus, the average value of X approaches

$$E(X) = x_1p(x_1) + x_2p(x_2) + \cdots + x_kp(x_k).$$

The expected value of X is also known as the **mean** value.

Example 15.1 ‡
Suppose that an insurance company has broken down yearly automobile claims for drivers from age 16 through 21 as shown in the following table.

Amount of claim	Probability
\$ 0	0.80
\$ 2000	0.10
\$ 4000	0.05
\$ 6000	0.03
\$ 8000	0.01
\$ 10000	0.01

How much should the company charge as its average premium in order to break even on costs for claims?

Solution.
Let X be the random variable of the amount of claim to be paid by the insurance. Finding the expected value of X, we have

$$E(X) = 0(.80)+2000(.10)+4000(.05)+6000(.03)+8000(.01)+10000(.01) = 760.$$

Since the average claim value is \$760, the average automobile insurance premium should be set at \$760 per year for the insurance company to break even ■

Example 15.2
Let A be a nonempty set. Consider the random variable I with range 0 and 1 and with pmf the indicator function I_A where

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(I)$.

Solution.
Since $I_A(I = 1) = P(A)$ and $I_A(I = 0) = P(A^c)$, we have

$$E(I) = 1 \cdot I_A(I = 1) + 0 \cdot I_A(I = 0) = 1 \cdot P(A) + 0 \cdot P(A^c) = P(A).$$

That is, the expected value of I is just the probability of A ■

Example 15.3
An insurance policy provides the policyholder with a payment of \$1,000 if a death occurs within 5 years. Let X be the random variable of the amount paid by an insurance company to the policyholder. Suppose that the probability of death of the policyholder within 5 years is estimated to be 0.15.
(a) Find the probability distribution of X.
(b) What is the most the policyholder should be willing to pay for this policy?

Solution.
(a) $P(X = 1,000) = 0.15,\ P(X = 0) = 0.85$ and 0 otherwise.
(b) $E(X) = 1000 \times 0.15 + 0 \times 0.85 = 150$. Thus, the policyholder expected payout is \$150, so he/she should not be willing to pay more than \$150 for the policy ■

Example 15.4
You have a fancy car video system in your car and you feel you want to insure it against theft. An insurance company offers you a \$2000 1-year coverage for a premium of \$225. The probability that the theft will occur is 0.1. What is your expected return from this policy?

Solution.
Let X be the random variable of the profit/loss from this policy to policyholder. Then either $X = 1,775$ with probability 0.1 or $X = -225$ with probability 0.9. Thus, the expected return of this policy is

$$E(X) = 1,775(0.1) + (-225)(0.9) = -\$25.$$

That is, by insuring the car video system for many years, and under the same circumstances, you will expect a net loss of $25 per year to the insurance company ■

Remark 15.1
The expected value (or mean) is related to the physical property of center of mass. If we have a weightless rod in which weights of mass $P(x)$ located at a distance x from the left endpoint of the rod then the point at which the rod is balanced is called the **center of mass.** If α is the center of mass then we must have $\sum_x (x - \alpha)p(x) = 0$. This equation implies that $\alpha = \sum_x xp(x) = E(X)$. Thus, the expected value tells us something about the center of the probability mass function.

Practice Problems

Problem 15.1
Consider the experiment of rolling two dice. Let X be the random variable representing the sum of the two faces. Find $E(X)$.

Problem 15.2
A game consists of rolling two dice. The sum of the two faces is a positive integer between 2 and 12. For each such a value, you win an amount of money as shown in the table below.

Score	2	3	4	5	6	7	8	9	10	11	12
$ won	4	6	8	10	20	40	20	10	8	6	4

Compute the expected value of this game.

Problem 15.3
A game consists of rolling two dice. The game costs \$2 to play. If a sum of 7 appears you win \$10 otherwise you lose your \$2. Would you be making money, losing money, or coming out about even if you keep playing this game? Explain.

Problem 15.4
A game consists of rolling two dice. The game costs \$8 to play. You get paid the sum of the numbers in dollars that appear on the dice. What is the expected value of this game (long-run average gain or loss per game)?

Problem 15.5
A storage company provides insurance coverage for items stored on its premises. For items valued at \$800, the probability that \$400 worth of items of being stolen is 0.01 while the probability the whole items being stolen is 0.0025. Assume that these are the only possible kinds of expected loss. How much should the storage company charge for people with this coverage in order to cover the money they pay out and to make an additional \$20 profit per person on the average?

Problem 15.6
A game consists of spinning a spinner with payoff as shown in Figure 15.1. The cost of playing is \$2 per spin. What is the expected return to the owner

of this game?

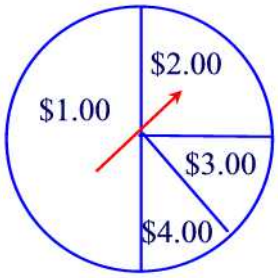


Figure 15.1

Problem 15.7
Consider a game that costs $1 to play. The probability of losing is 0.7. The probability of winning $50 is 0.1, and the probability of winning $35 is 0.2. Would you expect to win or lose if you play this game 10 times?

Problem 15.8
A lottery type game consists of matching the correct three numbers that are selected from the numbers 1 through 12. The cost of one ticket is $1. If your ticket matched the three selected numbers, you win $100. What are your expected earnings?

Problem 15.9 ‡
Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96 .
What is the expected excess of premiums over claims, given that the husband survives at least ten years?

Problem 15.10
An urn contains 30 marbles of which 8 are black, 12 are red, and 10 are blue. Randomly, select four marbles without replacement. Let X be the number black marbles in the sample of four.
(a) What is the probability that no black marble was selected?
(b) What is the probability that exactly one black marble was selected?
(c) Compute $E(X)$.

Problem 15.11
The distribution function of a discrete random variable X is given by

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.2 & -2 \leq x < 0 \\ 0.5 & 0 \leq x < 2.2 \\ 0.6 & 2.2 \leq x < 3 \\ 0.6 + q & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Suppose that $P(X > 3) = 0.1$.
(a) Determine the value of q?
(b) Compute $P(X^2 > 2)$.
(c) Find $p(0),\ p(1)$ and $p(P(X \leq 0))$.
(d) Find the formula of the probability mass function $p(x)$.
(e) Compute $E(X)$.

Problem 15.12
A computer store specializes in selling used laptops. The laptops can be classified as either in good condition or in fair condition. Assume that the store salesperson is able to tell whether a laptop is in good or fair condition. However, a buyer in the store can not tell the difference. Suppose that buyers are aware that the probability of a laptop of being in good condition is 0.4. A laptop in good condition costs the store \$400 and a buyer is willing to pay \$525 for it whereas a laptop in fair condition costs the store \$200 and a buyer is willing to pay for \$300 for it.
(a) Find the expected value of a used laptop to a buyer who has no extra information.
(b) Assuming that buyers will not pay more than their expected value for a used laptop, will sellers ever sell laptops in good condition?

Problem 15.13
An urn contains 10 marbles in which 3 are black. Four of the marbles are selected at random (without replacement) and are tested for the black color. Define the random variable X to be the number of the selected marbles that are not black.
(a) Find the probability mass function of X.
(b) What is the cumulative distribution function of X?
(c) Find the expected value of X.

Problem 15.14 ‡
An auto insurance company is implementing a new bonus system. In each month, if a policyholder does not have an accident, he or she will receive a 5.00 cash-back bonus from the insurer.
Among the 1,000 policyholders of the auto insurance company, 400 are classified as low-risk drivers and 600 are classified as high-risk drivers.
In each month, the probability of zero accidents for high-risk drivers is 0.80 and the probability of zero accidents for low-risk drivers is 0.90.
Calculate the expected bonus payment from the insurer to the 1000 policyholders in one year.

Problem 15.15
Suppose that X is a discrete random variable with probability mass function

$$p(x) = cx^2, \quad x = 1, 2, 3, 4$$

and 0 otherwise.
(a) Find the value of c.
(b) Find $E(X)$.

Problem 15.16
A random variable X has the following probability mass function defined in tabular form

x	-1	1	2
$p(x)$	2c	3c	4c

and 0 otherwise.
(a) Find the value of c.
(b) Compute $p(-1), p(1)$, and $p(2)$.
(c) Find $E(X)$.

Problem 15.17
Let X be a random variable with range $\{1, 2, 3, 4, 5, 6\}$. Suppose that $p(x) = kx$ for some positive constant k and 0 otherwise.
(a) Determine the value of k.
(b) Find $P(X = x)$ for x even.
(c) Find the expected value of X.

Problem 15.18
Consider a random variable X whose probability mass function is given by

$$p(x) = \begin{cases} 0.2 & x = -1 \\ 0.3 & x = 0 \\ 0.1 & x = 0.2 \\ 0.1 & x = 0.5 \\ 0.3 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X)$.

Problem 15.19
A box contains 7 marbles of which 3 are red and 4 are blue. Randomly select two marbles without replacement. If the marbles are of the same color then you win \$2, otherwise you lose \$1. Let X be the random variable representing your net winnings.
(a) Find the probability mass function of X.
(b) Compute $E(X)$.

Problem 15.20
A probability distribution of the claim size X for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

Find $E(X)$.

Problem 15.21 ‡
At a polling booth, ballots are cast by ten voters, of whom three are Republicans, two are Democrats, and five are Independents. A local journalist interviews two of these voters, chosen randomly.
Calculate the expectation of the absolute value of the difference between the number of Republicans interviewed and the number of Democrats interviewed.

Problem 15.22 ‡
The number of claims X on a health insurance policy is a random variable with $E(X^2) = 61$ and $E[(X-1)^2] = 47$. Calculate the standard deviation of the number of claims.

16 Expected Value of a Function of a Discrete Random Variable

If we apply a function $g(\cdot)$ to a random variable X, the result is another random variable $Y = g(X)$. For example, $X^2, \log X, \frac{1}{X}$ are all random variables derived from the original random variable X.
In this section, we are interested in finding the expected value of this new random variable. But first we look at an example.

Example 16.1
Let X be a discrete random variable with range $\{-1, 0, 1\}$ and probabilities $P(X = -1) = 0.2, P(X = 0) = 0.5,\ P(X = 1) = 0.3$ and 0 otherwise. Compute $E(X^2)$.

Solution.
Let $Y = X^2$. Then the range of Y is $\{0, 1\}$. Also, $P(Y = 0) = P(X = 0) = 0.5$ and $P(Y = 1) = P(X = -1) + P(X = 1) = 0.2 + 0.3 = 0.5$ Thus, $E(X^2) = 0(0.5)+1(0.5) = 0.5$. Note that $E(X) = -1(0.2)+0(0.5)+1(0.3) = 0.1$ so that $E(X^2) \neq (E(X))^2$ ■

Now, if X is a discrete random variable and $g(x) = x$ then we know that

$$E(g(X)) = E(X) = \sum_{x \in D} xp(x)$$

where D is the range of X and $p(x)$ is its probability mass function. This suggests the following result for finding $E(g(X))$.

Theorem 16.1
If X is a discrete random variable with range D and pmf $p(x)$, then the expected value of the random variable $g(X)$ is computed by

$$E(g(X)) = \sum_{x \in D} g(x)p(x)$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a real function.

Proof.
Let D be the range of X and D' be the range of $g(X)$. Then,

$$D' = g(D) = \{y = g(x) : x \in D\}.$$

Let p_Y denote the probability mass function of the random variable $Y = g(X)$. Then the expected value of Y is

$$E(Y) = \sum_{y \in D'} y p_Y(y) = \sum_{y \in D'} y P(Y = y) = \sum_{y \in D'} y P\left(\{s \in S : g(X)(s) = y\}\right).$$

For $y \in D'$, we let $A_y = \{x \in D : g(x) = y\}$. Since g may not be one-to-one, A_y might have more than one element. We will show that $\{s \in S : g(X)(s) = y\} = \bigcup_{x \in A_y} \{s \in S : X(s) = x = g^{-1}(y)\}$. The prove is by double inclusions. Let $s \in S$ be such that $g(X)(s) = g(X(s)) = y$. Since $X(s) \in D$, there is an $x \in D$ such that $x = X(s)$ and $g(x) = y$. This shows that $s \in \cup_{x \in A_y} \{s \in S : X(s) = x\}$. For the converse, let $s \in \cup_{x \in A_y} \{s \in S : X(s) = x\}$. Then there exists $x \in D$ such that $g(x) = y$ and $X(s) = x$. Hence, $g(X)(s) = g(x) = y$ and this implies that $s \in \{s \in S : g(X)(s) = y\}$. Next, if x_1 and x_2 are two distinct elements of A_y and $w \in \{s \in S : X(s) = x_1\} \cap \{t \in S : X(t) = x_2\}$ then this leads to $x_1 = x_2$, a contradiction. Hence, $\{s \in S : X(s) = x_1\} \cap \{t \in S : X(t) = x_2\} = \emptyset$.
From, the above discussion, we have

$$p_Y(y) = \sum_{x \in A_y} P(X = x) = \sum_{x \in A_y} p(x) \tag{16.1}$$

and

$$E(g(X)) = E(Y) = \sum_{y \in D'} \sum_{x \in A_y} y p(x) = \sum_{y \in D'} \sum_{x \in A_y} g(x) p(x) = \sum_{x \in D} g(x) p(x) \blacksquare$$

Remark 16.1
In case D is infinite, the convergence of the sum is absolutely convergent.

Example 16.2
Let X be the number of points on the side that comes up when rolling a fair die. Find the expected value of $g(X) = 2X^2 + 1$.

Solution.
Since each possible outcome has the probability $\frac{1}{6}$, we get

$$\begin{aligned} E[g(X)] =& \sum_{i=1}^{6}(2i^2+1)\cdot\frac{1}{6} \\ =& \frac{1}{6}\left(6+2\sum_{i=1}^{6} i^2\right) \\ =& \frac{1}{6}\left(6+2\frac{6(6+1)(2\cdot 6+1)}{6}\right) = \frac{94}{3} \end{aligned}$$ ■

As a consequence of the above theorem we have the following result.

Corollary 16.1
If X is a discrete random variable, then for any constants a and b we have

$$E(aX+b) = aE(X)+b.$$

Proof.
Let D denote the range of X. Then

$$\begin{aligned} E(aX+b) =& \sum_{x\in D}(ax+b)p(x) \\ =& a\sum_{x\in D} xp(x) + b\sum_{x\in D} p(x) \\ =& aE(X)+b \end{aligned}$$ ■

A similar argument establishes

$$E(aX^2+bX+c) = aE(X^2)+bE(X)+c.$$

Example 16.3
Let X be a random variable with $E(X) = 6$ and $E(X^2) = 45$, and let $Y = 20-2X$. Find $E(Y)$ and $E(Y^2)-[E(Y)]^2$.

Solution.
By the properties of expectation,

$$\begin{aligned} E(Y) =& E(20-2X) = 20 - 2E(X) = 20 - 12 = 8 \\ E(Y^2) =& E(400 - 80X + 4X^2) = 400 - 80E(X) + 4E(X^2) = 100 \end{aligned}$$

$$E(Y^2) - (E(Y))^2 = 100 - 64 = 36 \blacksquare$$

We conclude this section with the following definition. If $g(x) = x^n$ then we call $E(X^n) = \sum_x x^n p(x)$ the n^{th} **moment about the origin** of X or the n^{th} **raw moment**. Thus, $E(X)$ is the first moment of X.

Example 16.4
Show that $E(X^2) = E(X(X-1)) + E(X)$.

Solution.
Let D be the range of X. We have

$$\begin{aligned} E(X^2) &= \sum_{x \in D} x^2 p(x) \\ &= \sum_{x \in D} (x(x-1) + x) p(x) \\ &= \sum_{x \in D} x(x-1)p(x) + \sum_{x \in D} x p(x) = E(X(X-1)) + E(X) \blacksquare \end{aligned}$$

Remark 16.2
In our definition of expectation the set D can be countably infinite. It is possible to have a random variable with undefined expectation as seen in the next example.

Example 16.5
The probability distribution of X is given by

$$p(x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3, \cdots$$

and 0 otherwise. Show that $E(2^X)$ does not exist.

Solution.
We have

$$E(2^X) = (2^1)\frac{1}{2^1} + (2^2)\frac{1}{2^2} + \cdots = \sum_{n=1}^{\infty} 1$$

The series on the right is divergent so that $E(2^X)$ does not exist $\blacksquare$

Practice Problems

Problem 16.1

Suppose that X is a discrete random variable with probability mass function

$$p(x) = cx^2, \quad x = 1, 2, 3, 4$$

and 0 otherwise.
(a) Find the value of c.
(b) Find $E(X)$.
(c) Find $E(X(X-1))$.

Problem 16.2

A random variable X has the following probability mass function defined in tabular form

x	-1	1	2
$p(x)$	2c	3c	4c

and 0 otherwise.
(a) Find the value of c.
(b) Compute $p(-1), p(1)$, and $p(2)$.
(c) Find $E(X)$ and $E(X^2)$.

Problem 16.3

Let X be a random variable with range $\{1, 2, 3, 4, 5, 6\}$. Suppose that $p(x) = kx$ for some positive constant k and 0 otherwise.
(a) Determine the value of k.
(b) Find $P(X = x)$ for x even.
(c) Find the expected value of X.

Problem 16.4

Let X be a discrete random variable. Show that $E(aX^2+bX+c) = aE(X^2)+bE(X)+c$.

Problem 16.5

Consider a random variable X whose probability mass function is given by

$$p(x) = \begin{cases} 0.1 & x = -3 \\ 0.2 & x = 0 \\ 0.3 & x = 2.2 \\ 0.1 & x = 3 \\ 0.3 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(x)$ be the corresponding cdf. Find $E(F(X))$.

Problem 16.6 ‡
An insurance policy pays 100 per day for up to 3 days of hospitalization and 50 per day for each day of hospitalization thereafter.
The number of days of hospitalization, X, is a discrete random variable with probability function

$$p(k) = \begin{cases} \frac{6-k}{15} & k = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the expected payment for hospitalization under this policy.

Problem 16.7 ‡
An insurance company sells a one-year automobile policy with a deductible of 2 . The probability that the insured will incur a loss is 0.05 . If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \cdots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur.
Determine the net premium for this policy.

Problem 16.8
Consider a random variable X whose probability mass function is given by

$$p(x) = \begin{cases} 0.2 & x = -1 \\ 0.3 & x = 0 \\ 0.1 & x = 0.2 \\ 0.1 & x = 0.5 \\ 0.3 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(p(x))$.

Problem 16.9
A box contains 7 marbles of which 3 are red and 4 are blue. Randomly select two marbles without replacement. If the marbles are of the same color then you win \$2, otherwise you lose \$1. Let X be the random variable representing your net winnings.
(a) Find the probability mass function of X.
(b) Compute $E(2^X)$.

Problem 16.10
Three kinds of tickets are sold at a movie theater: children (for \$3), adult (for \$8), and seniors (for \$5). Let C denote the number of children tickets sold, A number of adult tickets, and S number of senior tickets. You are given: $E[C] = 45,\ E[A] = 137,\ E[S] = 34$. Assume the number of tickets sold is independent.
Any particular movie costs \$300 to show, regardless of the audience size.
(a) Write a formula relating C, A, and S to the theater's profit P for a particular movie.
(b) Find $E(P)$.

Problem 16.11
A probability distribution of the claim size X for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

Find $E(X^2) - (E(X))^2$.

Problem 16.12
A discrete random variable, X, has probability mass function

$$p(x) = c(x-3)^2, \quad x = -2, -1, 0, 1, 2$$

and 0 otherwise.
(a) Find the value of the constant c.
(b) Find $E(X^2) - (E(X))^2$.

Problem 16.13
An urn contains 10 marbles in which 3 are black. Four of the marbles are selected at random and are tested for the black color. Define the random variable X to be the number of the selected marbles that are not black.
(a) Find the probability mass function of X.
(b) Find $E(X^2) - (E(X))^2$.

Problem 16.14
Suppose that X is a discrete random variable with probability mass function

$$p(x) = cx^2, \quad x = 1, 2, 3, 4$$

and 0 otherwise.
(a) Find the value of c.
(b) Find $E(X)$ and $E(X(X-1))$.
(c) Find $E(X^2) - (E(X))^2$.

Problem 16.15
Suppose X is a random variable with $E(X) = 4$ and $E(X^2) - (E(X))^2 = 9$. Let $Y = 4X + 5$. Compute $E(Y^2) - (E(Y))^2$.

Problem 16.16
A box contains 3 red and 4 blue marbles. Two marbles are randomly selected without replacement. If they are the same color then you win \$2. If they are of different colors then you lose \$ 1. Let X denote the amount you win.
(a) Find the probability mass function of X.
(b) Compute $E(X^2) - (E(X))^2$.

Problem 16.17
Let X be a discrete random variable with probability mass function is given by

x	-4	1	4
$p(x)$	0.3	0.4	0.3

and 0 otherwise. Find $E(X^2) - (E(X))^2$.

Problem 16.18
Let X be a discrete random variable with probability distribution $p(0) = 1 - p, p(1) = p$, and 0 otherwise, where $0 < p < 1$. Find $E(X^2) - (E(X))^2$.

Problem 16.19
Let X be a discrete random variable with probability mass function

$$p(r) = C(n, r)p^r q^{n-r}, \ r = 0, 1, 2, \cdots, n$$

and 0 otherwise, where n is a positive integer and $p+q = 1$. Find $E(X), E(X(X-1))$ and $E(X^2)$. Hint: Use the Binomial Theorem.

Problem 16.20
Let X be a discrete random variable. Prove that $E[(X-E(X))^2] = E(X^2) - (E(X))^2$.

Problem 16.21 ‡
A policy covers a gas furnace for one year. During that year, only one of three problems can occur:
i) The igniter switch may need to be replaced at a cost of 60. There is a 0.10 probability of this.
ii) The pilot light may need to be replaced at a cost of 200. There is a 0.05 probability of this.
iii) The furnace may need to be replaced at a cost of 3000. There is a 0.01 probability of this.
Calculate the deductible that would produce an expected claim payment of 30.

17 Variance and Standard Deviation of a Discrete Random Variable

In the previous section we learned how to find the expected values of various functions of random variables. The most important of these are the variance and the standard deviation which give an idea about how spread out the probability mass function is about its expected value.
The expected squared distance between the random variable and its mean is called the **variance** of the random variable. The positive square root of the variance is called the **standard deviation** of the random variable. If σ_X denotes the standard deviation then the variance is given by the formula

$$\text{Var(X)} = \sigma_X^2 = E\left[(X - E(X))^2\right].$$

If $\{x_1, x_2, \cdots, x_n\}$ is the range of X and $p(x)$ is the pmf of X then

$$\text{Var}(X) = \sum_{i=1}^{n} (x_i - E(X))^2 p(x_i) \geq 0.$$

The variance of a random variable is typically calculated using the following formula

$$\begin{aligned}
\text{Var}(X) =& E[(X - E(X))^2] \\
=& E[X^2 - 2XE(X) + (E(X))^2] \\
=& E(X^2) - 2E(X)E(X) + (E(X))^2 \\
=& E(X^2) - (E(X))^2
\end{aligned}$$

where we have used the result of Problem 16.4.

Example 17.1
Find the variance of the random variable X with probability distribution $P(X = 1) = P(X = -1) = \frac{1}{2}$ and 0 otherwise.

Solution.
Since $E(X) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$ and $E(X^2) = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$, we find $\text{Var}(X) = 1 - 0 = 1$ ■

A useful identity is given in the following result.

Theorem 17.1
If X is a discrete random variable then for any constants a and b we have

$$\text{Var}(aX+b) = a^2\text{Var}(X).$$

Proof.
Since $E(aX+b) = aE(X)+b$, we have

$$\begin{aligned}\text{Var}(aX+b) =& E\left[(aX+b-E(aX+b))^2\right]\\ =& E[a^2(X-E(X))^2]\\ =& a^2E((X-E(X))^2)\\ =& a^2\text{Var}(X)\ \blacksquare\end{aligned}$$

Remark 17.1
Note that the units of Var(X) is the square of the units of X. This motivates the definition of the standard deviation $\sigma_X = \sqrt{\text{Var}(X)}$ which is measured in the same units as X.

Example 17.2
In a recent study, it was found that tickets cost to the Dallas Cowboys football games averages \$80 with a variance of 105 square dollar. What will be the variance of the cost of tickets if 3% tax is charged on all tickets?

Solution.
Let X be the current ticket price and Y be the new ticket price. Then $Y = 1.03X$. Hence,

$$\text{Var}(Y) = \text{Var}(1.03X) = 1.03^2\text{Var}(X) = (1.03)^2(105) = 111.3945\ \blacksquare$$

Example 17.3
In the experiment of rolling one die, let X be the number on the face that comes up. Find the variance and standard deviation of X.

Solution.
We have

$$E(X) = (1+2+3+4+5+6)\cdot\frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

and

$$E(X^2) = (1^2+2^2+3^2+4^2+5^2+6^2)\cdot\frac{1}{6} = \frac{91}{6}.$$

Thus,

$$\text{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

The standard deviation is

$$\sigma_X = \sqrt{\frac{35}{12}} \approx 1.7078 \blacksquare$$

Theorem 17.2
For any discrete random variable X, we have

$$\text{Var}(X) = E(X) + E(X(X-1)) - [E(X)]^2.$$

Proof.
From the identity $\text{Var}(X) = E(X^2) - [E(X)]^2$, we find $E(X^2) = \text{Var}(X) + [E(X)]^2$. Thus,

$$\begin{aligned} E(X(X-1)) =& E(X^2 - X) = E(X^2) - E(X) \\ =& \text{Var}(X) + [E(X)]^2 - E(X). \end{aligned}$$

Solving for $\text{Var}(X)$, we find $\text{Var}(X) = E(X) + E(X(X-1)) - [E(X)]^2 \blacksquare$

Practice Problems

Problem 17.1 ‡
A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

Problem 17.2 ‡
The annual cost of maintaining and repairing a car averages 200 with a variance of 260. what will be the variance of the annual cost of maintaining and repairing a car if 20% tax is introduced on all items associated with the maintenance and repair of cars?

Problem 17.3
A discrete random variable, X, has probability mass function

$$p(x) = c(x-3)^2, \quad x = -2, -1, 0, 1, 2$$

and 0 otherwise.
(a) Find the value of the constant c.
(b) Find the mean and variance of X.

Problem 17.4
An urn contains 10 marbles in which 3 are black. Four of the marbles are selected at random and are tested for the black color. Define the random variable X to be the number of the selected marbles that are not black.
(a) Find the probability mass function of X.
(b) Find the variance of X.

Problem 17.5
Suppose that X is a discrete random variable with probability mass function

$$p(x) = cx^2, \quad x = 1, 2, 3, 4$$

and 0 otherwise.
(a) Find the value of c.
(b) Find $E(X)$ and $E(X(X-1))$.
(c) Find $\text{Var}(X)$.

Problem 17.6
Suppose X is a random variable with $E(X) = 4$ and $\text{Var}(X) = 9$. Let $Y = 4X + 5$. Compute $E(Y)$ and $\text{Var}(Y)$.

Problem 17.7
A box contains 3 red and 4 blue marbles. Two marbles are randomly selected without replacement. If they are the same color then you win \$2. If they are of different colors then you lose \$ 1. Let X denote the amount you win.
(a) Find the probability mass function of X.
(b) Compute $E(X)$ and $E(X^2)$.
(c) Find $\text{Var}(X)$.

Problem 17.8
Let X be a discrete random variable with probability mass function is given by

x	-4	1	4
$p(x)$	0.3	0.4	0.3

and 0 otherwise. Find the variance and the standard deviation of X.

Problem 17.9
Let X be a random variable with probability distribution $p(0) = 1-p, p(1) = p$, and 0 otherwise, where $0 < p < 1$. Find $E(X)$ and $Var(X)$.

Problem 17.10
Let X be a discrete random variable with probability mass function

$$p(r) = P(X = r) = C(n, r)p^r q^{n-r}, \ r = 0, 1, 2, \cdots, n$$

and 0 otherwise, where n is a positive integer and $p + q = 1$. Find $\text{Var}(X)$.

Problem 17.11
A random variable X is said to be a **Poisson** random variable with parameter $\lambda > 0$ if its probability mass function has the form

$$p(k) = P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

and 0 otherwise.
(a) Find $E(X), E(X(X-1))$ and $E(X^2)$.
(b) Find Var(X).

Problem 17.12
A **geometric** random variable X with parameter $p, 0 < p < 1$ has a probability mass function

$$p(n) = P(X = n) = p(1-p)^{n-1}, \quad n = 1, 2, \cdots$$

and 0 otherwise.
(a) Using the geometric series $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, find $f'(1-p)$ and $f''(1-p)$.
(b) Find $E(X)$, $E(X(X-1))$, and $E(X^2)$.
(c) Find Var(X).

Problem 17.13
The cumulative distribution function of X is given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < -4 \\ 0.3 & -4 \leq x < 1 \\ 0.7 & 1 \leq x < 4 \\ 1 & x \geq 4. \end{cases}$$

(a) Find the probability mass function.
(b) Find the variance and the standard deviation of X.

Problem 17.14
The probability mass function of a discrete random variable X is given by

$$p(x) = \begin{cases} 0 & x < 4 \\ 0.1 & x = 4 \\ 0.3 & x = 5 \\ 0.3 & x = 6 \\ 0.2 & x = 8 \\ 0.1 & x = 9 \\ 0 & x > 9. \end{cases}$$

Find $\text{Var}(X)$.

Problem 17.15
Let X be the random variable of the previous problem. Define $Y = 5 - 2X$. Find $\text{Var}(Y)$.

Problem 17.16
A **uniform discrete random variable** X with parameters n is a random variable with probability mass function given by

$$p(x) = \frac{1}{n}, \quad x = 1, 2, \cdots, n$$

and 0 otherwise. Find $E(X)$ and $\text{Var}(X)$. An example of such a random variable is the roll of a fair die where X denotes the face of the die.

Problem 17.17
In the experiment of rolling one die, let X be the number on the face that comes up. Find the mean and the variance of X using the previous problem.

Problem 17.18
A **uniform discrete random variable** X with parameters a and b, where $a, b \in \mathbb{N},\ a < b$, is a random variable with probability mass function given by

$$p(x) = \frac{1}{b - a + 1}, \quad x = a, a + 1, \cdots, b$$

and 0 otherwise, where $b - a + 1$ is the number of values that X may take. Find $E(X), E(X^2)$ and $\text{Var}(X)$.

Problem 17.19
Let X be a discrete random variable and $g : \mathbb{R} \to \mathbb{R}$ be a real function. Show that $\text{Var}(ag(X) + b) = a^2\text{Var}(g(X))$ where a and b are constants.

Problem 17.20
Let X be a random variable. Define $Z = \frac{X - E(X)}{\sigma_X}$. Find $E(Z)$ and $\text{Var}(Z)$.

Problem 17.21 ‡
An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. For every full ten inches of snow in excess of 40 inches during the winter season, the insurer pays the airport 300 up to a policy maximum of 700. The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	$[90,\infty)$
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Problem 17.22 ‡

An insurance company has an equal number of claims in each of three territories. In each territory, only three claim amounts are possible: 100, 500, and 1000. Based on the companys data, the probabilities of each claim amount are:

	Claim Amount		
	100	500	1000
Territory 1	0.90	0.08	0.02
Territory 2	0.80	0.11	0.09
Territory 3	0.70	0.20	0.10

Calculate the standard deviation of a randomly selected claim amount.

Commonly Used Discrete Random Variables

In this chapter, we consider the discrete random variables listed in the exam's syllabus: uniform, binomial, negative binomial, geometric, hyper-geometric, and Poisson.

18 Uniform Discrete Random Variable

Uniform discrete random variables are the simplest discrete random variables. Let X be a discrete random variable defined on a sample space S such that $\text{Im}(X) = \{x_1, x_2, \cdots, x_n\}$. Suppose that $P(X = x_i) = \alpha$ for all $i = 1, 2, \cdots, n$. This and the fact that $\sum_{i=1}^{n} P(X = x_i) = 1$ imply that $a = \frac{1}{n}$. Hence, the the probability mass function of X is

$$p(x_i) = \frac{1}{n}, \quad i = 1, 2, \cdots, n$$

and 0 otherwise. X is called a **uniform discrete random variable**. For example, consider the experiment of rolling a dice. Let X denote the number of dots on the face that comes up. Then the range of X is the set $\text{Im}(X) = \{1, 2, 3, 4, 5, 6\}$. Moreover, $p(x) = \frac{1}{6}$ for $x \in \text{Im}(X)$.
The expected value of X is

$$E(X) = \sum_{i=1}^{n} x_i p(x_i) = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and the second moment of X is

$$E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i) = \sum_{i=1}^{n} \frac{x_i^2}{n} = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}.$$

The variance of X is

$$\text{Var}(X) = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^2}.$$

Example 18.1
Let a and h be a real numbers and n a positive integer. Let X be a uniform discrete random variable that takes the integer values $a, a+h, a+2h, \cdots, a+(n-1)h$. We assume that all these values are equally likely so that the probability mass function of X is

$$p(x) = \frac{1}{n}, \quad x = a, a+h, a+2h, \cdots, a+(n-1)h$$

and 0 otherwise. Find $E(X), E(X^2)$, and $\text{Var}(X)$.

Solution.
The expected value of X is

$$\begin{aligned} E(X) =& \frac{\sum_{k=0}^{n-1}(a+kh)}{n} = \frac{na + h\sum_{k=0}^{n-1} k}{n} \\ =& a + \frac{h}{n} \cdot \frac{n(n-1)}{2} \\ =& a + \frac{h(n-1)}{2}. \end{aligned}$$

The second moment of X is

$$\begin{aligned} E(X^2) =& \sum_{k=0}^{n-1}(a+kh)^2 p(a+kh) \\ =& \frac{1}{n}\sum_{k=0}^{n-1}(a^2 + 2akh + k^2h^2) \\ =& \frac{1}{n}\left[na^2 + 2ah\sum_{k=0}^{n-1} k + h^2\sum_{k=0}^{n-1} k^2\right] \\ =& \frac{1}{n}\left[na^2 + 2ah\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right] \\ =& a^2 + ah(n-1) + h^2\frac{(n-1)(2n-1)}{6} \end{aligned}$$

and the variance of X is

$$\begin{aligned}
\text{Var}(X) =& a^2 + ah(n-1) + h^2\frac{(n-1)(2n-1)}{6} - \left(a + \frac{1}{2}h(n-1)\right)^2 \\
=& a^2 + ah(n-1) + h^2\frac{(n-1)(2n-1)}{6} - a^2 - ah(n-1) - \frac{1}{4}h^2(n-1)^2 \\
=& h^2(n-1)\left[\frac{2n-1}{6} - \frac{(n-1)}{4}\right] \\
=& h^2(n-1)\left[\frac{4n-2}{12} - \frac{3n-3}{12}\right] \\
=& \frac{h^2(n^2-1)}{12} \blacksquare
\end{aligned}$$

Example 18.2
Let a and b be two positive integers with $a < b$. Let X be the uniform discrete random variable that takes the integer values $a, a+1, a+2, \cdots, b$. We assume that all these values are equally likely. Find $E(X), E(X^2)$, and $\text{Var}(X)$.

Solution.
Note that $b = a + (b-a)$. Using the previous example with $h = 1$ and $n = b - a + 1$, we find

$$\begin{aligned}
E(X) =& a + \frac{b-a}{2} = \frac{a+b}{2} \\
E(X^2) =& a^2 + a(b-a) + \frac{(b-a)(2b-2a+1)}{6}
\end{aligned}$$

and the variance of X is

$$\text{Var}(X) = \frac{(b-a+1)^2 - 1}{12} \blacksquare$$

Practice Problems

Problem 18.1
Roll two dice and let X be the sum of the two dice. Is X a uniform discrete random variable?

Problem 18.2
A roulette wheel consists of the numbers $\{0, 1, \cdots, 36\}$. Roll the wheel and let X be number that lands. Is X uniform discrete random variable?

Problem 18.3
Sketch the probability histogram of the uniform discrete random variable X on the values $x = 1, 2, \cdots, 6$.

Problem 18.4
Find the mean and the variance of the uniform discrete random variable X whose range is $\text{Im}(X) = \{3, 4, 5, 6, 7, 8, 9, 10\}$.

Problem 18.5
Let X be a uniform discrete random variable on the values $x = a, a+1, \cdots, b$. Define the function $F(x) = P(X \leq x)$. Find a formula of $F(x)$ for $x = a, a + 1, \cdots, b$.

Problem 18.6
Let X be a uniform discrete random variable with parameters $x = 1, 2, \cdots, n$. Find $E(X)$ and $\text{Var}(X)$.

Problem 18.7
Suppose that X is a discrete random variable that is uniformly distributed on the odd integers $x = 1, 3, 5, ..., 21$. Find $E(X)$ and $\text{Var}(X)$.

Problem 18.8
Let X be a uniform discrete random variable on the even integers $x = 0, 2, 4, \cdots, 22$. Find $P(X = 18)$.

Problem 18.9
Let X be a uniform discrete random variable on the values $x = 0, 1, 2, 3, 4, 5, 6$. Find $P(X < E(X))$.

Problem 18.10
Let X be a discrete uniform distribution on the values $x = a, a+1, \cdots, b$. Find a formula for $M_X(t) = E(e^{tX})$.

Problem 18.11
The first digit of a product's serial number is likely to be any one of the digits 0 to 9. If one product is selected randomly and X is the first digit of serial number, then X has a discrete uniform distribution. Find $E(X)$ and $\text{Var}(X)$.

Problem 18.12
Let X be a discrete uniform distribution on the values $x = a, a+1, \cdots, b$ such that $E(X) = 6.5$ and $\text{Var}(X) = 5.25$. Find the values of a and b.

Problem 18.13
Let X be a discrete uniform distribution on the values $x = -4, -3, \cdots, 4$. Find the probability mass function $p(x)$ and the cumulative distribution function $F(x)$.

Problem 18.14
Let X be the random variable that shows the face that comes up when rolling a dice. Graph $p(x)$ and $F(x)$.

Problem 18.15
Let X be a discrete uniform random variable on the values $x = -2, -1, 0, 1, 2$. Let $Y = |X|$. Find $p_Y(y)$ using 16.1.

Problem 18.16
Let X be a discrete uniform distribution on the values $x = 1, 2, \cdots, n$. Find the cumulative distribution function $F(x)$.

Problem 18.17
Let X be a discrete uniform distribution on the set $\{1, 2, \cdots, 10\}$. Let $Y = 5X + 3$. Find $E(Y)$ and $\text{Var}(Y)$.

Problem 18.18
Let X and Y be two discrete uniform random variables on the set $\{1, 2, \cdots, n\}$. Suppose that X and Y satisfy the condition $P(X = i, Y = j) = P(X = i)P(Y = j)$ where $i, j \in \{1, 2, \cdots, n\}$. Show that the random variable $Z = X + Y$ is not uniform.

Problem 18.19
Let X be uniformly distributed on the set $\{-3, -1, 1, 3\}$. Find $E(X^2)$.

Problem 18.20
Let X be uniformly distributed on the set $\{-3, -1, 1, 3\}$. Find $E[(2X+10)^2]$.

Problem 18.21 ‡
A theme park conducts a study of families that visit the park during a year. The fraction of such families of size m is $\frac{8-m}{28}$, $m = 1, 2, 3, 4, 5, 6$, and 7.
For a family of size m that visits the park, the number of members of the family that ride the roller coaster follows a discrete uniform distribution on the set $\{1, \cdots, m\}$.
Calculate the probability that a family visiting the park has exactly six members, given that exactly five members of the family ride the roller coaster

19 Bernoulli Trials and Binomial Distributions

A **Bernoulli trial**[6] is an experiment with exactly two outcomes: **Success** and **failure.** The probability of a success is denoted by p and that of a failure by q. Moreover, p and q are related by the formula

$$p + q = 1, \quad 0 < p, q < 1.$$

Example 19.1
Consider the experiment of rolling a fair die where a success is the face that comes up shows a number divisible by 2. Find p and q.

Solution.
The numbers on the die that are divisible by 2 are 2,4, and 6. Thus, $p = \frac{3}{6} = \frac{1}{2}$ and $q = 1 - \frac{1}{2} = \frac{1}{2}$ ■
A **Bernoulli experiment** is a sequence of independent[7] Bernoulli trials.
Let X represent the number of successes that occur in n independent Bernoulli trials. Then X is said to be a **binomial random variable** with parameters (n, p). If $n = 1$ then X is said to be a **Bernoulli random variable**.
The central question of a binomial experiment is to find the probability of r successes out of n trials. In the next paragraph we'll see how to compute such a probability.

Example 19.2
We roll a fair die 5 times. A success is when the face that comes up shows a prime number. We are interested in the probability of obtaining three prime numbers. What are p, q, n, and r?

Solutions.
This is a Binomial experiment with 5 trials. The prime numbers on the die are $2, 3, 5$ so that $p = q = \frac{1}{2}$. Also, we have $n = 5$ and $r = 3$ ■

Binomial Distribution Function
As mentioned above, the central problem of a Binomial experiment is to find the probability of r successes out of n independent trials. The sample

[6]The prefix bi in Binomial experiment refers to the fact that there are two possible outcomes (e.g., head or tail, true or false, working or defective) to each trial.

[7]That is what happens to one trial does not affect the probability of a success in any other trial.

space in a Binomial experiment of n independent Bernoulli trials is the set of sequences of the form $S_1S_2\cdots S_n$ where $S_i = S$ (a success) or $S_i = F$ (a failure). Let X denotes the number of successes in each outcome. For example, if $n = 3$ then $X(SFS) = 2$ whereas $X(FFS) = 1$ and $X(FFF) = 0$. Let r be a non-negative integer and A_r be the event whose outcomes are those sequences with exactly r successes and $n - r$ failures. But the number of all outcomes with r successes and $n - r$ failures is ${}_nC_r$. That is, $\#(A_r) =_n C_r$. Hence, $P(A_r)$ is a sum of ${}_nC_r$ terms of the form $S_1S_2\cdots S_n$ where $S_i = S$ for r values of i and $S_i = F$ for the remaining $n - r$ values. To illustrate, suppose $n = 3$ and $r = 2$. Then $A_2 = \{SSF, SFS, FSS\}$ and $\#(A_2) =_3 C_2$. Moreover,

$$P(A_2) = P(SSF) + P(SFS) + P(FSS).$$

Using independence, we have $P(SSF) = P(S)P(S)P(F) = p^2q = P(SFS) = P(FSS)$. Thus,

$$P(X = 2) = P(A_2) =_3 C_2p^2q.$$

In general, $P(A_r) =_n C_rp^rq^{n-r}$. That is, the probability of having r successes in any order is given by the **Binomial mass function**

$$p(r) = P(X = r) = {}_nC_rp^rq^{n-r}$$

and 0 otherwise. Note that by letting $a = p$ and $b = 1 - p$ in the Binomial formula we find

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} {}_nC_kp^k(1-p)^{n-k} = (p + 1 - p)^n = 1.$$

The histogram of the Binomial distribution is given in Figure 19.1.

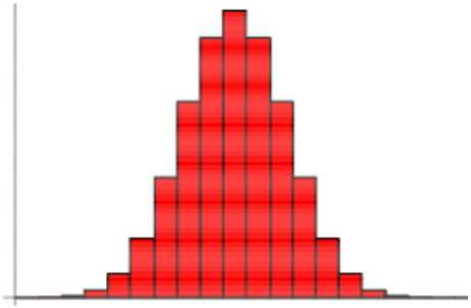


Figure 19.1

The **cumulative distribution function** is given by

$$F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} {}_nC_k p^k (1-p)^{n-k}, & 0 \le x \le n \\ 1, & x > n \end{cases}$$

where $\lfloor k \rfloor$ is the **floor function**[8].

Example 19.3
Suppose that in a box of 100 computer chips, the probability of a chip to be defective is 3%. Inspection process for defective chips consists of selecting with replacement 5 randomly chosen chips in the box and to send the box for shipment if none of the five chips is defective. Write down the random variable, the corresponding probability distribution and then determine the probability that the box described here will be allowed to be shipped.

Solution.
Let X be the number of defective chips in the box. Then X is a Binomial random variable with probability distribution

$$p(x) = P(X = x) = {}_5C_x (0.03)^x (0.97)^{5-x}, \ x = 0, 1, 2, 3, 4, 5$$

and 0 otherwise. Now,

$$P(\text{box allowed to be shipped}) = P(X = 0) = (0.97)^5 = 0.859 \ \blacksquare$$

Example 19.4
Suppose that 40% of the voters in a city are in favor of a ban of smoking in public buildings. Suppose 5 voters are to be randomly sampled. Find the probability that
(a) 2 favor the ban.
(b) less than 4 favor the ban.
(c) at least 1 favor the ban.

Solution.
(a) $P(X = 2) = {}_5C_2 (0.4)^2 (0.6)^3 \approx 0.3456$.
(b) $P(X < 4) = p(0) + p(1) + p(2) + p(3) = {}_5C_0 (0.4)^0 (0.6)^5 + {}_5C_1 (0.4)^1 (0.6)^4 + {}_5C_2 (0.4)^2 (0.6)^3 + {}_5C_3 (0.4)^3 (0.6)^2 \approx 0.913$.
(c) $P(X \ge 1) = 1 - P(X < 1) = 1 - {}_5C_0 (0.4)^0 (0.6)^5 \approx 0.922 \ \blacksquare$

[8] $\lfloor x \rfloor$ = the largest integer less than or equal to x.

Example 19.5
A student takes a test consisting of 10 true-false questions.
(a) What is the probability that the student answers at least six questions correctly?
(b) What is the probability that the student answers at most two questions correctly?

Solution.
(a) Let X be the number of correct responses. Then X is a Binomial random variable with parameters $n = 10$ and $p = \frac{1}{2}$. So, the desired probability is

$$\begin{aligned} P(X \geq 6) =& P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) \\ =& \sum_{x=6}^{10} {}_{10}C_x (0.5)^x (0.5)^{10-x} \approx 0.3769. \end{aligned}$$

(b) We have

$$P(X \leq 2) = \sum_{x=0}^{2} {}_{10}C_x (0.5)^x (0.5)^{10-x} \approx 0.0547 \blacksquare$$

Example 19.6
A study shows that 30 percent of people aged 50-60 in a certain town have high blood pressure. What is the probability that in a sample of fourteen individuals aged between 50 and 60 tested for high blood pressure, more than six will have high blood pressure?

Solution.
Let X be the number of people in the town aged 50-60 with high blood pressure. Then X is a Binomial random variable with $n = 14$ and $p = 0.3$. Thus,

$$\begin{aligned} P(X > 6) =& 1 - P(X \leq 6) \\ =& 1 - \sum_{i=0}^{6} {}_{14}C_i (0.3)^i (0.7)^{14-i} \\ \approx& 0.0933 \blacksquare \end{aligned}$$

Practice Problems

Problem 19.1
Mark is a car salesman with a 10% chance of persuading a randomly selected customer to buy a car. Out of 8 customers that were serviced by Mark, what is the probability that exactly one agreed to buy a car?

Problem 19.2
The probability of a newly born child to get a genetic disease is 0.25. If a couple carry the disease and wish to have four children then what is the probability that 2 of the children will get the disease?

Problem 19.3
A skyscraper has three elevators. Each elevator has a 50% chance of being down, independently of the others. Let X be the number of elevators which are down at a particular time. Find the probability mass function (pmf) of X.

Problem 19.4 ‡
A hospital receives 1/5 of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials.
For Company X shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective.
What is the probability that this shipment came from Company X?

Problem 19.5 ‡
A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee.
Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

Problem 19.6 ‡
A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05 .
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

Problem 19.7
The probability of winning a game is $\frac{1}{300}$. If you play this game 200 times, what is the probability that you win at least twice?

Problem 19.8
Suppose a local bus service accepted 12 reservations for a commuter bus with 10 seats. Seven of the ten reservations went to regular commuters who will show up for sure. The other 5 passengers will show up with a 50% chance, independently of each other.
(a) Find the probability that the bus will be overbooked.
(b) Find the probability that there will be empty seats.

Problem 19.9
Suppose that 3% of flashlight batteries produced by a certain machine are defective. The batteries are put into packages of 20 batteries for distribution to retailers.
What is the probability that a randomly selected package of batteries will contain at least 2 defective batteries?

Problem 19.10
The probability of late arrival of flight 701 in any day is 0.20 and is independent of the late arrival in any other day. The flight can be late only once per day. Calculate the probability that the flight is late two or more times in ten days.

Problem 19.11
Ashley finds that she beats Carla in tennis 70% of the time. The two play 3 times in a particular month. Assuming independence of outcomes, what is the probability Ashley wins at least 2 of the 3 matches?

Problem 19.12
The probability of a computer chip to be defective is 0.05. Consider a package of 6 computer chips.
(a) What is the probability one chip will be defective?
(b) What is the probability at least one chip will be defective?
(c) What is the probability that more than one chip will be defective, given at least one is defective?

Problem 19.13
In a promotion, a popcorn company inserts a coupon for a free Red Box movie in 10% of boxes produced. Suppose that we buy 10 boxes of popcorn, what is the probability that we get at least 2 coupons?

Problem 19.14
A quiz for a Math class consists of 5 questions. The questions are multiple-choice with 4 possible answers each in which one is the correct answer. A student randomly guesses on all 5 questions.
What is the probability of that student getting a passing grade (Assume 60% correct is passing)?

Problem 19.15
Let X be a discrete random variable with probability mass function

$$p(r) = P(X = r) = C(n, r)p^r q^{n-r},\ r = 0, 1, 2, \cdots, n$$

and 0 otherwise, where n is a positive integer and $p + q = 1$. Find $E(X)$ and $E(X(X - 1))$.

Problem 19.16
Let X be a discrete random variable with probability mass function

$$p(r) = P(X = r) = C(n, r)p^r q^{n-r},\ r = 0, 1, 2, \cdots, n$$

and 0 otherwise, where n is a positive integer and $p + q = 1$. Find $\mathrm{Var}(X)$.

Problem 19.17
Suppose 80% of students who register for MATH 3153 complete the course successfully. And suppose you took many, many samples of six people, with replacement. What would be the expected mean and standard deviation of the number of people that would finish successfully?

Problem 19.18
When randomly guessing on a multiple choice test with 8 questions, where each question has 4 options, what is the expected number of questions a student will get correct without studying for the exam?

Problem 19.19
The probability of a student passing an exam is 0.2. Ten students took the exam.
(a) What is the probability that at least two students passed the exam?
(b) What is the expected number of students who passed the exam?

Problem 19.20
Let X be a Binomial random variable with parameters (n, p). Show that

$$p(k) = \frac{p}{1-p}\frac{n-k+1}{k}p(k-1),\ k = 1, 2, 3, \cdots, n.$$

Problem 19.21 ‡
An actuary has done an analysis of all policies that cover two cars. 70% of the policies are of type A for both cars, and 30% of the policies are of type B for both cars. The number of claims on different cars across all policies are mutually independent. The distributions of the number of claims on a car are given in the following table.

Number of Claims	Type A	Type B
0	40%	25%
1	30%	25%
2	20%	25%
3	10%	25%

Four policies are selected at random. Calculate the probability that exactly one of the four policies has the same number of claims on both covered cars.

Problem 19.22 ‡
A factory tests 100 light bulbs for defects. The probability that a bulb is defective is 0.02. The occurrences of defects among the light bulbs are mutually independent events. Calculate the probability that exactly two are defective given that the number of defective bulbs is two or fewer.

20 The Expected Value and Variance of the Binomial Distribution

In this section, we find the expected value and the variance of a Binomial random variable X with parameters (n, p).

The expected value is found as follows.

$$\begin{aligned} E(X) =& \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\ =& np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j} = np(p+1-p)^{n-1} = np \end{aligned}$$

where we used the Binomial theorem and the substitution $j = k - 1$. Also, we have

$$\begin{aligned} E(X(X-1)) =& \sum_{k=0}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ =& n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k} \\ =& n(n-1)p^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} p^j (1-p)^{n-2-j} \\ =& n(n-1)p^2 (p+1-p)^{n-2} = n(n-1)p^2. \end{aligned}$$

This implies $E(X^2) = E(X(X-1)) + E(X) = n(n-1)p^2 + np$. The variance of X is then

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p).$$

Example 20.1
The probability of a student passing an exam is 0.2. Ten students took the exam.
(a) What is the probability that at least two students passed the exam?
(b) What is the expected number of students who passed the exam?
(c) How many students must take the exam to make the probability at least 0.99 that a student will pass the exam?

Solution.
Let X be the number of students who passed the exam. Then, X has a Binomial distribution with $n = 10$ and $p = 0.2$.
(a) The event that at least two students passed the exam is $\{X \geq 2\}$. So,

$$\begin{aligned} P(X \geq 2) =& 1 - P(X < 2) = 1 - p(0) - p(1) \\ =& 1 - {}_{10}C_0(0.2)^0(0.8)^{10} - {}_{10}C_1(0.2)^1(0.8)^9 \\ \approx& 0.6242. \end{aligned}$$

(b) $E(X) = np = 10 \cdot (0.2) = 2$.
(c) Suppose that n students are needed to make the probability at least 0.99 that a student will pass the exam. Let A denote the event that a student pass the exam. Then, A^c means that all the students fail the exam. We have,

$$P(A) = 1 - P(A^c) = 1 - (0.8)^n \geq 0.99.$$

Solving the inequality, we find that $n \geq \frac{\ln(0.01)}{\ln(0.8)} \approx 20.6$. So, the required number of students is 21 ■

Example 20.2
Let X be a Binomial random variable with parameters $(12, 0.5)$. Find the variance and the standard deviation of X.

Solution.
We have $n = 12$ and $p = 0.5$. Thus, $\text{Var}(X) = np(1-p) = 6(1-0.5) = 3$. The standard deviation is $\sigma_X = \sqrt{3}$ ■

Example 20.3
A multiple choice exam consists of 25 questions each with five choices with once choice is correct. Randomly select an answer for each question. Let X be the random variable representing the total number of correctly answered questions.
(a) What is the probability that you get exactly 16, or 17, or 18 of the questions correct?
(b) What is the probability that you get at least one of the questions correct.
(c) Find the expected value of the number of correct answers.

Solution.
(a) Let X be the number of correct answers. We have

$$\begin{aligned} P(X = 16 \text{ or } X = 17 \text{ or } X = 18) =& {}_{25}C_{16}(0.2)^{16}(0.8)^9 + {}_{25}C_{17}(0.2)^{17}(0.8)^8 \\ &+ {}_{25}C_{18}(0.2)^{18}(0.8)^7 = 2.06 \times 10^{-6}. \end{aligned}$$

(b) $P(X \geq 1) = 1 - P(X = 0) = 1 - {}_{25}C_0(0.8)^{25} = 0.9962$.
(c) We have $E(X) = 25(0.2) = 5$ ■

A useful fact about the Binomial distribution is a recursion for calculating the probability mass function.

Theorem 20.1
Let X be a Binomial random variable with parameters (n, p). Then for $k = 1, 2, 3, \cdots, n$

$$p(k) = \frac{p}{1-p}\frac{n-k+1}{k}p(k-1).$$

Proof.
We have

$$\begin{aligned}\frac{p(k)}{p(k-1)} =& \frac{{}_nC_k p^k(1-p)^{n-k}}{{}_nC_{k-1}p^{k-1}(1-p)^{n-k+1}} \\ =& \frac{\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}}{\frac{n!}{(k-1)!(n-k+1)!}p^{k-1}(1-p)^{n-k+1}} \\ =& \frac{(n-k+1)p}{k(1-p)} = \frac{p}{1-p}\frac{n-k+1}{k}\end{aligned}$$ ■

Binomial Random Variable Histogram
The histogram of a Binomial random variable is constructed by putting the r values on the horizontal axis and $p(r)$ values on the vertical axis. The width of the bar is 1 and its height is $p(r)$. The bars are centered at the r values.

Example 20.4
Construct the Binomial distribution for the total number of heads in four flips of a balanced coin. Make a histogram.

Solution.
The Binomial distribution is given by the following table

r	0	1	2	3	4
$p(r)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

and 0 otherwise. The corresponding histogram is shown in Figure 20.1 ■

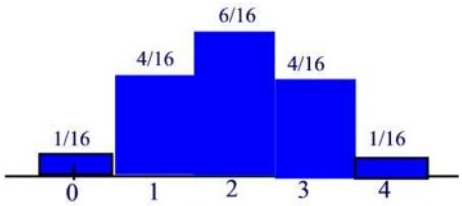


Figure 20.1

The following theorem sheds information about the histogram of a Binomial distribution.

Theorem 20.2
Let X be a Binomial random variable with parameters (n, p). Then $p(k)$ has a unique global maximum at $k = \lfloor (n+1)p \rfloor$ if $(n+1)p$ is not an integer. If $(n+1)p$ is an integer then p has maxima at $(n+1)p$ and $(n+1)p - 1$.

Proof.
From Theorem 20.1, we have

$$\frac{p(k)}{p(k-1)} = \frac{p}{1-p}\frac{n-k+1}{k} = 1 + \frac{(n+1)p - k}{k(1-p)}$$

for $k \in \{1, 2, \cdots, n\}$. Accordingly, $p(k) > p(k-1)$ when $k < (n+1)p$ and $p(k) < p(k-1)$ when $k > (n+1)p$. Suppose that $(n+1)p$ is not an integer and let $m = \lfloor (n+1)p \rfloor$. Then $m \in \{1, 2, \cdots, n\}$ and $m < (n+1)p < m+1$. If $k < m < (n+1)p$ then $p(k)$ is increasing. If $k > m$ then $k+1 > (n+1)p$ and therefore $p(k+1) < p(k)$. That is, $p(k)$ is decreasing. Hence, $p(k)$ has a global maximum at m.
Now, if $(n+1)p = m$ is an integer then $p(m) = p(m-1)$ In this case, the histogram has two maxima at $(n+1)p$ and $(n+1)p - 1$ as shown in Figure 20.2 ■

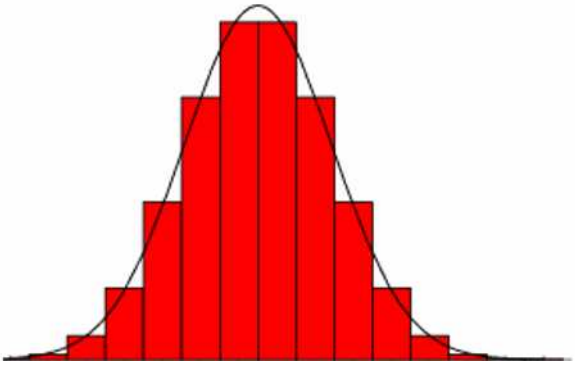


Figure 20.2

Practice Problems

Problem 20.1
If X is the number of "6"'s that turn up when 72 ordinary dice are independently thrown, find the expected value of X^2.

Problem 20.2 ‡
A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists.
Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist.
What is the expected revenue of the tour operator?

Problem 20.3
Let Y be a Binomial random variable with parameters $(n, 0.2)$. Define the random variable

$$S = 100 + 50Y - 10Y^2.$$

Give the expected value of S when $n = 1$, 2, and 3.

Problem 20.4
A recent study shows that the probability of a marriage will end in a divorce within 10 years is 0.4. Letting a divorce be a success, find the mean and the standard deviation for the Binomial distribution X involving 1000 marriages.

Problem 20.5
The probability of a person contracting the flu on exposure is 0.4. Let a success be a person contracting the flu. Consider the Binomial distribution for a group of 5 people that has been exposed.
(a) Find the probability mass function.
(b) Compute $p(x)$ for $x = 0, 1, 2, 3, 4, 5$.
(c) Draw a histogram for the distribution.
(d) Find the mean and the standard deviation.

Problem 20.6
A fair die is rolled twice. A success is when the face that comes up shows 3

or 6.
(a) Write the function defining the distribution.
(b) Construct a table for the distribution.
(c) Construct a histogram for the distribution.
(d) Find the mean and the standard deviation for the distribution.

Problem 20.7
A motorist makes three driving errors, each independently resulting in an accident with probability 0.25. Calculate the expected value and the variance of the number of accidents.

Problem 20.8
A student takes a multiple-choice test with 40 questions. The probability that the student answers a given question correctly is 0.5, independent of all other questions. Calculate the expected value and the variance of the number of correct answers.

Problem 20.9 ‡
An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail.
(a) Find the expected value and the variance of the number of failed components.
(b) Calculate the probability that the system will overheat.

Problem 20.10
In a group of 15 health insurance policyholders diagnosed with cancer, each policyholder has probability 0.90 of receiving radiation and probability 0.40 of receiving chemotherapy. Radiation and chemotherapy treatments are independent events for each policyholder.
(a) Find the expected value and the variance of the number of policyholders who undergo radiation.
(b) Find the expected value and the variance of the number of policyholders who undergo chemotherapy.

Problem 20.11
Let X be a Binomial random variable with parameters (n, p). Find a formula for the function $M_X(t) = E(e^{tX})$ where t is a real number.

Problem 20.12
Let $M_X(t)$ be as defined in Problem 20.11. Show that $E(X) = M'_X(0)$.

Problem 20.13
Let $M_X(t)$ be as defined in Problem 20.11. Show that $E(X^2) = M''_X(0)$.

Problem 20.14
Let X be a Binomial random variable with $n = 10$ and $p = 0.4$. Find $P[E(X) - \sigma_X \leq X \leq E(X) + \sigma_X]$.

Problem 20.15
A laser production facility is known to have a 75% yield; that is, 75% of the lasers manufactured by the facility pass the quality test. Suppose that today the facility is scheduled to produce 15 lasers.
(a) What is the expected number of lasers to pass the test?
(b) What is the variance?

Problem 20.16
A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 100 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year. The events of different employees achieving a high performance level during the coming year are mutually independent. What is the expected number of employees with high performance level?

Problem 20.17
A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05 .
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the expected number of hurricanes in a 20-year period.

Problem 20.18
A study is being conducted in which the health of a group of ten policyholders is being monitored over a one-year period of time. Individual participants

in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the expected number of participants who complete the study?

Problem 20.19
A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is 0.60. The numbers of accidents that occur in different months are mutually independent. In a period of five months, what is the expected number of months with no accidents?

Problem 20.20
A certain type of cables is breakable with a probability 0.1. What is the expected number of breakable cables in a random sample of 400 cables?

21 Poisson Random Variable

The Poisson random variable is most commonly used to model the number of random occurrences of some phenomenon in a fixed interval of space or time. For example, the number of phone calls received by a telephone operator in a 10-minute period or the number of typos per page made by a secretary.
Consider the following problem: A customer service center receives inquiries from customers by phone only. We are interested in finding the probability of a certain number of phone calls received in a time interval $[0, \omega]$. Let X denote the number of phone calls received in the time interval $[0, \omega]$. Then $\text{Im}(X) = \{0, 1, 2, \cdots\}$. Thus, X is a discrete random variable. Divide the interval $[0, \omega]$ into n equal sub-intervals each of length $\frac{\omega}{n}$. A sub-interval is considered a success if a phone call is received in that sub-interval. Let X_n be the number of sub-intervals in which a phone call is received. Then $X_n$is a Binomial random variable with parameters (n, p). Note that in the time interval $[0, \omega]$, X is fixed whereas X_n changes with n. Clearly,

$$P(X(\omega) = k) = \lim_{n\to\infty} P(X_n = k).$$

Now, let λ be the average number of phone calls received per unit time. Then the probability that a sub-interval takes a call is $p \approx \frac{\omega}{n}\lambda$. Hence,

$$\begin{aligned} P(X(\omega) = k) &= \lim_{n\to\infty} {}_nC_k p^k (1-p)^{n-k} \\ &= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{(\lambda\omega)^k}{n^k} \left(1 - \frac{\lambda\omega}{n}\right)^{n-k} \\ &= \lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \frac{(\lambda\omega)^k}{k!} \left(1 - \frac{\lambda\omega}{n}\right)^{n} \left(1 - \frac{\lambda\omega}{n}\right)^{-k} \\ &= e^{-\lambda\omega} \frac{(\lambda\omega)^k}{k!} \end{aligned}$$

where we use the calculus result

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

A random variable X is said to be a **Poisson** random variable with parameter $\lambda > 0$ if its probability mass function has the form

$$p(k) = P(X(\omega) = k) = e^{-\lambda\omega} \frac{(\lambda\omega)^k}{k!}, \quad \omega > 0,\ k = 0, 1, 2, \cdots \tag{21.1}$$

and 0 otherwise, where λ indicates the average number of successes per unit time or space.
Note that $p(k) \geq 0$ and

$$\sum_{k=0}^{\infty} p(k) = e^{-\lambda\omega} \sum_{k=0}^{\infty} \frac{(\lambda\omega)^k}{k!} = e^{-\lambda\omega} e^{\lambda\omega} = 1$$

so that $p(k)$ as defined above is a valid probability mass function.

Remark 21.1
A Poisson random variable can take on any positive integer value. In contrast, the Binomial distribution always has a finite upper limit.

Example 21.1
The number of car accidents on a certain section of highway I-40 averages 2.1 per day. Assuming that the number of accidents in a given day follows a Poisson distribution, what is the probability that 4 accidents will occur on a given day?

Solution.
Let X be the number of accidents on a given day. Then X follows a Poisson distribution with $\lambda = 2.1$. Letting $\omega = 1$ in Equation (21.1), the probability that 4 accidents will occur on a given day is given by

$$P(X = 4) = e^{-2.1} \frac{(2.1)^4}{4!} \approx 0.0992 \blacksquare$$

Example 21.2
The number of people entering a movie theater averages one every two minutes. Assuming that a Poisson distribution is appropriate.
(a) What is the probability that no people enter between 12:00 and 12:05?
(b) Find the probability that at least 4 people enter during [12:00,12:05].

Solution.
(a) Let X be the number of people that enter between 12:00 and 12:05. We model X as a Poisson random variable with parameter λ, the average number of people that arrive per minute. But, if 1 person arrives every 2 minutes

then on average 0.5 person arrives per minute. Thus, $\lambda = 0.5$. Letting $\omega = 5$ in Equation (21.1), we find

$$P(X = 0) = e^{-2.5}\frac{2.5^0}{0!} = e^{-2.5} \approx 0.0821.$$

(b)

$$\begin{aligned} P(X \geq 4) =& 1 - P(X \leq 3) \\ =& 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) \\ =& 1 - e^{-2.5}\frac{2.5^0}{0!} - e^{-2.5}\frac{2.5^1}{1!} - e^{-2.5}\frac{2.5^2}{2!} - e^{-2.5}\frac{2.5^3}{3!} \\ \approx& 0.2424 \ \blacksquare \end{aligned}$$

Example 21.3
The number of weekly life insurance sold by an insurance agent averages 3 per week. Assuming that this number follows a Poisson distribution, calculate the probability that in a given week the agent will sell
(a) some policies
(b) 2 or more policies but less than 5 policies.
(c) Assuming that there are 5 working days per week, what is the probability that in a given day the agent will sell one policy?

Solution.
(a) Let X be the number of policies sold in a week. Then X follows a Poisson distribution with $\lambda = 3$ and $\omega = 1$. Thus,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-3}3^0}{0!} \approx 0.9502.$$

(b) We have

$$\begin{aligned} P(2 \leq X < 5) =& P(X = 2) + P(X = 3) + P(X = 4) \\ =& \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} \approx 0.6161. \end{aligned}$$

(c) Let Y be the number of policies sold per day. Then Y follows a Poisson distribution with $\lambda = \frac{3}{5} = 0.6$ policy per day. Thus,

$$P(Y = 1) = \frac{e^{-0.6}(0.6)}{1!} \approx 0.3293 \ \blacksquare$$

Example 21.4
The number X of people in a certain town who get the influenza follows a Poisson distribution. The proportion of people who did not get the flu is 0.01. Find the probability mass function of X.

Solution.
The probability mass function of X is

$$p(k) = P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

and 0 otherwise. Since $P(X = 0) = 0.01$, we can write $e^{-\lambda} = 0.01$. Thus, $\lambda = 4.605$ and the exact Poisson distribution is

$$p(k) = P(X = k) = (4.605)^k\frac{e^{-4.605}}{k!}, \quad k = 0, 1, 2, \cdots$$

and 0 otherwise ■
If X has a Poisson distribution, its expected value is found as follows.

$$\begin{aligned} E(X) =& \sum_{k=1}^{\infty} ke^{-\lambda\omega}\frac{(\lambda\omega)^k}{k!} = \lambda\omega e^{-\lambda\omega}\sum_{k=1}^{\infty}\frac{(\lambda\omega)^{k-1}}{(k-1)!} \\ =& \lambda\omega e^{-\lambda\omega}e^{\lambda\omega} = \lambda\omega. \end{aligned}$$

To find the variance, we first compute $E(X^2)$. From

$$\begin{aligned} E(X(X-1)) =& \sum_{k=2}^{\infty} k(k-1)e^{-\lambda\omega}\frac{(\lambda\omega)^k}{k!} = (\lambda\omega)^2 e^{-\lambda\omega}\sum_{k=2}^{\infty}\frac{(\lambda\omega)^{k-2}}{(k-2)!} \\ =& (\lambda\omega)^2 e^{-\lambda\omega}\sum_{k=0}^{\infty}\frac{(\lambda\omega)^k}{k!} = (\lambda\omega)^2 e^{-\lambda\omega}e^{\lambda\omega} = (\lambda\omega)^2 \end{aligned}$$

we find $E(X^2) = E(X(X-1)) + E(X) = (\lambda\omega)^2 + \lambda\omega$. Thus, $\text{Var}(X) = E(X^2) - (E(X))^2 = \lambda\omega$.

Example 21.5
Misprints in a book averages one misprint per 10 pages. Suppose that the number of misprints per page is a random variable having Poisson distribution. Let X denote the number of misprints in a stack of 50 pages. Find the mean and the standard deviation of X.

Solution.
X is a Poisson random variable with $\lambda = 0.1$ and $\omega = 50$. Hence, $E(X) = \lambda t = 5$ and $\sigma_X = \sqrt{5}$ ■

Poisson Approximation to the Binomial Distribution
Letting $\omega = 1$ in the earlier discussion of this section, we find $\lambda \approx np$ and

$$\lim_{n\to\infty} {}_nC_k p^k (1-p)^{n-k} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Thus, for large n (and hence small p) the Poisson distribution can be used as an approximation to the Binomial distribution. Now, how large n should be and how small p should be so that the approximation is acceptable? In general, Poisson distribution will provide a good approximation to binomial probabilities when $np < 10$. In this case, we let $\lambda = pn$.

Example 21.6
In a group of 100 individuals, let X be the random variable representing the total number of people in the group with a birthday on Thanksgiving day. Then X is a binomial random variable with parameters $n = 100$ and $p = \frac{1}{365}$. What is the probability at least one person in the group has a birthday on Thanksgiving day?

Solution.
We have

$$P(X \geq 1) = 1 - P(X = 0) = 1 - {}_{100}C_0 \left(\frac{1}{365}\right)^0 \left(\frac{364}{365}\right)^{100} \approx 0.2399.$$

Using the Poisson approximation, with $\lambda = 100 \times \frac{1}{365} = \frac{100}{365} = \frac{20}{73}$ we find

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{(20/73)^0}{0!}e^{\left(-\frac{20}{73}\right)} \approx 0.2396$$ ■

Example 21.7
Consider a contest where a participant fires at a small can placed on the top of box. Each time the can is hit, it is replaced by another can. Suppose that the probability of a participant hitting the can is $\frac{1}{32}$. Assume that the participant shoots 96 times, and that all shoots are independent.
(a) Find the probability mass function of the number of shoots that hit a

can.
(b) Give an approximation for the probability of the participant hitting no more than one can.

Solution.
(a) Let X denote the number of shoots that hit a can. Then X follows a binomial distribution:

$$P(X = k) = {}_nC_k p^k (1-p)^{n-k}, \quad n = 96,\ p = \frac{1}{32}.$$

(b) Since n is large, and p small, we can use the Poisson approximation, with parameter $\lambda = np = 3$. Thus,

$$P(X \leq 1) = P(X = 0) + P(X = 1) \approx e^{-\lambda} + \lambda e^{-\lambda} = 4e^{-3} \approx 0.199 \ \blacksquare$$

Practice Problems

Problem 21.1
The number of accidents on a certain section of a highway averages 4 per day. Assuming that this number follows a Poisson distribution, what is the probability of no car accident in one day? What is the probability of 1 car accident in two days?

Problem 21.2
A phone operator receives calls on average of 2 calls per minute. Assuming that the number of calls follows a Poisson distribution, what is the probability of receiving 10 calls in 5 minutes?

Problem 21.3
In the first draft of a book on probability theory, there are an average of 15 spelling errors per page. Suppose that the number of errors per page follows a Poisson distribution. What is the probability of having no errors on a page?

Problem 21.4
Suppose that the number of people admitted to an emergency room each day is a Poisson random variable with parameter $\lambda = 3$.
(a) Find the probability that 3 or more people admitted to the emergency room today.
(b) Find the probability that no people were admitted to the emergency room today.

Problem 21.5
At a reception event guests arrive at an average of 2 per minute. Assume that the number of guests arriving per minute has a Poisson distribution. Find the probability that
(a) at most 4 will arrive at any given minute
(b) at least 3 will arrive during an interval of 2 minutes
(c) 5 will arrive in an interval of 3 minutes.

Problem 21.6
Suppose that the number of car accidents on a certain section of a highway can be modeled by a random variable having Poisson distribution with standard deviation $\sigma = 2$. What is the probability that there are at least three accidents?

Problem 21.7
A Geiger counter is monitoring the leakage of alpha particles from a container of radioactive material. It is found that, an average of 50 particles per minute is leaked. Assume the number of particles leaked in any given minute is modeled by a Poisson distribution.
(a) Compute the probability that at least one particle is leaked in a particular 1-second period.
(b) Compute the probability that at least two particles is leaked in a particular 2-second period.

Problem 21.8 ‡
An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims.
If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

Problem 21.9 ‡
A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and $10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5 .
What is the expected amount paid to the company under this policy during a one-year period?

Problem 21.10 ‡
A baseball team has scheduled its opening game for April 1. If it rains on April 1, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed.
The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6 .
What is the standard deviation of the amount the insurance company will have to pay?

Problem 21.11
The average number of trains arriving on any one day at a train station in a certain city is known to be 12. Assuming that the number of trains arriving on any given day follows a Poisson distribution, what is the probability that on a given day fewer than nine trains will arrive at this station?

Problem 21.12
In the inspection sheet metals produced by a machine, five defects per 10 square feet were spotted, on average. If we assume a Poisson distribution, what is the probability that a 15-square feet sheet of the metal will have at least six defects?

Problem 21.13
Let X be a Poisson random variable with mean λ. If $P(X = 1|X \leq 1) = 0.8$, what is the value of λ?

Problem 21.14
The number of trucks arriving at a truck depot on a given day has a Poisson distribution with a mean of 2.5 per day.
(a) What is the probability a day goes by with no more than one truck arriving?
(b) Give the mean and standard deviation of the number of trucks arriving in an 8-day period.

Problem 21.15 ‡
Let X represent the number of customers arriving during the morning hours and let Y represent the number of customers arriving during the afternoon hours at a diner. You are given:
(i) X and Y are Poisson distributed.
(ii) The first moment of X is less than the first moment of Y by 8.
(iii) The second moment of X is 60% of the second moment of Y.
Calculate the variance of Y.

Problem 21.16
Let X be a Poisson random variable with expected value λ and $g : \{0, 1, 2, \cdots\} \to \mathbb{R}$ be a bounded function. Calculate $E[\lambda g(X + 1) - Xg(X)]$.

Problem 21.17
The manager of an industrial plant is planning to buy a new machine. For each day's operation, the number of repairs X, that the machine needs is a Poisson random variable with mean 0.96 repairs per day. The daily cost of operating the machine is $C = 160 + 40X^2$. Find the expected value of the daily cost of operating the machine.

Problem 21.18
It is believed that the number of bookings taken per hour at an online travel agency follows a Poisson distribution. Past records indicate that the hourly number of bookings has a mean of 15 and a standard deviation of 2.5. Comment on the suitability of the Poisson distribution for this problem?

Problem 21.19
Find a formula for $M_X(t) = E(e^{tX})$ when X is a Poisson random variable with parameter λ.

Problem 21.20
There are 300 misprints in a 100 page document. Suppose the number of misprints on a given page follows a Poisson distribution. What is the probability there at least two misprints on page 57?

Problem 21.21
Let X be a binomial distribution with parameters $n = 200$ and $p = 0.02$. We want to calculate $P(X \geq 2)$. Explain why a Poisson distribution can be expected to give a good approximation of $P(X \geq 2)$ and then find the value of this approximation.

Problem 21.22
In a TV plant, the probability of manufacturing a defective TV is 0.03. Using Poisson approximation, find the probability of obtaining exactly one defective TV set out of a group of 20.

Problem 21.23
Suppose that 1 out of 400 tires are defective. Let X denote the number of defective tires in a group of 200 tires. What is the probability that at least three of them are defective?

Problem 21.24
1000 cancer patients are receiving a clinical trial drug for cancer. Side effects are being studied. The probability that a patient experiences side effects to the drug is found to be 0.001. Find the probability that none of the patients administered the trial drug experienced any side effect.

Problem 21.25
From a group of 120 engineering students, 3% are not in favor of studying

differential equations. Use the Poisson approximation to estimate the probability that
(a) exactly 2 students are not in favor of studying differential equations;
(b) at least two students are not in favor of studying differential equations.

Problem 21.26
Suppose 8% of the tires manufactured at a particular plant are defective. Find the probability that exactly one tire is defective in a sample of 20 tires.

Problem 21.27
When given the parameters of a binomial distribution, n and p, what are the mean and variance of the Poisson distribution used for approximating the binomial?

Problem 21.28
The probability that a person will develop an infection even after taking a vaccine that was supposed to prevent the infection is 0.03. In a random sample of 200 people in a community who got the vaccine, what is the probability that six or fewer people will be infected?

Problem 21.29
Given that 5% of a population are left-handed, use the Poisson distribution to estimate the probability that a random sample of 100 people contains 2 or more left-handed people.

Problem 21.30 ‡
The number of traffic accidents per week at intersection Q has a Poisson distribution with mean 3. The number of traffic accidents per week at intersection R has a Poisson distribution with mean 1.5.
Let A be the probability that the number of accidents at intersection Q exceeds its mean. Let B be the corresponding probability for intersection R. Calculate $B - A$.

Problem 21.31 ‡
Company XYZ provides a warranty on a product that it produces. Each year, the number of warranty claims follows a Poisson distribution with mean c. The probability that no warranty claims are received in any given year is 0.60.
Company XYZ purchases an insurance policy that will reduce its overall

warranty claim payment costs. The insurance policy will pay nothing for the first warranty claim received and 5000 for each claim thereafter until the end of the year.
Calculate the expected amount of annual insurance policy payments to Company XYZ.

Problem 21.32 ‡
A life insurance company has found there is a 3% probability that a randomly selected application contains an error. Assume applications are mutually independent in this respect. An auditor randomly selects 100 applications. Calculate the probability that 95% or less of the selected applications are error-free.

Problem 21.33 ‡
The number of traffic accidents occurring on any given day in Coralville is Poisson distributed with mean 5. The probability that any such accident involves an uninsured driver is 0.25, independent of all other such accidents. Calculate the probability that on a given day in Coralville there are no traffic accidents that involve an uninsured driver.

Problem 21.34 ‡
A company has purchased a policy that will compensate for the loss of revenue due to severe weather events. The policy pays 1000 for each severe weather event in a year after the first two such events in that year. The number of severe weather events per year has a Poisson distribution with mean 1. Calculate the expected amount paid to this company in one year.

Problem 21.35 ‡
A company provides each of its employees with a death benefit of 100. The company purchases insurance that pays the cost of total death benefits in excess of 400 per year. The number of employees who will die during the year is a Poisson random variable with mean 2. Calculate the expected annual cost to the company of providing the death benefits, excluding the cost of the insurance.

Problem 21.36 ‡
The number of boating accidents X a policyholder experiences this year is modeled by a Poisson random variable with variance 0.10. An insurer reimburses only the first accident. Let Y be the number of unreimbursed

accidents the policyholder experiences this year and let p be the probability function of Y. Determine $p(y)$.

22 Geometric Random Variable

A geometric random variable models the number of successive independent Bernoulli trials that must be performed to obtain the "first" success. For example, the number of flips of a fair coin until the first head appears follows a geometric distribution.
Let X be number of trials needed to achieve the first success. Then the probability mass function of X is

$$\begin{aligned} p(n) =& P(X = n) = P(X_1 = F, X_2 = F, \cdots, X_{n-1} = F, X_n = S) \\ =& P(X_1 = F)P(X_2 = F) \cdots P(X_{n-1} = F)P(X_n = S) \\ =& p(1-p)^{n-1} \end{aligned}$$

and 0 otherwise. Note that the second line follows from the fact that the events $\{X_1 = F\}, \cdots, \{S\}$ are independent.
Note that $p(n) \geq 0$ and

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = \frac{p}{1-(1-p)} = 1$$

so that $p(n)$ is a valid probability mass function.
We call X a **geometric** random variable with parameter p.

Example 22.1
Consider the experiment of rolling a pair of fair dice.
(a) What is the probability of getting a sum of 11?
(b) If you roll the dice repeatedly, what is the probability that the first 11 occurs on the 8^{th} roll?

Solution.
(a) A sum of 11 occurs when the pair of dice show either $(5, 6)$ or $(6, 5)$ so that the required probability is $\frac{2}{36} = \frac{1}{18}$.
(b) Let X be the number of rolls on which the first 11 occurs. Then X is a geometric random variable with parameter $p = \frac{1}{18}$. Thus,

$$P(X = 8) = \left(\frac{1}{18}\right)\left(1 - \frac{1}{18}\right)^7 = 0.0372 \blacksquare$$

To find the expected value and variance of a geometric random variable we proceed as follows. First we recall from calculus the geometric series

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ |x| < 1.$$

Differentiating $f(x)$ twice we find

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = (1-x)^{-2} \quad \text{and} \quad f''(x) = \sum_{n=1}^{\infty} n(n-1)x^{n-2} = 2(1-x)^{-3}.$$

Evaluating $f'(x)$ and $f''(x)$ at $x = 1-p$, we find

$$f'(1-p) = \sum_{n=1}^{\infty} n(1-p)^{n-1} = p^{-2}$$

and

$$f''(1-p) = \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} = 2p^{-3}.$$

We next apply these equalities in finding $E(X)$ and $E(X^2)$. Indeed, we have

$$E(X) = \sum_{n=1}^{\infty} n(1-p)^{n-1}p = p\sum_{n=1}^{\infty} n(1-p)^{n-1} = p \cdot p^{-2} = p^{-1}$$

and

$$\begin{aligned} E(X(X-1)) =& \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-1}p \\ =& p(1-p)\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} \\ =& p(1-p)\cdot(2p^{-3}) = 2p^{-2}(1-p). \end{aligned}$$

Thus,

$$E(X^2) = E[X(X-1)] + E(X) = 2p^{-2}(1-p) + p^{-1} = (2-p)p^{-2}.$$

The variance is then given by

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = (2-p)p^{-2} - p^{-2} = \frac{1-p}{p^2}.$$

Next, observe that for $k = 1, 2, \cdots$ we have

$$P(X \geq k) = \sum_{n=k}^{\infty} p(1-p)^{n-1} = p(1-p)^{k-1} \sum_{n=0}^{\infty} (1-p)^n = \frac{p(1-p)^{k-1}}{1-(1-p)} = (1-p)^{k-1}$$

and

$$P(X \leq k) = 1 - P(X \geq k+1) = 1 - (1-p)^k.$$

From this, one can find the cdf X given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & x \geq 1. \end{cases}$$

Example 22.2
Used watch batteries are tested one at a time until a good battery is found. Let X denote the number of batteries that need to be tested in order to find the first good one. Find the expected value of X, given that $P(X > 3) = 0.5$.

Solution.
X has geometric distribution, so $P(X > 3) = P(X \geq 4) = (1-p)^3$. Setting this equal to $1/2$ and solving for p gives $p = 1 - 2^{-\frac{1}{3}}$. Therefore,

$$E(X) = \frac{1}{p} = \frac{1}{1 - 2^{-\frac{1}{3}}} \approx 4.8473 \blacksquare$$

Example 22.3
From past experience it is noted that 3% of customers at an ATM machine make deposits at the start of a new business day.
(a) What is the probability that the first deposit was made with the 5^{th} customer who used the ATM?
(b) What is the probability that the first deposit was made when 5 customers used the ATM?

Solution.
(a) Let X be the number of customers who used the ATM before the first deposit was made. Then X is a geometric random variable with $p = 0.03$. If 5 customers used the ATM before a deposit was made, then the first four customers did not make a deposit and the fifth customer made the deposit. Hence,

$$P(X = 5) = 0.03(0.97)^4 = 0.02656.$$

(b)

$$P(X \leq 5) = 1 - 0.97^5 \approx 0.1413 \blacksquare$$

Example 22.4
An archer shoots arrows at a circular target where the central portion of the target inside is called the bull. Suppose that an archer hits the target 70% of the time. Let X be the number of shots until the first hit. Find the expected value and the standard deviation of X.

Solution.
X is a geometric random variable with $p = 0.7$. Thus, $E(X) = \frac{1}{p} = \frac{1}{0.7} \approx 1.4286$ and $\sigma_X = \sqrt{\frac{1-p}{p^2}} = \sqrt{\frac{0.3}{0.7^2}} \approx 0.7825$ ■

Practice Problems

Problem 22.1
A box of candies contains 5 KitKat, 4 M&M, and 1 Crunch. Candies are drawn, with replacement, until a Crunch is found. If X is the random variable counting the number of trials until a Crunch appears, then
(a) What is the probability that the Crunch appears on the first trial?
(b) What is the probability that the Crunch appears on the second trial?
(c) What is is the probability that the Crunch appears on the n^{th} trial.

Problem 22.2
The probability that a computer chip is defective is 0.10. Each computer is checked for inspection as it is produced. Find the probability that at least 10 computer chips must be checked to find one that is defective.

Problem 22.3
Suppose a certain exam is classified as either difficult (with probability 90/92) or fair (with probability 2/92). Exams are taken one after the other. What is the probability that at least 4 difficult exams will occur before the first fair one?

Problem 22.4
Assume that every time you eat hot salsa, there is a 0.001 probability that you will get heartburn, independent of all other times you eaten hot salsa.
(a) What is the probability you will eat hot salsa two or less times until your first heartburn?
(b) What is the expected number of times you will eat hot salsa until you get your first heartburn?

Problem 22.5
Consider the experiment of flipping three coins simultaneously. Let a success be when the three outcomes are the same. What is the probability that
(a) exactly three rounds of flips are needed for the first success?
(b) more than four rounds are needed?

Problem 22.6
You roll a fair die repeatedly. Let a success be when the die shows either a 1 or a 3. Let X be the number of times you roll the die.
(a) What is $P(X = 3)$? What is $P(X = 50)$?
(b) Find the expected value of X.

Problem 22.7
Fifteen percent of the cars in a dealer's showroom have a sunroof. A salesperson starts showing you cars at random, one after the other. Let X be the number of cars with a sunroof that you see, before the first car that has no sunroof.
(a) What is the probability distribution function of X
(b) What is the probability distribution of $Y = X + 1$?

Problem 22.8
A study of car batteries shows that 3% of car batteries produced by a certain machine are defective. The batteries are put into packages of 20 batteries for distribution to retailers.
(a) What is the probability that a randomly selected package of batteries will contain at least 2 defective batteries?
(b) Suppose we continue to select packages of batteries randomly from the production site. What is the probability that it will take fewer than five packages to find a package with at least 2 defective batteries?

Problem 22.9
Show that the Geometric distribution with parameter p satisfies the equation

$$P(X > i + j | X > i) = P(X > j).$$

This says that the Geometric distribution satisfies the **memoryless property**

Problem 22.10 ‡
As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represent the number of tests completed when the first person with high blood pressure is found. The expected value of X is 12.5.
Calculate the probability that the sixth person tested is the first one with high blood pressure.

Problem 22.11
Suppose that the probability for an applicant to get a job offer after an interview is 0.1. An applicant plans to keep trying out for more interviews until she gets offered. Assume outcomes of interviews are independent.
(a) How many interviews does she expect to have to take in order to get a job offer?
(b) What is the probability she will need to attend more than 2 interviews?

Problem 22.12
In each of the following you are to determine whether the problem is a binomial type problem or a geometric type. In each case, find the probability mass function $p(x)$. Assume outcomes of individual trials are independent with constant probability of success.
(a) A arch shooter will aim at the target until one successfully hits it. The underlying probability of success is 0.40.
(b) A clinical trial enrolls 20 patients with a rare disease. Each patient is given an experimental therapy, and the number of patients showing marked improvement is observed. The true underlying probability of success is 0.60.

Problem 22.13
A lab network consisting of 20 computers was attacked by a computer virus. This virus enters each computer with probability 0.4, independently of other computers. A computer manager checks the lab computers, one after another, to see if they were infected by the virus. What is the probability that she has to test at least 6 computers to find the first infected one?

Problem 22.14
A representative from the National Football League's Marketing Division randomly selects people on a random street in Kansas City, Kansas until he finds a person who attended the last home football game. Let p, the probability that he succeeds in finding such a person, equal 0.20. And, let X denote the number of people he selects until he finds his first success. How many people should we expect the marketing representative needs to select before he finds one who attended the last home football game?

Problem 22.15
A plane's engines start successfully at a given attempt with a probability of 0.75. Any time that the mechanics are unsuccessful in starting the engines, they must wait five minutes before trying again. Find probability that the plane is launched within 10 minutes.

Problem 22.16
An urn contains N white and M black balls. Balls are randomly selected with replacement, one at a time, until a black one is obtained. What is the probability that exactly n draws are needed?

Problem 22.17
Let X be a geometric distribution with parameter p. Find a formula for $M_X(t) = E(e^{tX})$.

Problem 22.18
Let X be a geometric random variable with parameters p. Let Y be the random variable representing the number of failures before a first success. Then $Y = X - 1$. Find $E(Y)$ and $\text{Var}(Y)$.

Problem 22.19
I am new to basketball and so I am practicing to shoot the ball through the basket. If the probability of success at each throw is 0.2, how many times would I fail on average before a success occurs.

Problem 22.20
At an orchard in Maine, "20-lb" bags of apples are weighed. Suppose that four percent of the bags are underweight and that each bag weighed is independent. Let X be the number of bags observed to find the first underweight bag. Find $E(X)$ and σ_X.

Problem 22.21 ‡
An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0 . If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4.
What is the expected benefit under this policy?

Problem 22.22 ‡
A company has five employees on its health insurance plan. Each year, each employee independently has an 80% probability of no hospital admissions. If an employee requires one or more hospital admissions, the number of admissions is modeled by a geometric distribution with a mean of 1.50. The numbers of hospital admissions of different employees are mutually independent. Each hospital admission costs 20,000.
Calculate the probability that the company's total hospital costs in a year are less than 50,000.

Problem 22.23 ‡
A group of 100 patients is tested, one patient at a time, for three risk factors for a certain disease until either all patients have been tested or a patient tests positive for more than one of these three risk factors. For each risk factor, a patient tests positive with probability p, where $0 < p < 1$. The outcomes of the tests across all patients and all risk factors are mutually independent.
Determine an expression for the probability that exactly n patients are tested, where n is a positive integer less than 100.

Problem 22.24 ‡
A representative of a market research firm contacts consumers by phone in order to conduct surveys. The specific consumer contacted by each phone call is randomly determined. The probability that a phone call produces a completed survey is 0.25.
Calculate the probability that more than three phone calls are required to produce one completed survey.

Problem 22.25 ‡
Patients in a study are tested for sleep apnea, one at a time, until a patient is found to have this disease. Each patient independently has the same probability of having sleep apnea. Let r represent the probability that at least four patients are tested.
Determine the probability that at least twelve patients are tested given that at least four patients are tested.

23 Negative Binomial Random Variable

The geometric distribution is the distribution of the number of Bernoulli trials needed to get the first success. In this section, we consider an extension of this distribution. We will study the distribution of the number of independent Bernoulli trials needed to get the r^{th} success.

Consider a Bernoulli experiment where a success occurs with probability p and a failure occurs with probability $q = 1 - p$. Assume that the experiment continues, that is the trials are performed independently, until the r^{th} success occurs. For example, in the rolling of a fair die, let a success be when the die shows a 5. We roll the die repeatedly until the fourth time the face 5 appears. In this case, $p = \frac{1}{6}$ and $r = 4$.

Let X be the random variable representing the number of trials needed to get the r^{th} success. Then X is called a **negative binomial** distribution with parameters r and p. It is worth mentioning the difference between the binomial distribution and the negative binomial distribution: In the binomial distribution, X is the number of success in a fixed number of independent Bernoulli trials n. In the negative binomial distribution, X is the number of trials needed to get a fixed number of successes r.

For the r^{th} success to occur on the n^{th} trial, there must have been $r - 1$ successes among the first $n - 1$ trials. The number of ways of distributing $r - 1$ successes among $n - 1$ trials is ${}_{n-1}C_{r-1}$. But the probability of having $r - 1$ successes and $n - r$ failures is $p^{r-1}(1 - p)^{n-r}$. Since the probability of the r^{th} success is p, the product of these three terms (using independence) is the probability that there are r successes and $n - r$ failures in the n trials, with the r^{th} success occurring on the n^{th} trial. Hence, the probability mass function of X is

$$p(n) = P(X = n) = {}_{n-1}C_{r-1}p^r(1 - p)^{n-r},$$

and 0 otherwise, where $n = r, r + 1, \cdots$ (In order to have r successes there must be at least r trials.)

Note that if $r = 1$ then X is a geometric random variable with parameter p. The negative binomial distribution is sometimes defined in terms of the random variable $Y =$ number of failures before the r^{th} success. This formulation is statistically equivalent to the one given above in terms of $X =$ number of trials at which the r^{th} success occurs, since $Y = X - r$. The alternative form of the negative binomial distribution is

$$p_Y(y) = P(Y = y) = {}_{y+r-1}C_y p^r(1-p)^y = {}_{y+r-1}C_{r-1}p^r(1-p)^y, \;\; y = 0, 1, 2, \cdots .$$

In this form, the negative binomial distribution is used when the number of successes is fixed and we are interested in the number of failures before reaching the fixed number of successes.
Note that the binomial coefficient

$$_{r+y-1}C_y = \frac{(r+y-1)!}{y!(r-1)!} = \frac{(y+r-1)(y+r-2)\cdots(r+1)r}{y!}$$

can be alternatively written in the following manner, justifying the name "negative binomial:

$$_{r+y-1}C_y = (-1)^y \frac{(-r)(-r-1)\cdots(-r-y+1)}{y!} = (-1)^y {}_{-r}C_y$$

which is the defining equation for binomial coefficient with negative integers. Now, recalling the Taylor series expansion of the function $f(t) = (1-t)^{-r}$ at $t = 0$,

$$\begin{aligned}(1-t)^{-r} &= \sum_{k=0}^{\infty} (-1)^k {}_{-r}C_k t^k \\ &= \sum_{k=0}^{\infty} {}_{r+k-1}C_k t^k, \quad -1 < t < 1\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{y=0}^{\infty} P(Y=y) &= \sum_{y=0}^{\infty} {}_{r+y-1}C_y p^r (1-p)^y \\ &= p^r \sum_{y=0}^{\infty} {}_{r+y-1}C_y (1-p)^y \\ &= p^r \cdot p^{-r} = 1\end{aligned}$$

This shows that $p_Y(y)$ is a valid probability mass function.

Example 23.1
A research study is concerned with the side effects of a new drug. The drug is given to patients, one at a time, until two patients develop side effects. If the probability of getting a side effect from the drug is $\frac{1}{6}$, what is the probability that eight patients are needed?

Solution.
Let Y be the number of patients who do not show side effects. Then Y follows a negative binomial distribution with $r = 2,\ y = 6$, and $p = \frac{1}{6}$. Thus,

$$P(Y = 6) = {}_{2+6-1}C_6 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 \approx 0.0651 \blacksquare$$

Example 23.2
A person is conducting a phone survey. Define "success" as the event a person completes the survey and let Y be the number of failures before the third success. What is the probability that there are 10 failures before the third success? Assume that 1 out of 6 people contacted completed the survey.

Solution.
The probability that there are 10 failures before the third success is given by

$$P(Y = 10) = {}_{3+10-1}C_{10} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{10} \approx 0.0493 \blacksquare$$

Example 23.3
A four-sided die is rolled repeatedly. A success is when the die shows a 1. What is the probability that the tenth success occurs in the 40^{th} attempt?

Solution.
Let X number of attempts at which the tenth success occurs. Then X is a negative binomial random variable with parameters $r = 10$ and $p = 0.25$. Thus,

$$P(X = 40) = {}_{40-1}C_{10-1}(0.25)^{10}(0.75)^{30} \approx 0.0360911 \blacksquare$$

Expected value and Variance

The expected value of Y is

$$\begin{aligned}
E(Y) =& \sum_{y=0}^{\infty} {}_{r+y-1}C_y p^r (1-p)^y y \\
=& \sum_{y=1}^{\infty} \frac{(r+y-1)!}{(y-1)!(r-1)!} p^r (1-p)^y \\
=& r(1-p)p^r \sum_{y=1}^{\infty} {}_{r+y-1}C_{y-1} (1-p)^{y-1} \\
=& r(1-p)p^r \sum_{z=0}^{\infty} {}_{(r+1)+z-1}C_z (1-p)^z \\
=& r(1-p)p^r p^{-(r+1)} = \frac{r(1-p)}{p}.
\end{aligned}$$

It follows that

$$E(X) = E(Y+r) = E(Y) + r = \frac{r}{p}.$$

Similarly,

$$\begin{aligned}
E[Y(Y-1)] =& \sum_{y=0}^{\infty} {}_{r+y-1}C_y p^r (1-p)^y y(y-1) \\
=& \sum_{y=2}^{\infty} \frac{(r+y-1)!}{(y-2)!(r-1)!} p^r (1-p)^y \\
=& r(r+1)(1-p)^2 p^r \sum_{y=2}^{\infty} {}_{r+y-1}C_{y-2} (1-p)^{y-2} \\
=& r(r+1)(1-p)^2 p^r \sum_{z=0}^{\infty} {}_{(r+2)+z-1}C_z (1-p)^z \\
=& r(r+1)(1-p)^2 p^r p^{-(r+2)} = \frac{r(r+1)(1-p)^2}{p^2}.
\end{aligned}$$

Hence,

$$E(Y^2) = E(Y(Y-1)) + E(Y) = \frac{r(r+1)(1-p)^2}{p^2} + \frac{r(1-p)}{p} = \frac{r^2(1-p)^2}{p^2} + \frac{r(1-p)}{p^2}.$$

The variance of Y is

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{r(1-p)}{p^2}.$$

Since $X = Y + r$,

$$\text{Var}(X) = \text{Var}(Y) = \frac{r(1-p)}{p^2}.$$

Example 23.4
A person is conducting a phone survey. Suppose that 1 of 6 people contacted will complete the survey. Define "success" as the event a person completes the survey and let Y be the number of failures before the third success. Find $E(Y)$ and $\text{Var}(Y)$.

Solution.
The expected value of Y is

$$E(Y) = \frac{r(1-p)}{p} = \frac{3(1-1/6)}{(1/6)} = 15$$

and the variance is

$$\text{Var}(Y) = \frac{r(1-p)}{p^2} = \frac{3(1-1/6)}{(1/6)^2} = 90 \blacksquare$$

Practice Problems

Problem 23.1
Consider a biased coin with the probability of getting heads is 0.1. Let X be the number of flips needed to get the 8^{th} heads.
(a) What is the probability of getting the 8^{th} heads on the 50^{th} toss?
(b) Find the expected value and the standard deviation of X.

Problem 23.2
Recently it is found that the bottom of the Mediterranean sea near Cyprus has potential of oil discovery. Suppose that a well oil drilling has 20% chance of striking oil. Find the probability that the third oil strike comes on the 5^{th} well drilled.

Problem 23.3
Consider a 52-card deck. Repeatedly draw a card with replacement and record its face value. Let X be the number of trials needed to get three kings.
(a) What is the distribution of X?
(b) What is the probability that $X = 39$?

Problem 23.4
Repeatedly roll a fair die until the outcome 3 has occurred for the 4^{th} time. Let X be the number of times needed in order to achieve this goal. Find $E(X)$ and $\text{Var}(X)$.

Problem 23.5
Find the probability of getting the fourth head on the ninth flip of a fair coin.

Problem 23.6
There is 75% chance to pass the written test for a driver's license. What is the probability that a person will pass the test on the second try?

Problem 23.7 ‡
A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $\frac{3}{5}$.
The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months.

Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.

Problem 23.8
Somehow waiters at a cafe are extremely distracted today and are mixing orders giving customers decaf coffee when they ordered regular coffee. Suppose that there is 60% chance of making such a mistake in the order. What is the probability of getting the second decaf on the seventh order of regular coffee?

Problem 23.9
A machine that produces computer chips produces 3 defective chips out of 100. Computer chips are delivered to retailers in packages of 20 chips each.
(a) A package is selected randomly. What is the probability that the package will contain at least 2 defective chips?
(b) What is the probability that the tenth package selected is the third to contain at least two defective chips?

Problem 23.10
Let X be a negative binomial distribution with $r = 2$ and $p = 0.1$. Find $E(X)$ and σ_X.

Problem 23.11
Suppose that the probability of a child exposed to the flu will catch the flu is 0.40. What is the probability that the tenth child exposed to the flu will be the third to catch it?

Problem 23.12
In rolling a fair die repeatedly (and independently on successive rolls), find the probability of getting the third "3" on the n^{th} roll.

Problem 23.13 ‡
Each time a hurricane arrives, a new home has a 0.4 probability of experiencing damage. The occurrences of damage in different hurricanes are independent. Calculate the mode of the number of hurricanes it takes for the home to experience damage from two hurricanes. Hint: The mode of X is the number that maximizes the probability mass function of X.

Problem 23.14
A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. The probability that a couple agree to participate in the study is 0.2. What is the probability that 15 couples must be asked before 5 are found who agree to participate?

Problem 23.15
The probability that a basketball player makes a free-throw shots is 60%. The player was asked not to leave practice unless he makes 10 shots. Let Y be the number of free-throws missed prior to the 10^{th} shots. Find the mean and the variance of Y.

Problem 23.16
An oil company conducts a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. What is the probability that the third strike comes on the seventh well drilled?

Problem 23.17
An oil company conducts a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. Find the mean and the variance of the number of wells that must be drilled if the oil company wants to set up three producing wells?

Problem 23.18
Let X be a negative binomial distribution with parameters r and p. Find a formula for $M_X(t) = E(e^{tX})$.

Problem 23.19
Find $E(X)$ by using the function $M_X(t)$ of the previous problem.

Problem 23.20
Find the variance of X by using the function $M_X(t)$.

Problem 23.21 ‡
On any given day, a certain machine has either no malfunctions or exactly one malfunction. The probability of malfunction on any given day is 0.40. Machine malfunctions on different days are mutually independent. Calculate the probability that the machine has its third malfunction on the fifth day, given that the machine has not had three malfunctions in the first three days.

24 Hyper-geometric Random Variable

Consider a population of N objects where the objects can be divided exactly into two types: Type A and Type B. Suppose that the number of objects of type A is n. Then the number objects of Type B is $N - n$. For example, the sex gender of students in a certain college.
A random sample of size r is selected without replacement in such a way that each subset of size r is equally likely to be chosen. The **Hyper-geometric random variable** X counts the total number of objects of Type A in the sample.
If $r \leq n$ then the sample there could have at most r objects of Type A . If $r > n$, then there can be at most n objects of Type A in the sample. Thus, the value $\min\{r, n\}$ is the maximum possible number of objects of Type A in the sample.
On the other hand, if $r \leq N - n$, then the smallest number of objects of Type A is 0. If $r > N - n$, then the smallest number of objects of Type A is $r-(N-n)$. Thus, the value $\max\{0, r-(N-n)\}$ is the least possible number of objects of Type A in the sample.
What is the probability of having exactly k objects of Type A in the sample, where $\max\{0, r - (N - n)\} \leq k \leq \min\{r, n\}$? This is a type of problem that we have done before: In a group of N people there are n men (and the rest women). If we appoint a committee of r persons from this group at random, what is the probability there are exactly k men on it? The number of subsets of the group with cardinality r is ${}_NC_r$. The number of subsets of the men with cardinality k is ${}_nC_k$ and the number of subsets of the women with cardinality $r - k$ is ${}_{N-n}C_{r-k}$. Thus, the probability of getting exactly k men on the committee is

$$p(k) = P(X = k) = \frac{{}_nC_k \cdot {}_{N-n}C_{r-k}}{{}_NC_r}, \ k = 0, 1, \cdots, r.$$

This is the probability mass function of X. Note that $p(k) \geq 0$ and

$$\sum_{k=0}^{r} \frac{{}_nC_k \cdot {}_{N-n}C_{r-k}}{{}_NC_r} = 1.$$

The proof of this result follows from

Theorem 24.1 *(**Vendermonde's identity**)*

$$_{n+m}C_r = \sum_{k=0}^{r} {}_nC_k \cdot {}_mC_{r-k}.$$

Proof.
Suppose a committee consists of n men and m women. In how many ways can a subcommittee of r members be formed? The answer is $_{n+m}C_r$. But on the other hand, the answer is the sum over all possible values of k, of the number of subcommittees consisting of k men and $r-k$ women ■

Example 24.1
An urn contains 70 red marbles and 30 green marbles. If we draw out 20 without replacement, what is the probability of getting exactly 14 red marbles?

Solution.
If X is the number of red marbles, then X is a hyper-geometric random variable with parameters $N = 100, r = 20, n = 70$. Thus,

$$P(X = 14) = \frac{{}_{70}C_{14} \cdot {}_{30}C_6}{{}_{100}C_{20}} \approx 0.21 \text{ ■}$$

Example 24.2
A barn consists of 13 cows, 12 pigs and 8 horses. A group of 8 will be selected to participate in the city fair. What is the probability that exactly 5 of the group will be cows?

Solution.
Let X be the number of cows in the group. Then X is hyper-geometric random variable with parameters $N = 33, r = 8, n = 13$. Thus,

$$P(X = 5) = \frac{{}_{13}C_5 \cdot {}_{20}C_3}{{}_{33}C_8} \approx 0.10567 \text{ ■}$$

Expected Value and Variance
Next, we find the expected value of a hyper-geometric random variable with parameters N, n, r. First, we notice the following

$$_nC_k = \frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k}{}_{n-1}C_{k-1}$$

and

$$ {}_NC_r = \frac{N!}{r!(N-r)!} = \frac{N}{r}\frac{(N-1)!}{(r-1)!(N-r)!} = \frac{N}{r}{}_{N-1}C_{r-1}. $$

Thus,

$$ \begin{aligned} E(X) =& \sum_{k=0}^{r} kP(X=k) \\ =& \sum_{k=1}^{r} k\frac{{}_nC_k \cdot {}_{N-n}C_{r-k}}{{}_NC_r} \\ =& \frac{rn}{N}\sum_{k=1}^{r}\frac{({}_{n-1}C_{k-1})({}_{N-1-(n-1)}C_{r-1-(k-1)})}{{}_{N-1}C_{r-1}} \\ =& \frac{rn}{N}\sum_{i=0}^{r-1}\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_{N-1}C_{r-1}}. \end{aligned} $$

The sum in this equation is 1 as it is the sum over all probabilities of a hyper-geometric distribution with parameters $(N-1, r-1, n-1)$. Therefore we have

$$ E(X) = \frac{rn}{N}. $$

Next, we find the second moment of X. We have

$$ \begin{aligned} E(X^2) =& \sum_{k=0}^{r} k^2 P(X=k) \\ =& \sum_{k=1}^{r} k^2\frac{{}_nC_k \cdot {}_{N-n}C_{r-k}}{{}_NC_r} \\ =& n\sum_{k=1}^{r}\frac{k({}_{n-1}C_{k-1})({}_{N-1-(n-1)}C_{r-1-(k-1)})}{{}_NC_r} \\ =& n\sum_{i=0}^{r-1}(i+1)\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_NC_r} \\ =& n\left[\sum_{i=0}^{r-1} i\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_NC_r} + \sum_{i=0}^{r-1}\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_NC_r}\right] \\ =& n\left[\frac{r}{N}\sum_{i=0}^{r-1} i\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_{N-1}C_{r-1}} + \frac{r}{N}\sum_{i=0}^{r-1}\frac{({}_{n-1}C_{i})({}_{N-1-(n-1)}C_{r-1-i})}{{}_{N-1}C_{r-1}}\right]. \end{aligned} $$

The first sum is the expected value of a hyper-geometric random variable with parameters $(N-1, n-1, r-1)$ whereas the second sum is the sum over all probabilities of this random variable. Thus,

$$\sum_{i=0}^{r-1} i \frac{({}_{n-1}C_i)({}_{N-1-(n-1)}C_{r-1-i})}{{}_{N-1}C_{r-1}} = \frac{(r-1)(n-1)}{N-1}$$

and

$$\sum_{i=0}^{r-1} \frac{({}_{n-1}C_i)({}_{N-1-(n-1)}C_{r-1-i})}{{}_{N-1}C_{r-1}} = 1.$$

Hence,

$$E(X^2) = n\left(\frac{r}{N} \cdot \frac{(r-1)(n-1)}{N-1} + \frac{r}{N}\right) = \frac{nr}{N}\left[\frac{(r-1)(n-1)}{N-1} + 1\right].$$

The variance of X is

$$\begin{aligned} Var(X) =& E(X^2) - [E(X)]^2 = \frac{nr}{N}\left[\frac{(n-1)(r-1)}{N-1} + 1 - \frac{nr}{N}\right] \\ =& \frac{nr}{N} \cdot \frac{N-r}{N} \cdot \frac{N-n}{N-1}. \end{aligned}$$

Example 24.3
The faculty senate of a certain college has 20 members. Suppose there are 12 men and 8 women. A committee of 10 senators is selected at random.
(a) What is the probability that there will be 6 men and 4 women on the committee?
(b) What is the expected number of men on this committee?
(c) What is the variance of the number of men on this committee?

Solution.
Let X be the number of men of the committee of 10 selected at random. Then X is a hyper-geometric random variable with $N = 20, r = 10$, and $n = 12$.
(a) $P(X = 6) = \frac{{}_{12}C_6 \cdot {}_8C_4}{{}_{20}C_{10}} \approx 0.3501$
(b) $E(X) = \frac{nr}{N} = \frac{12(10)}{20} = 6$
(c) $\text{Var}(X) = n \cdot \frac{r}{N} \cdot \frac{N-r}{N} \cdot \frac{N-n}{N-1} \approx 1.2632$ ■

Example 24.4
A package of 15 computer chips contains 6 defective chips and 9 non-defective. Five chips are randomly selected without replacement.
(a) What is the probability that there are 2 defective and 3 non-defective chips in the sample?
(b) What is the probability that there are at least 3 non-defective chips in the sample?
(c) What is the expected number of defective chips in the sample?

Solution.
(a) Let X be the number of defective chips in the sample. Then, X has a hyper-geometric distribution with $n = 6, N = 15, r = 5$. The desired probability is

$$P(X = 2) = \frac{{}_6C_2 \cdot {}_9C_3}{{}_{15}C_5} = \frac{420}{1001}$$

(b) Note that the event that there are at least 3 non-defective chips in the sample is equivalent to the event that there are at most 2 defective chips in the sample, i.e. $\{X \leq 2\}$. So, we have

$$\begin{aligned} P(X \leq 2) =& P(X = 0) + P(X = 1) + P(X = 2) \\ =& \frac{{}_6C_0 \cdot {}_9C_5}{{}_{15}C_5} + \frac{{}_6C_1 \cdot {}_9C_4}{{}_{15}C_5} + \frac{{}_6C_2 \cdot {}_9C_3}{{}_{15}C_5} \\ =& \frac{714}{1001} \end{aligned}$$

(c) $E(X) = \frac{rn}{N} = 5 \cdot \frac{6}{15} = 2$ ■

Practice Problems

Problem 24.1
Five cards are drawn randomly and without replacement from a deck of 52 playing cards. Find the probability of getting exactly two black cards.

Problem 24.2
An urn contains 15 red marbles and 10 blue ones. Seven marbles were randomly drawn without replacement. Find the probability of picking exactly 3 red marbles.

Problem 24.3
A lottery game consists of matching 6 numbers from the official six drawn numbers out of 53 numbers. Let X equal the number of matches. Find the probability distribution function.

Problem 24.4
A package of 20 computer chips contains 4 defective chips. Randomly select 10 chips without replacement. Compute the probability of obtaining exactly 3 defective chips.

Problem 24.5
A wallet contains 10 \$50 bills and 190 \$1 bills. You randomly choose 10 bills without replacement. What is the probability that you will choose exactly 2 \$50 bills?

Problem 24.6
A batch of 8 components contains 2 defective components and 6 good ones. Randomly select four components without replacement.
(a) What is the probability that all four components are good?
(b) What are the mean and variance for the number of good components?

Problem 24.7
In Texas all vehicles are subject to annual inspection. A transportation company has a fleet of 20 trucks in which 7 do not meet the standards for passing inspection. Five trucks are randomly selected for inspection. What is the probability of no more than 2 trucks that fail to have the standards for passing inspection being selected?

Problem 24.8
A recent study shows that in a certain city 2,477 cars out of 123,850 are stolen. The city police are trying to find the stolen cars. Suppose that 100 randomly chosen cars are checked by the police. Find the expression that gives the probability that exactly 3 of the chosen cars are stolen. You do not need to give the numerical value of this expression.

Problem 24.9
Consider a suitcase with 7 shirts and 3 pants. Suppose we draw 4 items without replacement from the suitcase. Let X be the total number of shirts we get. Compute $P(X \leq 1)$.

Problem 24.10
A group consists of 4 women and 20 men. A committee of six is to be formed. Using the appropriate hyper-geometric distribution, what is the probability that none of the women are on the committee?

Problem 24.11
A jar contains 10 white balls and 15 black balls. Let X denote the number of white balls in a sample of 10 balls selected at random and without replacement. Find $\frac{\text{Var}(X)}{E(X)}$.

Problem 24.12
Among the 48 applicants for an actuarial position, 30 have a college degree in actuarial science. Ten of the applicants are randomly chosen for interviews. Let X be the number of applicants among these ten who have a college degree in actuarial science. Find $P(X \leq 8)$.

Problem 24.13
Suppose that a lot of 25 machine parts is delivered, where a part is considered acceptable only if it passes tolerance. We sample 10 parts and find that none are defective (all are within tolerance). What is the probability of this event if there are 6 defectives in the lot of 25?

Problem 24.14
A crate contains 50 light bulbs of which 5 are defective and 45 are not. A Quality Control Inspector randomly samples 4 bulbs without replacement. Let X be the number of defective bulbs selected. Find the probability mass function, $p(x)$, of the discrete random variable X.

Problem 24.15
A population of 70 registered voters contains 40 in favor of Proposition 134 and 30 opposed. An opinion survey selects a random sample without replacement of 10 voters from this population.
(a) What is the probability that there will be no one in favor of Proposition 134 in the sample?
(b) What is the probability that there will be at least one person in favor?

Problem 24.16
There are 9 men and 11 women in a group. 7 are chosen at random. What is the probability you get more women than men?

Problem 24.17
Five individuals from an animal population thought to be near extinction in a certain region have been caught, tagged, and released to mix into the population. After they have had an opportunity to mix, a random sample of 10 of these animals is selected. Let X be the number of tagged animals in the second sample. If there are actually 25 animals of this type in the region, what is the $E(X)$ and σ_X?

Problem 24.18
A Statistics department purchased 24 hand calculators from a dealer in order to have a supply on hand for tests for use by students who forget to bring their own. Although the department was not aware of this, five of the calculators were defective and gave incorrect answers to calculations. When a test is being written, students who have forgotten their own calculators are allowed to select one of the Department's (at random).
Suppose at the first test of the term, four students forgot to bring their calculators. What is the probability that exactly one of these students selects a defective calculator?

Problem 24.19 ‡
In a group of 25 factory workers, 20 are low-risk and five are high-risk. Two of the 25 factory workers are randomly selected without replacement.
Calculate the probability that exactly one of the two selected factory workers is low-risk.

Problem 24.20 ‡
In a casino game, a gambler selects four different numbers from the first

twelve positive integers. The casino then randomly draws nine numbers without replacement from the first twelve positive integers. The gambler wins the jackpot if the casino draws all four of the gambler's selected numbers. Calculate the probability that the gambler wins the jackpot.

Problem 24.21 ‡

A state is starting a lottery game. To enter this lottery, a player uses a machine that randomly selects six distinct numbers from among the first 30 positive integers. The lottery randomly selects six distinct numbers from the same 30 positive integers. A winning entry must match the same set of six numbers that the lottery selected. The entry fee is 1, each winning entry receives a prize amount of 500,000, and all other entries receive no prize. Calculate the probability that the state will lose money, given that 800,000 entries are purchased.

Cumulative and Survival Distribution Functions

In this chapter, we study the properties of two important functions in probability theory related to random variables: the cumulative distribution function and the survival distribution function.

25 The Cumulative Distribution Function

In this section, we will discuss properties of the cumulative distribution function that are valid to a random variable of type discrete, continuous or mixed. Recall from Section 14 that if X is a random variable then the **cumulative distribution function** (abbreviated c.d.f) is the function

$$F(t) = P(X \leq t).$$

First, we prove that a probability function P is a continuous set function. In order to do that, we need the following definitions.
A sequence of sets $\{E_n\}_{n=1}^{\infty}$ is said to be **increasing** if

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots$$

whereas it is said to be a **decreasing** sequence if

$$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots$$

If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of events we define a new event

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n.$$

For a decreasing sequence, we define

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n.$$

Proposition 25.1
If $\{E_n\}_{n\geq 1}$ is either an increasing or decreasing sequence of events then

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$$

(a) that is

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$

for an increasing sequence and
(b)

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$$

for a decreasing sequence.

Proof.
(a) Suppose first that $E_n \subset E_{n+1}$ for all $n \geq 1$. Define the events

$$\begin{aligned} F_1 =& E_1 \\ F_n =& E_n \cap E_{n-1}^c, \quad n > 1 \end{aligned}$$

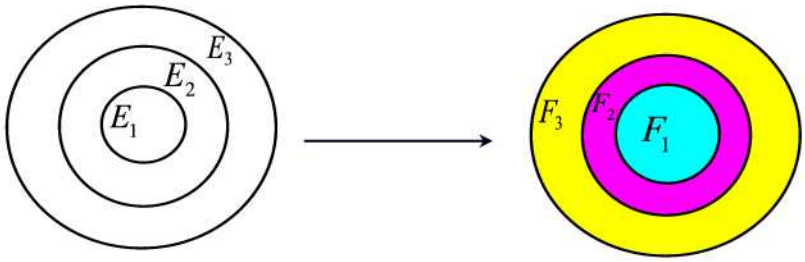


Figure 25.1

These events are shown in the Venn diagram of Figure 25.1. Note that for $n > 1$, F_n consists of those outcomes in E_n that are not in any of the earlier $E_i,\ i < n$. Clearly, for $i \neq j$ we have $F_i \cap F_j = \emptyset$. Also, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$ and for $n \geq 1$ we have $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. From these properties we have

$$\begin{aligned} P(\lim_{n\to\infty} E_n) =& P(\bigcup_{n=1}^{\infty} E_n) \\ =& P(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} P(F_n) \\ =& \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} F_i) \\ =& \lim_{n\to\infty} P(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty} P(E_n). \end{aligned}$$

(b) Now suppose that $\{E_n\}_{n\geq 1}$ is a decreasing sequence of events. Then $\{E_n^c\}_{n\geq 1}$ is an increasing sequence of events. Hence, from part (a), we have

$$P(\bigcup_{n=1}^{\infty} E_n^c) = \lim_{n\to\infty} P(E_n^c)$$

By De Morgan's Law we have $\bigcup_{n=1}^{\infty} E_n^c = \left(\bigcap_{n=1}^{\infty} E_n\right)^c$. Thus,

$$P\left(\left(\bigcap_{n=1}^{\infty} E_n\right)^c\right) = \lim_{n\to\infty} P(E_n^c).$$

Equivalently,

$$1 - P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} [1 - P(E_n)] = 1 - \lim_{n\to\infty} P(E_n)$$

or

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n) \blacksquare$$

Proposition 25.2
F is an non-decreasing function; that is, if $a \leq b$ then $F(a) \leq F(b)$.

Proof.
Suppose that $a \leq b$. Then $\{s : X(s) \leq a\} \subseteq \{s : X(s) \leq b\}$. This implies that $P(X \leq a) \leq P(X \leq b)$. Hence, $F(a) \leq F(b)$ ■

Example 25.1
Determine whether the given values can serve as the values of a cumulative distribution function of a random variable with the range $x = 1, 2, 3, 4$.

$$F(1) = 0.5,\ F(2) = 0.4,\ F(3) = 0.7, \text{ and } F(4) = 1.0.$$

Solution.
Since $F(2) < F(1)$, F is not increasing and therefore F can not be a cdf ■

Proposition 25.3
F is continuous from the right. That is, $\lim_{t \to b^+} F(t) = F(b)$.

Proof.
Let $\{b_n\}$ be a decreasing sequence that converges to b with $b_n \geq b$ for all $n \in \mathbb{N}$. Define $E_n = \{s : X(s) \leq b_n\}$. Then $\{E_n\}_{n \geq 1}$ is a decreasing sequence of events such that $\bigcap_{n=1}^{\infty} E_n = \{s : X(s) \leq b\}$. By Proposition 25.1, we have

$$\lim_{t \to b^+} F(t) = \lim_{n \to \infty} F(b_n) = \lim_{n \to \infty} P(E_n) = P\left(\bigcap_{n=1}^{\infty} E_n\right) = P(X \leq b) = F(b)$$ ■

Proposition 25.4
(a) $\lim_{b \to -\infty} F(b) = 0$
(b) $\lim_{b \to \infty} F(b) = 1$.

Proof.
(a) Note that $\lim_{x \to -\infty} F(x) = \lim_{n \to \infty} F(x_n)$ where (x_n) is a decreasing sequence such that $\lim_{n \to \infty} x_n = -\infty$. Define $E_n = \{s \in S : X(s) \leq x_n\}$. Then we have the nested chain $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$. Moreover,

$$\emptyset = \bigcap_{n=1}^{\infty} E_n.$$

By Proposition 25.1, we find

$$\lim_{x \to -\infty} F(x) = \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} P(E_n) = P\left(\bigcap_{n=1}^{\infty} E_n\right) = P(\emptyset) = 0.$$

(b) Note that $\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(x_n)$ where (x_n) is an increasing sequence such that $\lim_{n\to\infty} x_n = \infty$. Define $E_n = \{s \in S : X(s) \leq x_n\}$. Then we have the nested chain $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$. Moreover,

$$S = \bigcup_{n=1}^{\infty} E_n.$$

By Proposition 25.1, we find

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} P(E_n) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = P(S) = 1 \blacksquare$$

Example 25.2
Determine whether the given values can serve as the values of a cumulative distribution function of a random variable with the range $x = 1, 2, 3, 4$.

$$F(1) = 0.3,\ F(2) = 0.5,\ F(3) = 0.8, \text{ and } F(4) = 1.2.$$

Solution.
No, because $F(4)$ exceeds 1 $\blacksquare$

Remark 25.1
Any non-negative function $F(x)$ that satisfies Propositions 25.2 - 25.4 serves as a cumulative distribution function.

All probability questions can be answered in terms of the c.d.f.

Proposition 25.5
For any random variable X and any real number a, we have

$$P(X > a) = 1 - F(a).$$

Proof.
Let $A = \{x \in S : X(s) \leq a\}$. Then $A^c = \{s \in S : X(s) > a\}$. We have $P(X > a) = P(A^c) = 1 - P(A) = 1 - P(X \leq a) = 1 - F(a)$ $\blacksquare$

Example 25.3
Let X have probability mass function (pmf) $p(x) = \frac{1}{8}$ for $x = 1, 2, \cdots, 8$. Find
(a) the cumulative distribution function (cdf) of X;
(b) $P(X > 5)$.

Solution.
(a) The cdf is given by

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{\lfloor x \rfloor}{8} & 1 \leq x \leq 8 \\ 1 & x > 8 \end{cases}$$

where $\lfloor x \rfloor$ is the floor function of x.
(b) We have $P(X > 5) = 1 - F(5) = 1 - \frac{5}{8} = \frac{3}{8}$ ■

Proposition 25.6
For any random variable X and any real number a, we have

$$P(X < a) = \lim_{n \to \infty} F\left(a - \frac{1}{n}\right) = F(a^-).$$

Proof.
For each positive integer n, define $E_n = \{s \in S : X(s) \leq a - \frac{1}{n}\}$. Then $\{E_n\}$ is an increasing sequence of sets such that

$$\bigcup_{n=1}^{\infty} E_n = \{s \in S : X(s) < a\}.$$

We have

$$\begin{aligned} P(X < a) =& P\left(\bigcup_{n=1}^{\infty} E_n\right) = \left(\lim_{n \to \infty} E_n\right) \\ &= \lim_{n \to \infty} P\left(X \leq a - \frac{1}{n}\right) \\ &= \lim_{n \to \infty} F\left(a - \frac{1}{n}\right) = F(a^-) \ \blacksquare \end{aligned}$$

Note that $P(X < a)$ does not necessarily equal $F(a)$, since $F(a)$ also includes the probability that X equals a.

Corollary 25.1

$$P(X \geq a) = 1 - \lim_{n \to \infty} F\left(a - \frac{1}{n}\right) = 1 - F(a^-).$$

Proposition 25.7
If $a < b$ then $P(a < X \leq b) = F(b) - F(a)$.

Proof.
Let $A = \{s : X(s) > a\}$ and $B = \{s : X(s) \leq b\}$. Note that $P(A \cup B) = 1$. Then

$$\begin{aligned} P(a < X \leq b) =& P(A \cap B) \\ =& P(A) + P(B) - P(A \cup B) \\ =& (1 - F(a)) + F(b) - 1 = F(b) - F(a) \end{aligned}$$ ■

Proposition 25.8
If $a < b$ then $P(a \leq X < b) = F(b^-) - F(a^-)$.

Proof.
Let $A = \{s : X(s) \geq a\}$ and $B = \{s : X(s) < b\}$. Note that $P(A \cup B) = 1$. We have,

$$\begin{aligned} P(a \leq X < b) =& P(A \cap B) \\ =& P(A) + P(B) - P(A \cup B) \\ =& 1 - F(a^-) + F(b^-) - 1 \\ =& F(b^-) - F(a^-) \end{aligned}$$ ■

Proposition 25.9
If $a < b$ then $P(a \leq X \leq b) = F(b) - F(a^-)$.

Proof.
Let $A = \{s : X(s) \geq a\}$ and $B = \{s : X(s) \leq b\}$. Note that $P(A \cup B) = 1$. Then

$$\begin{aligned} P(a \leq X \leq b) =& P(A \cap B) \\ =& P(A) + P(B) - P(A \cup B) \\ =& 1 - F(a^-) + F(b) - 1 \\ =& F(b) - F(a^-) \end{aligned}$$ ■

Example 25.4
Show that $P(X = a) = F(a) - F(a^-)$.

Solution.
Applying the previous result we can write

$$P(X = a) = P(a \leq x \leq a) = F(a) - F(a^-) \blacksquare$$

Proposition 25.10
If $a < b$ then $P(a < X < b) = F(b^-) - F(a)$.

Proof.
Let $A = \{s : X(s) > a\}$ and $B = \{s : X(s) < b\}$. Note that $P(A \cup B) = 1$. Then

$$\begin{aligned} P(a < X < b) =& P(A \cap B) \\ =& P(A) + P(B) - P(A \cup B) \\ =& (1 - F(a)) + F(b^-) - 1 \\ =& F(b^-) - F(a) \blacksquare \end{aligned}$$

Figure 25.2 illustrates a typical F for a discrete random variable X. Note that for a discrete random variable the cumulative distribution function will always be a step function with jumps at each value of x that has probability greater than 0 and the size of the step at any of the values $x_1, x_2, x_3, \cdots$ is equal to the probability that X assumes that particular value.

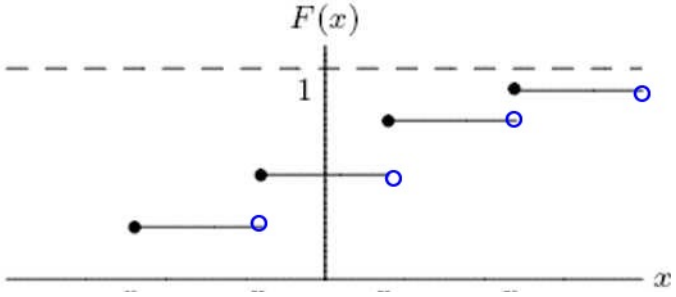


Figure 25.2

Example 25.5 *(*Mixed RV*)*
The distribution function of a random variable X, is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \leq x < 1 \\ \frac{2}{3}, & 1 \leq x < 2 \\ \frac{11}{12}, & 2 \leq x < 3 \\ 1, & 3 \leq x. \end{cases}$$

(a) Graph $F(x)$.
(b) Compute $P(X < 3)$.
(c) Compute $P(X = 1)$.
(d) Compute $P(X > \frac{1}{2})$
(e) Compute $P(2 < X \leq 4)$.

Solution.
(a) The graph is given in Figure 25.3.
(b) $P(X < 3) = F(3^-) = \frac{11}{12}$.
(c) $P(X = 1) = F(1) - F(1^-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.
(d) $P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$.
(e) $P(2 < X \leq 4) = F(4) - F(2) = \frac{1}{12}$ ■

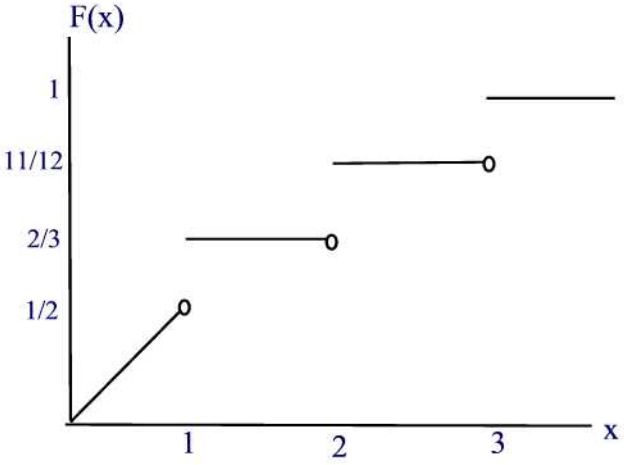


Figure 25.3

Example 25.6
If X has the cdf

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}, & -1 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 3 \\ \frac{3}{4}, & 3 \leq x < 5 \\ 1, & x \geq 5. \end{cases}$$

find
(a) $P(X \leq 3)$
(b) $P(X = 3)$
(c) $P(X < 3)$

(d) $P(X \geq 1)$
(e) $P(-0.4 < X < 4)$
(f) $P(-0.4 \leq X < 4)$
(g) $P(-0.4 < X \leq 4)$
(h) $P(-0.4 \leq X \leq 4)$
(i) $P(X = 5)$.

Solution.
(a) $P(X \leq 3) = F(3) = \frac{3}{4}$.
(b) $P(X = 3) = F(3) - F(3^-) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$
(c) $P(X < 3) = F(3^-) = \frac{1}{2}$
(d) $P(X \geq 1) = 1 - F(1^-) = 1 - \frac{1}{4} = \frac{3}{4}$
(e) $P(-0.4 < X < 4) = F(4^-) - F(-0.4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$
(f) $P(-0.4 \leq X < 4) = F(4^-) - F(-0.4^-) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$
(g) $P(-0.4 < X \leq 4) = F(4) - F(-0.4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$
(h) $P(-0.4 \leq X \leq 4) = F(4) - F(-0.4^-) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$
(i) $P(X = 5) = F(5) - F(5^-) = 1 - \frac{3}{4} = \frac{1}{4}$ ■

Practice Problems

Problem 25.1
In your pocket, you have 1 dime, 2 nickels, and 2 pennies. You select 2 coins at random (without replacement). Let X represent the amount (in cents) that you select from your pocket.
(a) Give (explicitly) the probability mass function for X.
(b) Give (explicitly) the cdf, $F(x)$, for X.
(c) How much money do you expect to draw from your pocket?

Problem 25.2
We are inspecting a lot of 25 batteries which contains 5 defective batteries. We randomly choose 3 batteries. Let $X =$ the number of defective batteries found in a sample of 3. Give the cumulative distribution function as a table.

Problem 25.3
Suppose that the cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{4} & 0 \le x < 1 \\ \frac{1}{2} + \frac{x-1}{4} & 1 \le x < 2 \\ \frac{11}{12} & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

(a) Find $P(X = i), i = 1, 2, 3$.
(b) Find $P(\frac{1}{2} < X < \frac{3}{2})$.

Problem 25.4
If the cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 2 \\ \frac{4}{5} & 2 \le x < 3 \\ \frac{9}{10} & 3 \le x < 3.5 \\ 1 & 3.5 \le x \end{cases}$$

Calculate the probability mass function.

Problem 25.5
Consider a random variable X whose distribution function (cdf) is given by

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.1 & -2 \leq x < 1.1 \\ 0.3 & 1.1 \leq x < 2 \\ 0.6 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(a) Give the probability mass function, $p(x)$, of X, explicitly.
(b) Compute $P(2 < X < 3)$.
(c) Compute $P(X \geq 3)$.
(d) Compute $P(X \geq 3 | X \geq 0)$.

Problem 25.6
Consider a random variable X whose probability mass function is given by

$$p(x) = \begin{cases} p & x = -1.9 \\ 0.1 & x = -0.1 \\ 0.3 & x = 20p \\ p & x = 3 \\ 4p & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is p?
(b) Find $F(x)$ and sketch its graph.
(c) What is $F(0)$? What is $F(2)$? What is $F(F(3.1))$?
(d) What is $P(2X - 3 \leq 4 | X \geq 2.0)$?
(e) Compute $E(F(X))$.

Problem 25.7
The cdf of X is given by

$$F(x) = \begin{cases} 0 & x < -4 \\ 0.3 & -4 \leq x < 1 \\ 0.7 & 1 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

(a) Find the probability mass function.
(b) Find the variance and the standard deviation of X.

Problem 25.8
In the game of "dice-flip", each player flips a coin and rolls one die. If the coin comes up tails, his score is the number of dots showing on the die. If the coin comes up heads, his score is twice the number of dots on the die. (i.e., (tails,4) is worth 4 points, while (heads,3) is worth 6 points.) Let X be the first player's score.
(a) Find the probability mass function $P(x)$.
(b) Compute the cdf $F(x)$ for all numbers x.
(c) Find the probability that $X < 4$. Is this the same as $F(4)$?

Problem 25.9
A random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x < 1 \\ \frac{1+x}{4} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

(a) What is the probability that $X = 0$? What is the probability that $X = 1$? What is the probability that $X = 2$?
(b) What is the probability that $\frac{1}{2} < X \le 1$?
(c) What is the probability that $\frac{1}{2} \le X < 1$?
(d) What is the probability that $X > 1.5$?
Hint: You may find it helpful to sketch this function.

Problem 25.10
Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < \frac{1}{2} \\ \alpha & x = \frac{1}{2} \\ 1 - 2^{-2x} & x > \frac{1}{2} \end{cases}$$

(a) Find $P(X > \frac{3}{2})$.
(b) Find $P(\frac{1}{4} < X \le \frac{3}{4})$.
(c) Find α.
(d) Find $P(X = \frac{1}{2})$.
(e) Sketch the graph of $F(x)$.

Problem 25.11
Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

(a) Find $P\left(X > \frac{1}{2}\right)$.
(b) Find $P\left(-\frac{1}{2} < X \leq \frac{3}{4}\right)$.
(c) Find $P\left(|X| \leq \frac{1}{2}\right)$.

Problem 25.12
Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < -5 \\ \frac{1}{144}(x+5)^2 & -5 \leq x < 7 \\ 1 & x \geq 7. \end{cases}$$

Find a such that $P(X > a) = \frac{2}{3}$.

Problem 25.13
Determine whether or not the following function is a cumulative distribution function.

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 0.5 & 0 < x \leq 1 \\ 0.75 & 1 < x \leq 3 \\ 1 & x \geq 3. \end{cases}$$

Problem 25.14
Let X be a random variable with cumulative distribution function $F(x)$. Find $P(X^2 \leq a)$, where $a > 0$.

Problem 25.15
A discrete random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.1 & 0 \leq x < 1 \\ 0.3 & 1 \leq x < 2 \\ 0.5 & 2 \leq x < 4 \\ 0.8 & 4 \leq x < 5 \\ 1 & x \geq 5. \end{cases}$$

Find the probability mass function $p(x)$.

Problem 25.16 ‡
A man purchases a life insurance policy on his 40^{th} birthday. The policy will pay 5000 if he dies before his 50^{th} birthday and will pay 0 otherwise. The length of lifetime, in years from birth, of a male born the same year as the insured has the cumulative distribution function

$$F(t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{\frac{1-1.1^t}{1000}}, & t > 0. \end{cases}$$

Calculate the expected payment under this policy.

Problem 25.17 ‡
Individuals purchase both collision and liability insurance on their automobiles. The value of the insured's automobile is V. Assume the loss L on an automobile claim is a random variable with cumulative distribution function

$$F(\ell) = \begin{cases} \frac{3}{4}\left(\frac{\ell}{V}\right)^3, & 0 \leq \ell < V \\ 1 - \frac{1}{10}e^{-\frac{\ell - V}{V}}, & \ell \geq V. \end{cases}$$

Calculate the probability that the loss on a randomly selected claim is greater than the value of the automobile.

26 The Survival Distribution Function

Another key function describing a random variable is the survival distribution function. We describe this function through an example. Let X be the random variable representing the age at death of a person and let $F(x)$ be the corresponding cumulative distribution function. Thus, for a positive integer a, the number $F(a) = P(X \leq a)$ is the probability that a person will die by age a. However, the number $S(a) = 1 - F(a)$ gives the probability that the person will survive to age a. Thus, the term "survival".
The function $S(x)$ is called the **survival function** (abbreviated SDF), also known as a **reliability function**. It gives the probability that a patient, device, or other object of interest will survive beyond any given specified time. Thus, we define the survival distribution function by

$$S(x) = P(X > x) = 1 - F(x).$$

It follows from the properties of the cumulative distribution function $F(x)$, that any survival function satisfies the properties: $S(-\infty) = 1$, $S(\infty) = 0$, $S(x)$ is right-continuous, and that $S(x)$ is non-increasing. These four conditions are necessary and sufficient so that any non-negative function $S(x)$ that satisfies these conditions serves as a survival function.

Remark 26.1
For a discrete random variable, the survival function need not be left-continuous, that is, it is possible for its graph to jump down. When it jumps, the value is assigned to the bottom of the jump.

Example 26.1
Let X be a continuous random variable with survival distribution defined by $S(x) = e^{-0.34x}$ for $x \geq 0$ and 1 otherwise. Compute $P(5 < X \leq 10)$.

Solution.
We have

$$P(5 < X \leq 10) = F(10) - F(5) = S(5) - S(10) = e^{-0.34\times 5} - e^{-0.34\times 10} \approx 0.149 \blacksquare$$

Example 26.2
Let X be the continuous random variable representing the age of death of an

individual. The survival distribution function for an individual is determined to be

$$S(x) = \begin{cases} 1, & x < 0 \\ \frac{75-x}{75}, & 0 \leq x \leq 75 \\ 0, & x > 75. \end{cases}$$

(a) Find the probability that the person dies before reaching the age of 18.
(b) Find the probability that the person lives more than 55 years.
(c) Find the probability that the person dies between the ages of 25 and 70.

Solution.
(a) First, note that for a continuous random variable $P(X = a) = F(a) - F(a^-) = F(a) - F(a) = 0$. Thus, $P(X \leq a) = P(X < a) + P(X = a) = P(X < a)$. We have

$$P(X < 18) = P(X \leq 18) = F(18) = 1 - S(18) = 0.24.$$

(b) We have

$$P(X > 55) = S(55) = 0.267.$$

(c) We have

$$P(25 < X < 70) = F(70) - F(25) = S(25) - S(70) = 0.60 \blacksquare$$

Practice Problems

Problem 26.1
Consider the continuous random variable X with survival distribution defined by

$$S(x) = \begin{cases} 1, & x < 0 \\ \frac{1}{10}(100 - x)^{\frac{1}{2}}, & 0 \leq x \leq 100 \\ 0, & x > 100. \end{cases}$$

(a) Find the corresponding expression for the cumulative probability function.
(b) Compute $P(65 < X \leq 75)$.

Problem 26.2
Let X denote the age at death of an individual. The survival distribution is given by

$$S(x) = \begin{cases} 1, & x < 0 \\ 1 - \frac{x}{100}, & 0 \leq x \leq 100 \\ 0, & x > 100. \end{cases}$$

(a) Find the probability that a person dies before reaching the age of 30.
(b) Find the probability that a person lives more than 70 years.

Problem 26.3
If X is a continuous random variable then the survival distribution function is defined by

$$S(x) = \int_x^\infty f(t)dt$$

where $f(t)$ is called the **probability density function** of X. Show that $F'(x) = f(x)$.

Problem 26.4
Let X be a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

where $\lambda > 0$. Find the probability density function $f(x)$.

Problem 26.5
Given the cumulative distribution function

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Find $S(x)$.

Problem 26.6
A survival distribution function is defined by

$$S(x) = \begin{cases} 1, & x < 0 \\ ax^2 + b, & 0 \leq x \leq \omega \\ 0 & x > \omega. \end{cases}$$

Determine the values of a and b.

Problem 26.7
Consider an age-at-death random variable X with survival distribution defined by

$$S(x) = \begin{cases} 1, & x < 0 \\ \frac{1}{10}(100 - x)^{\frac{1}{2}}, & 0 \leq x \leq 100 \\ 0 & x > 100. \end{cases}$$

(a) Explain why this is a suitable survival function.
(b) Find the corresponding expression for the cumulative probability function.
(c) Compute the probability that a newborn with survival function defined above will die between the ages of 65 and 75.

Problem 26.8
Let

$$S(x) = \begin{cases} 1, & x < 0 \\ \left(1 - \frac{x}{120}\right)^{\frac{1}{6}}, & 0 \leq x \leq 120 \\ 0 & x > 120. \end{cases}$$

where X is the age-at-death random variable. Determine the probability that a newborn survives beyond age 25.

Problem 26.9
The SDF of a continuous random variable is given by

$$S(x) = \begin{cases} 1, & x < 0 \\ e^{-0.34x}, & x \geq 0. \end{cases}$$

Find $P(10 < X \leq 23)$.

Problem 26.10
Show that the function

$$S(x) = \begin{cases} 1, & x < 0 \\ e^{-0.34x}, & x \geq 0. \end{cases}$$

can serve as a survival distribution function, where $x \geq 0$.

Problem 26.11
Consider an age-at-death random variable X with survival distribution defined by

$$S(x) = \begin{cases} 1, & x < 0 \\ e^{-0.34x}, & x \geq 0. \end{cases}$$

Compute $P(5 < X < 10)$.

Problem 26.12
Find the cumulative distribution function corresponding to the survival function

$$S(x) = \begin{cases} 1, & x < 0 \\ 1 - \frac{x^2}{100}, & 0 \leq x \leq 10 \\ 0, & x > 10. \end{cases}$$

Problem 26.13
Which of the following is a SDF?
(I) $S(x) = (x+1)e^{-x},\ x \geq 0$ and $S(x) = 1$ for $x < 0$.
(II) $S(x) = \frac{x}{2x+1},\ x \geq 0$ and $S(x) = 1$ for $x < 0$.
(III) $S(x) = \frac{x+1}{x+2},\ x \geq 0$ and $S(x) = 1$ for $x < 0$.

Problem 26.14
The mortality pattern of a certain population may be described as follows: Out of every 108 lives born together one dies annually until there are no survivors. Find a simple function that can be used as $S(x)$ for this population.

Problem 26.15
The density function of a random variable X is given by $f(x) = xe^{-x}$ for $x \geq 0$. Find the survival distribution function of X.

Problem 26.16
Consider an age-at-death random variable X with survival distribution defined by

$$S(x) = \begin{cases} 1, & x < 0 \\ e^{-0.34x}, & x \geq 0. \end{cases}$$

Find the PDF and CDF of X.

Calculus Prerequisite I

In this chapter, we discuss very briefly the concept of double integrals which we will encounter later in the book.

27 Improper Integrals

A very common mistake among students is when evaluating the integral $\int_{-1}^{1} \frac{1}{x} dx$. A non careful student will just argue as follows

$$\int_{-1}^{1} \frac{1}{x} dx = [\ln |x|]_{-1}^{1} = 0.$$

Unfortunately, that's not the right answer as we will see below. The important fact ignored here is that the integrand is not continuous at $x = 0$. In fact, $f(x) = \frac{1}{x}$ has an infinite discontinuity at $x = 0$.
A (proper) definite integral, denoted by $\int_a^b f(x)dx$, exists when
(a) $f(x)$ is continuous on $[a, b]$,
(b) $[a, b]$ is of finite length.

Improper integrals are integrals in which one or both of the above conditions are not met, i.e.,
(1) The interval of integration is infinite:

$$[a, \infty), (-\infty, b], (-\infty, \infty).$$

For example,

$$\int_1^{\infty} \frac{1}{x} dx.$$

(2) The integrand has an infinite discontinuity at some point c in the interval $[a, b]$, i.e., the integrand is unbounded near c :

$$\lim_{x\to c} f(x) = \pm\infty.$$

For example,

$$\int_{-1}^{1} \frac{1}{x} dx.$$

An improper integral is defined in terms of limits so it may exist or may not exist. If the limit exists, we say that the improper integral is **convergent.** Otherwise, the integral is **divergent**.
We will consider only improper integrals with positive integrands since they are the most common in probability.

Infinite Intervals of Integration
The first type of improper integrals arises when the domain of integration is infinite but the integrand is still continuous in the domain of integration. In case one of the limits of integration is infinite, we define

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

or

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

If both limits are infinite, we write

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

Example 27.1
Does the integral $\int_1^{\infty} \frac{1}{x^2} dx$ converge or diverge?

Solution.
We have

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2} dx = \lim_{b\to\infty} [-\frac{1}{x}]_1^b = \lim_{b\to\infty} (-\frac{1}{b} + 1) = 1.$$

In terms of area, the given integral represents the area under the graph of $f(x) = \frac{1}{x^2}$ from $x = 1$ and extending infinitely to the right. The above improper integral says the following. Let $b > 1$ and obtain the area shown in Figure 27.1.

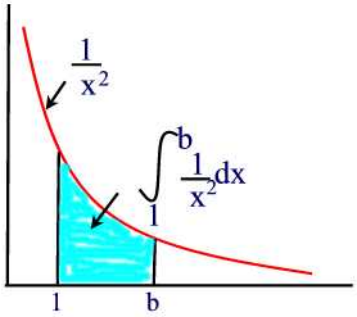


Figure 27.1

Then $\int_1^b \frac{1}{x^2}dx$ is the area under the graph of $f(x)$ from $x = 1$ to $x = b$. As b gets larger and larger this area is close to 1 ■

Example 27.2
Does the improper integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ converge or diverge?

Solution.
We have

$$\int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}[2\sqrt{x}]_1^b = \lim_{b\to\infty}(2\sqrt{b}-2) = \infty.$$

So the improper integral is divergent ■

The following example generalizes the results of the previous two examples.

Example 27.3
Determine for which values of p the improper integral $\int_1^\infty \frac{1}{x^p}dx$ diverges.

Solution.
Suppose first that $p = 1$. Then

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx$$
$$= \lim_{b\to\infty}[\ln|x|]_1^b = \lim_{b\to\infty}\ln b = \infty$$

so the improper integral is divergent.
Now, suppose that $p \neq 1$. Then

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b x^{-p}dx$$
$$= \lim_{b\to\infty}[\frac{x^{-p+1}}{-p+1}]_1^b$$
$$= \lim_{b\to\infty}(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}).$$

If $p < 1$ then $-p+1 > 0$ so that $\lim_{b\to\infty} b^{-p+1} = \infty$ and therefore the improper integral is divergent. If $p > 1$ then $-p+1 < 0$ so that $\lim_{b\to\infty} b^{-p+1} = 0$ and hence the improper integral converges:

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\ \blacksquare$$

Example 27.4
For what values of c is the improper integral $\int_0^\infty e^{cx}dx$ convergent?

Solution.
We have

$$\int_0^\infty e^{cx}dx = \lim_{b\to\infty}\int_0^b e^{cx}dx = \lim_{b\to\infty}\frac{1}{c}e^{cx}|_0^b$$
$$= \lim_{b\to\infty}\frac{1}{c}(e^{cb}-1) = -\frac{1}{c}$$

provided that $c < 0$. Otherwise, i.e. if $c \geq 0$, then the improper integral is divergent.$\blacksquare$

Remark 27.1
The previous two results are very useful and you may want to memorize them.

Example 27.5
Show that the improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ converges.

Solution.
We have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \lim_{a\to\infty}\int_{-a}^{a} \frac{1}{1+x^2}dx \\ &= \lim_{a\to\infty} \tan^{-1} x\Big]_{-a}^{c} + \lim_{a\to\infty} \tan^{-1} x\Big]_{c}^{a} \\ &= \lim_{a\to\infty}[\tan^{-1} a - \tan^{-1}(-a)] \\ &= \lim_{a\to\infty} 2\tan^{-1} a \\ &= 2\left(\frac{\pi}{2}\right) = \pi \ \blacksquare\end{aligned}$$

Integrands with infinite discontinuity
Suppose $f(x) > 0$ is unbounded at $x = a$ such that $\lim\limits_{x\to a^+} f(x) = \infty$. Then we define

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx.$$

Similarly, if $f(x)$ is unbounded at $x = b$ with $\lim_{x\to b^-} f(x) = \infty$ then we define

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx.$$

Now, if $f(x)$ is unbounded at an interior point $a < c < b$ then we define

$$\int_a^b f(x)dx = \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx.$$

If both limits exist then the integral on the left-hand side converges. If one of the limits does not exist then we say that the improper integral is divergent.

Example 27.6
Show that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges.

Solution.
Indeed,

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} 2\sqrt{x}|_t^1$$
$$= \lim_{t\to 0^+}(2-2\sqrt{t}) = 2.$$

In terms of area, we pick a $t > 0$ as shown in Figure 27.2. Then the shaded area is $\int_t^1 \frac{1}{\sqrt{x}}dx$. As t approaches 0 from the right, the area approaches the value 2 ■

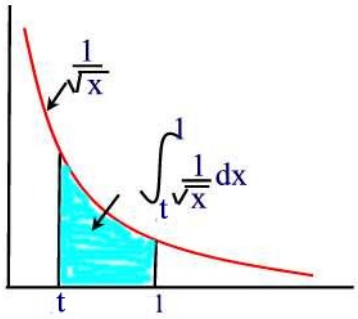


Figure 27.2

Example 27.7
Investigate the convergence of $\int_0^2 \frac{1}{(x-2)^2}dx$.

Solution.
We deal with this improper integral as follows

$$\int_0^2 \frac{1}{(x-2)^2}dx = \lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)^2}dx = \lim_{t\to 2^-} -\frac{1}{(x-2)}\Big]_0^t$$
$$= \lim_{t\to 2^-}\left(-\frac{1}{t-2}-\frac{1}{2}\right) = \infty,$$

so that the given improper integral is divergent ■

Example 27.8
Investigate the improper integral $\int_{-1}^1 \frac{1}{x}dx$.

Solution.
We first write

$$\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx.$$

On one hand, we have

$$\begin{aligned} \int_{-1}^{0} \frac{1}{x} dx &= \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x} dx = \lim_{t \to 0^-} \ln |x| \Big]_{-1}^{t} \\ &= \lim_{t \to 0^-} \ln (|t|) = \infty. \end{aligned}$$

This shows that the improper integral $\int_{-1}^{0} \frac{1}{x} dx$ is divergent and therefore the improper integral $\int_{-1}^{1} \frac{1}{x} dx$ is divergent ■

Improper Integrals of Mixed Type
Now, if the interval of integration is unbounded and the integrand is unbounded at one or more points inside the interval of integration we can evaluate the improper integral by decomposing it into a sum of an improper integral with finite interval but where the integrand is unbounded and an improper integral with an infinite interval. If each component integrals converges, then we say that the original integral converges to the sum of the values of the component integrals. If one of the component integrals diverges, we say that the entire integral diverges.

Example 27.9
Investigate the convergence of $\int_{0}^{\infty} \frac{1}{x^2} dx$.

Solution.
Note that the interval of integration is infinite and the function is undefined at $x = 0$. So we write

$$\int_{0}^{\infty} \frac{1}{x^2} dx = \int_{0}^{1} \frac{1}{x^2} dx + \int_{1}^{\infty} \frac{1}{x^2} dx.$$

But

$$\int_{0}^{1} \frac{1}{x^2} dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x^2} dx = \lim_{t \to 0^+} -\frac{1}{x} |_{t}^{1} = \lim_{t \to 0^+} (\frac{1}{t} - 1) = \infty.$$

Thus, $\int_{0}^{1} \frac{1}{x^2} dx$ diverges and consequently the improper integral $\int_{0}^{\infty} \frac{1}{x^2} dx$ diverges ■

Comparison Tests for Improper Integrals

Sometimes it is difficult to find the exact value of an improper integral by anti-differentiation, for instance the integral $\int_0^\infty e^{-x^2}dx$. However, it is still possible to determine whether an improper integral converges or diverges. The idea is to compare the integral to one whose behavior we already know, such us

- the p-integral $\int_1^\infty \frac{1}{x^p}dx$ which converges for $p > 1$ and diverges otherwise;
- the integral $\int_0^\infty e^{cx}dx$ which converges for $c < 0$ and diverges for $c \geq 0$;
- the integral $\int_0^1 \frac{1}{x^p}dx$ which converges for $p < 1$ and diverges otherwise.

The comparison method consists of the following:

Theorem 27.1

Suppose that f and g are continuous and $0 \leq g(x) \leq f(x)$ for all $x \geq a$. Then

(a) if $\int_a^\infty f(x)dx$ is convergent, so is $\int_a^\infty g(x)dx$

(b) if $\int_a^\infty g(x)dx$ is divergent, so is $\int_a^\infty f(x)dx$.

This is only common sense: if the curve $y = g(x)$ lies below the curve $y = f(x)$, and the area of the region under the graph of $f(x)$ is finite, then of course so is the area of the region under the graph of $g(x)$. Similar results hold for the other types of improper integrals.

Example 27.10

Determine whether $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ converges.

Solution.

For $x \geq 1$, we have that $x^3 + 5 \geq x^3$ so that $\sqrt{x^3+5} \geq \sqrt{x^3}$. Thus, $\frac{1}{\sqrt{x^3+5}} \leq \frac{1}{\sqrt{x^3}}$. Letting $f(x) = \frac{1}{\sqrt{x^3}}$ and $g(x) = \frac{1}{\sqrt{x^3+5}}$ then we have that $0 \leq g(x) \leq f(x)$. From the previous section we know that $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx$ is convergent, a p-integral with $p = \frac{3}{2} > 1$. By the comparison test, $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ is convergent ■

Example 27.11

Investigate the convergence of $\int_4^\infty \frac{dx}{\ln x - 1}$.

Solution.

For $x \geq 4$, we know that $\ln x - 1 < \ln x < x$. Thus, $\frac{1}{\ln x - 1} > \frac{1}{x}$. Let $g(x) = \frac{1}{x}$

and $f(x) = \frac{1}{\ln x - 1}$. Thus, $0 < g(x) \leq f(x)$. Since $\int_4^\infty \frac{1}{x} dx = \int_1^\infty \frac{1}{x} dx - \int_1^4 \frac{1}{x} dx$ and the integral $\int_1^\infty \frac{1}{x} dx$ is divergent being a p-integral with $p = 1$, the integral $\int_4^\infty \frac{1}{x} dx$ is divergent. By the comparison test $\int_4^\infty \frac{dx}{\ln x - 1}$ is divergent ■

Practice Problems

Problem 27.1
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0} \frac{dx}{\sqrt{3-x}}.$$

Problem 27.2
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-1}^{1} \frac{e^x}{e^x - 1} dx.$$

Problem 27.3
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{1}^{4} \frac{dx}{x-2}.$$

Problem 27.4
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{1}^{10} \frac{dx}{\sqrt{10-x}}.$$

Problem 27.5
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}.$$

Problem 27.6
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{dx}{x^2+4}.$$

Problem 27.7
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0} e^x dx.$$

Problem 27.8
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{dx}{(x-5)^{\frac{1}{3}}}.$$

Problem 27.9
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{2} \frac{dx}{(x-1)^2}.$$

Problem 27.10
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} \frac{x}{x^2+9} dx.$$

Problem 27.11
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{1} \frac{4dx}{\sqrt{1-x^2}}.$$

Problem 27.12
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} xe^{-x} dx.$$

Problem 27.13
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{1} \frac{x^2}{\sqrt{1-x^3}} dx.$$

Problem 27.14
Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_1^2 \frac{x}{x-1} dx.$$

Problem 27.15
Investigate the convergence of $\int_4^\infty \frac{dx}{\ln x - 1}$.

Problem 27.16
Investigate the convergence of the improper integral $\int_1^\infty \frac{\sin x + 3}{\sqrt{x}} dx$.

Problem 27.17
Investigate the convergence of $\int_1^\infty e^{-\frac{1}{2}x^2} dx$.

Continuous Random Variables

Discrete random variables are functions with domain the sample space and a countable range. In contrast, continuous random variables are functions with an uncountable infinite range such as an interval. For example, the height of a randomly selected tree. In this chapter, we discuss this type of random variables.

28 Distribution Functions

We say that a random variable X is **continuous** if there exists a non-negative function f (not necessarily continuous) defined for all real numbers and having the property that for any set B of real numbers we have

$$P(X \in B) = \int_B f(x)dx.$$

We call the function f the **probability density function** (abbreviated pdf) of the random variable X.
If we let $B = (-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)dx = P[X \in (-\infty, \infty)] = 1.$$

Now, if we let $B = [a, b]$ then

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

That is, areas under the probability density function represent probabilities as illustrated in Figure 28.1.

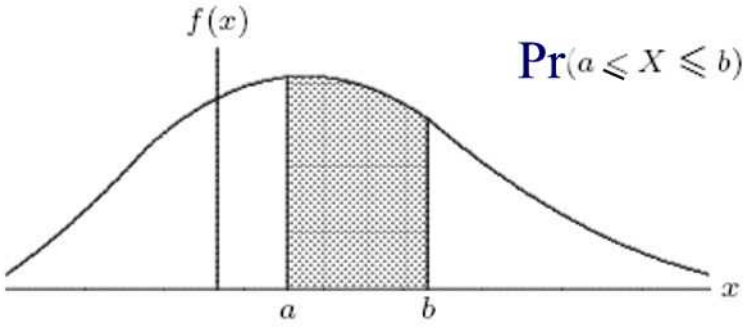


Figure 28.1

Now, if we let $a = b$ in the previous formula we find

$$P(X = a) = \int_a^a f(x)dx = 0.$$

It follows from this result that

$$P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = P(a \leq X \leq b)$$

and

$$P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)dx$$

and

$$P(X \geq a) = P(X > a) = \int_{a}^{\infty} f(x)dx.$$

The **cumulative distribution function** or simply the **distribution function** (abbreviated cdf) $F(t)$ of the random variable X is defined as follows

$$F(t) = P(X \leq t)$$

i.e., $F(t)$ is equal to the probability that the variable X assumes values, which are less than or equal to t. From this definition we can write

$$F(t) = \int_{-\infty}^{t} f(y)dy.$$

Geometrically, $F(t)$ is the area under the graph of f to the left of t.

Example 28.1
Find the distribution functions corresponding to the following density functions:
(a) $f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$
(b) $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty$
(c) $f(x) = \frac{a-1}{(1+x)^a}, \quad 0 < x < \infty$
(d) $f(x) = k\alpha x^{\alpha-1}e^{-kx^\alpha}, \quad 0 < x < \infty, k > 0, \alpha > 0.$

Solution.
(a)

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)} dy \\ &= \left[\frac{1}{\pi}\arctan y\right]_{-\infty}^{x} \\ &= \frac{1}{\pi}\arctan x - \frac{1}{\pi}\cdot\frac{-\pi}{2} \\ &= \frac{1}{\pi}\arctan x + \frac{1}{2}. \end{aligned}$$

(b)

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{e^{-y}}{(1+e^{-y})^2} dy \\ &= \left[\frac{1}{1+e^{-y}}\right]_{-\infty}^{x} \\ &= \frac{1}{1+e^{-x}} \end{aligned}$$

(c) For $x \geq 0$

$$\begin{aligned} F(x) &= \int_{0}^{x} \frac{a-1}{(1+y)^a} dy \\ &= \left[-\frac{1}{(1+y)^{a-1}}\right]_{0}^{x} \\ &= 1 - \frac{1}{(1+x)^{a-1}}. \end{aligned}$$

For $x < 0$ it is obvious that $F(x) = 0$, so we could write the result in full as

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{(1+x)^{a-1}} & x \geq 0. \end{cases}$$

(d) For $x \geq 0$

$$\begin{aligned} F(x) &= \int_0^x k\alpha y^{\alpha-1} e^{-ky^\alpha} dy \\ &= \left[-e^{-ky^\alpha}\right]_0^x \\ &= 1 - e^{-kx^\alpha}. \end{aligned}$$

For $x < 0$ we have $F(x) = 0$ so that

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-kx^\alpha} & x \geq 0 \end{cases} \blacksquare$$

Example 28.2
If the probability density of X is given by

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the probability density of $Y = X^3$.

Solution.
We have

$$\begin{aligned} F(y) =& P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{\frac{1}{3}}) \\ &= \int_0^{y^{\frac{1}{3}}} 6x(1-x)dx = 3y^{\frac{2}{3}} - 2y \end{aligned}$$

Hence, $f(y) = F'(y) = 2(y^{-\frac{1}{3}} - 1)$, for $0 < y < 1$ and 0 otherwise $\blacksquare$

Example 28.3
Suppose X is an exponential random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the density function of $Y = e^X$?

Solution.
For $y \geq 1$, we have

$$\begin{aligned} F(y) =& P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) \\ =& F_X(\ln y) = \int_0^{\ln y} \lambda e^{-\lambda x} dx \\ =& - e^{-\lambda x}\Big|_0^{\ln y} \\ =& 1 - y^{-\lambda}. \end{aligned}$$

This,

$$f_Y(y) = \lambda y^{-\lambda - 1}, \ y \geq 1$$

and 0 otherwise ■

Next, we list the properties of the cumulative distribution function $F(t)$ for a continuous random variable X.

Theorem 28.1
The cumulative distribution function of a continuous random variable X satisfies the following properties:
(a) $0 \leq F(t) \leq 1$.
(b) $F(t)$ is a non-decreasing function, i.e. if $a < b$ then $F(a) \leq F(b)$.
(c) $F(t) \to 0$ as $t \to -\infty$ and $F(t) \to 1$ as $t \to \infty$.
(d) $P(a < X \leq b) = F(b) - F(a)$.
(e) F is continuous.
(f) $F'(t) = f(t)$ whenever the derivative exists.

Proof.
Properties $(a) - (d)$ were established in Section 25. For part (e), we know that F is right continuous (See Proposition 25.3). Left-continuity follows from Example 25.4 and the fact that $P(X = a) = 0$. Part (f) is the result of applying the Second Fundamental Theorem of Calculus to the function $F(x) = \int_{-\infty}^{x} f(t)dt$ ■

Figure 28.2 illustrates a representative cdf.

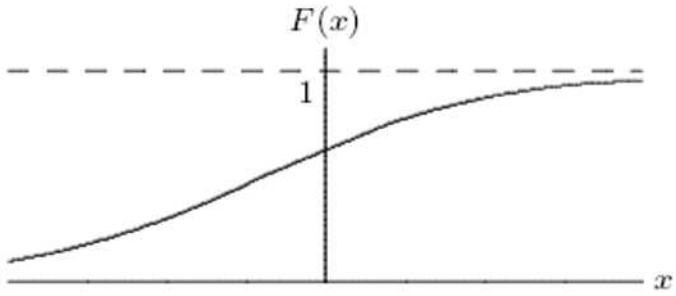


Figure 28.2

Remark 28.1
It is important to keep in mind that a pdf does not represent a probability. However, it can be used as a measure of how likely it is that the random variable will be near a. To see this, let $\epsilon > 0$ be a small positive number. Then

$$P(a \leq X \leq a+\epsilon) = F(a+\epsilon) - F(a) = \int_a^{a+\epsilon} f(t)dt \approx \epsilon f(a).$$

In particular,

$$\begin{aligned} P\left(a - \frac{\epsilon}{2} \leq X \leq a + \frac{\epsilon}{2}\right) =& P\left(a - \frac{\epsilon}{2} \leq X \leq a\right) + P\left(a \leq X \leq \frac{\epsilon}{2}\right) \\ \approx& \frac{\epsilon}{2} f(a) + \frac{\epsilon}{2} f(a) = \epsilon f(a). \end{aligned}$$

This means that the probability that X will be contained in an interval of length ϵ around the point a is approximately $\epsilon f(a)$.

Remark 28.2
From Theorem 28.1 (c) and the fact that the graph of $F(t)$ levels off when $t \to \pm\infty$, we find $\lim_{t\to\pm\infty} F'(t) = 0$. Using (f), we have $\lim_{t\to\pm\infty} f(t) = 0$.

Example 28.4
Suppose that the function $f(t)$ defined below is the density function of some random variable X.

$$f(t) = \begin{cases} e^{-t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Compute $P(-10 \leq X \leq 10)$.

Solution.

$$\begin{aligned} P(-10 \le X \le 10) &= \int_{-10}^{10} f(t)dt \\ &= \int_{-10}^{0} f(t)dt + \int_{0}^{10} f(t)dt \\ &= \int_{0}^{10} e^{-t}dt \\ &= -e^{-t}\Big|_0^{10} = 1 - e^{-10} \end{aligned}$$ ■

A pdf need not be continuous, as the following example illustrates.

Example 28.5
(a) Determine the value of c so that the following function is a pdf.

$$f(x) = \begin{cases} \frac{15}{64} + \frac{x}{64} & -2 \le x \le 0 \\ \frac{3}{8} + cx & 0 < x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) Determine $P(-1 \le X \le 1)$.
(c) Find $F(x)$.

Solution.
(a) Observe that f is discontinuous at the points -2 and 0, and is potentially also discontinuous at the point 3. We first find the value of c that makes f a pdf.

$$\begin{aligned} 1 &= \int_{-2}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{3} \left(\frac{3}{8} + cx\right) dx \\ &= \left[\frac{15}{64}x + \frac{x^2}{128}\right]_{-2}^{0} + \left[\frac{3}{8}x + \frac{cx^2}{2}\right]_{0}^{3} \\ &= \frac{30}{64} - \frac{2}{64} + \frac{9}{8} + \frac{9c}{2} \\ &= \frac{100}{64} + \frac{9c}{2} = \frac{25}{16} + \frac{9c}{2}. \end{aligned}$$

Solving for c we find $c = -\frac{1}{8}$.
(b) The probability $P(-1 \leq X \leq 1)$ is calculated as follows.

$$P(-1 \leq X \leq 1) = \int_{-1}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{1} \left(\frac{3}{8} - \frac{x}{8}\right) dx = \frac{69}{128}.$$

(c) For $-2 \leq x \leq 0$ we have

$$F(x) = \int_{-2}^{x} \left(\frac{15}{64} + \frac{t}{64}\right) dt = \frac{x^2}{128} + \frac{15}{64}x + \frac{7}{16}$$

and for $0 < x \leq 3$

$$F(x) = \int_{-2}^{0} \left(\frac{15}{64} + \frac{x}{64}\right) dx + \int_{0}^{x} \left(\frac{3}{8} - \frac{t}{8}\right) dt = \frac{7}{16} + \frac{3}{8}x - \frac{1}{16}x^2.$$

Hence the full cdf is

$$F(x) = \begin{cases} 0 & x < -2 \\ \frac{x^2}{128} + \frac{15}{64}x + \frac{7}{16} & -2 \leq x \leq 0 \\ \frac{7}{16} + \frac{3}{8}x - \frac{1}{16}x^2 & 0 < x \leq 3 \\ 1 & x > 3. \end{cases}$$

Observe that at all points of discontinuity of the pdf, the cdf is continuous. That is, even when the pdf is discontinuous, the cdf is continuous ■

Remark 28.3
For a continuous distribution, the graph of its CDF is continuous non-decreasing curve. For a discrete distribution, the graph of its CDF consists of a series of horizontal lines, with jumps between them. If you see both jumps and pieces of continuous increasing curves, you are looking at a CDF of a mixed distribution. For example, the random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2 - 2x + 2}{2} & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

is a mixed random variable with the discrete portion concentrated at $X = 1$. Now, since

$$P(1 < X \leq 2) = F(2) - F(1) = \frac{1}{2}$$

and $P(X > 2) = 1 - F(2) = 0$, we must have $P(X = 1) = \frac{1}{2}$. Hence, the pdf of X is

$$f(x) = F'(x) = \begin{cases} \frac{1}{2} & x = 1 \\ x - 1 & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Practice Problems

Problem 28.1
Determine the value of c so that the following function is a pdf.

$$f(x) = \begin{cases} \frac{c}{(x+1)^3} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 28.2
Let X denote the length of time (in minutes) of using a computer at a public library with pdf given by

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the probability of using a computer for more than 10 minutes.
(b) Find the probability of using a computer between 5 and 10 minutes.
(c) Find the cumulative distribution function of X.

Problem 28.3
A probability student is always late to class and arrives within ten minutes after the start of the class. Let X be the time that elapses between the start of the class and the time the student arrives to class with a probability density function

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

where $k > 0$ is a constant. Compute the value of k and then find the probability that the student arrives less than 3 minutes after the start of the class.

Problem 28.4
The lifetime X of a battery (in hours) has a density function given by

$$f(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ \frac{3}{4} & 2 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that a battery will last for more than 15 minutes?

Problem 28.5
Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1 \\ (x+2)/6 & 1 \le x < 4 \\ 1 & x \ge 4. \end{cases}$$

(a) Show that F satisfies conditions (a),(b),(c), and (e) of Theorem 30.1.
(b) Find the probability density function $f(x)$.

Problem 28.6
The amount of time X (in minutes) it takes a person standing in line at a post office to reach the counter is described by the continuous probability function:

$$f(x) = \begin{cases} kxe^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.
(a) Determine the value of k.
(b) What is the probability that a person takes more than 1 minute to reach the counter?

Problem 28.7
A mixed random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{e^x}{e^x+1} & x \ge 0. \end{cases}$$

(a) Find the probability density function.
(b) Find $P(0 < X \le 1)$.

Problem 28.8
A commercial water distributor supplies an office with gallons of water once a week. Suppose that the weekly supplies in tens of gallons is a random variable with pdf

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

At least how many gallons c should be delivered in one week so that the probability of the supply is 0.1? Round to a whole number of gallons.

Problem 28.9 ‡
The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & 0 < x < 20 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16 ?

Problem 28.10 ‡
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ with probability density function f, where $f(x)$ is proportional to $(10 + x)^{-2}$.
Calculate the probability that the lifetime of the machine part is less than 6.

Problem 28.11 ‡
A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by

$$V = 100000Y$$

where Y is a random variable with density function

$$f(y) = \begin{cases} k(1-y)^4 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.
What is the conditional probability that V exceeds 40,000, given that V exceeds 10,000?

Problem 28.12 ‡
An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2 ?

Problem 28.13 ‡
An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64 . Calculate C.

Problem 28.14
Let X have the density function

$$f(x) = \begin{cases} \frac{3x^2}{\theta^3} & 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $P(X > 1) = \frac{7}{8}$.
(a) Show that $\theta > 1$.
(b) Find the value of θ.

Problem 28.15
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & -2 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{\infty} xf(x)dx$.

Problem 28.16
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{4} & 8 \leq x \leq 12 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \frac{10}{X}$. Find $f_Y(y)$.

Problem 28.17
An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1 + x)^{-4}$, where $0 < x < \infty$ and 0 otherwise.
Calculate $\int_{-\infty}^{\infty} xf(x)dx$.

Problem 28.18
A continuous random variable X has a density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{\infty} \min(x, 4) f(x) dx$ and $\int_{-\infty}^{\infty} \min(x, 4)^2 f(x) dx$.

Problem 28.19 ‡
Let X be a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 1 - \left(\frac{2}{x}\right)^2 & x > 2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$. Find the density function of Y.

Problem 28.20 ‡
The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is twice that of Company I.
Let g be the density function for the distribution of the monthly profit of Company II.
Determine $g(y)$ where it is not zero.

Problem 28.21 ‡
Damages to a car in a crash are modeled by a random variable with density function

$$f(x) = \begin{cases} c(x^2 - 60x + 800), & 0 < x < 20 \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant. A particular car is insured with a deductible of 2. This car was involved in a crash with resulting damages in excess of the deductible. Calculate the probability that the damages exceeded 10.

Problem 28.22 ‡
The distribution of the size of claims paid under an insurance policy has probability density function

$$f(x) = \begin{cases} cx^a, & 0 < x < 5 \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0$ and $c > 0$.
For a randomly selected claim, the probability that the size of the claim is less than 3.75 is 0.4871. Calculate the probability that the size of a randomly selected claim is greater than 4.

Problem 28.23 ‡
A large university will begin a 13-day period during which students may register for that semester's courses. Of those 13 days, the number of elapsed days before a randomly selected student registers has a continuous distribution with density function $f(t)$ that is symmetric about $t = 6.5$ and proportional to $\frac{1}{t+1}$ between days 0 and 6.5.
A student registers at the 60^{th} percentile of this distribution. Calculate the number of elapsed days in the registration period for this student.

Problem 28.24 ‡
Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$. Let Y be the smallest integer greater than or equal to X. Determine the probability function of Y.

29 Expectation and Variance

As with discrete random variables, the expected value of a continuous random variable is a measure of location. It defines the balancing point of the distribution.

Suppose that a continuous random variable X has a density function $f(x)$ defined in $[a, b]$. Let's try to estimate $E(X)$ by cutting $[a, b]$ into n equal sub-intervals, each of width Δx, so $\Delta x = \frac{(b-a)}{n}$. Let $x_i = a + i\Delta x, i = 0, 1, ..., n$, be the partition points between the sub-intervals. Then, the probability of X assuming a value in $[x_i, x_{i+1}]$ is

$$P(x_i \leq X \leq x_{i+1}) = \int_{x_i}^{x_{i+1}} f(x)dx \approx \Delta x f\left(\frac{x_i + x_{i+1}}{2}\right)$$

where we used the midpoint rule to estimate the integral. An estimate of the desired expectation is approximately

$$E(X) \approx \sum_{i=0}^{n-1} \left(\frac{x_i + x_{i+1}}{2}\right) \Delta x f\left(\frac{x_i + x_{i+1}}{2}\right).$$

A better estimate is obtained by letting $n \to \infty$. Thus, we obtain

$$E(X) = \int_a^b xf(x)dx.$$

The above argument applies if either a or b are infinite. In this case, one has to make sure that all improper integrals in question converge.

Since the domain of f consists of all real numbers, we define the **expected value** of X by the improper integral

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

provided that the improper integral converges.

Example 29.1

A continuous random variable has the pdf

$$f(x) = \begin{cases} \frac{600}{x^2}, & 100 < x < 120 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the mean and variance of X.

(b) Find $P(X > 110)$.

Solution.
(a) We have

$$E(X) = \int_{100}^{120} x \cdot 600x^{-2}dx = 600 \ln x|_{100}^{120} \approx 109.39$$

and

$$\sigma_X^2 = E(X^2) - (E(X))^2 = \int_{100}^{120} x^2 \cdot 600x^{-2}dx - 109.39^2 \approx 33.19.$$

(b) The desired probability is

$$P(X > 110) = \int_{110}^{120} 600x^{-2}dx = \frac{5}{11} \blacksquare$$

Sometimes for technical purposes the following theorem is useful. It expresses the expectation in terms of an integral of probabilities. It is most often used for random variables X that have only positive values; in that case the second term is of course zero.

Theorem 29.1
Let X be a continuous random variable with probability density function f. Then

$$E(X) = \int_0^{\infty} P(X > y)dy - \int_{-\infty}^{0} P(X < y)dy.$$

Proof.
From the definition of $E(X)$ we have

$$\begin{aligned} E(X) &= \int_0^{\infty} xf(x)dx + \int_{-\infty}^{0} xf(x)dx \\ &= \int_0^{\infty} \int_{y=0}^{y=x} dyf(x)dx - \int_{-\infty}^{0} \int_{y=x}^{y=0} dyf(x)dx \end{aligned}$$

Interchanging the order of integration as shown in Figure 29.1 we can write

$$\int_0^{\infty} \int_{y=0}^{y=x} dyf(x)dx = \int_0^{\infty} \int_y^{\infty} f(x)dxdy$$

and

$$\int_{-\infty}^{0} \int_{y=x}^{y=0} dyf(x)dx = \int_{-\infty}^{0} \int_{-\infty}^{y} f(x)dxdy.$$

The result follows by putting the last two equations together and recalling that

$\int_y^\infty f(x)dx = P(X > y)$ and $\int_{-\infty}^y f(x)dx = P(X < y)$ ■

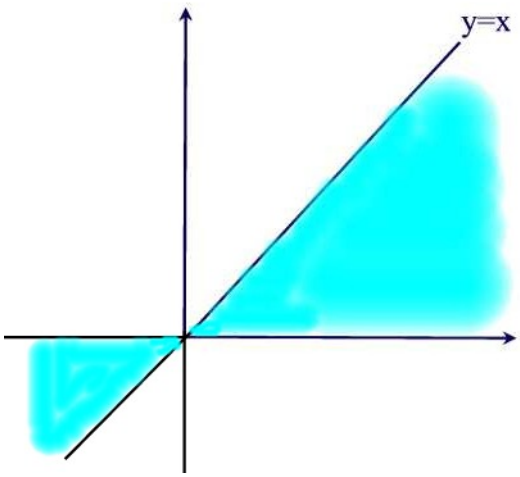


Figure 29.1

Note that if X is a continuous random variable and g is a function defined for the values of X and with real values, then $Y = g(X)$ is also a random variable. The following theorem is particularly important and convenient. If a random variable $Y = g(X)$ is expressed in terms of a continuous random variable, then this theorem gives the expectation of Y in terms of probabilities associated to X.

Theorem 29.2
If X is continuous random variable with a probability density function $f(x)$, and if $Y = g(X)$ is a function of the random variable, then the expected value of the function $g(X)$ is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Proof.
By the previous theorem we have

$$E(g(X)) = \int_0^\infty P[g(X) > y]dy - \int_{-\infty}^0 P[g(X) < y]dy.$$

If we let $B_y = \{x : g(x) > y\}$ then from the definition of a continuous random variable we can write

$$P[g(X) > y] = \int_{B_y} f(x)dx = \int_{\{x:g(x)>y\}} f(x)dx.$$

Thus,

$$E(g(X)) = \int_0^\infty \left[\int_{\{x:g(x)>y\}} f(x)dx\right] dy - \int_{-\infty}^0 \left[\int_{\{x:g(x)<y\}} f(x)dx\right] dy.$$

Now we can interchange the order of integration to obtain

$$\begin{aligned} E(g(X)) &= \int_{\{x:g(x)>0\}} \int_0^{g(x)} f(x)dydx - \int_{\{x:g(x)<0\}} \int_{g(x)}^0 f(x)dydx \\ &= \int_{\{x:g(x)>0\}} g(x)f(x)dx + \int_{\{x:g(x)<0\}} g(x)f(x)dx = \int_{-\infty}^\infty g(x)f(x)dx. \end{aligned}$$

Figure 29.2 helps understanding the process of interchanging the order of integration that we used in the proof above ■

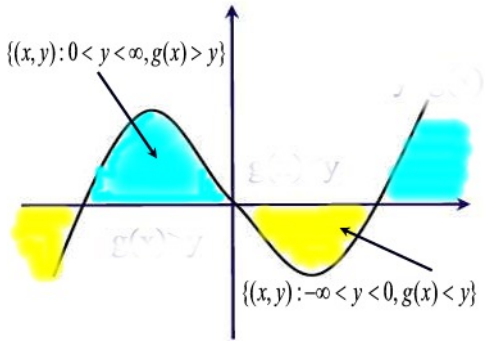


Figure 29.2

Example 29.2
Let T be a continuous random variable with pdf

$$f(t) = \begin{cases} \frac{1}{10}e^{-\frac{t}{10}}, & t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Define the continuous random variable by

$$X = \begin{cases} 100 & 0 < T \leq 1 \\ 50 & 1 < T \leq 3 \\ 0 & T > 3. \end{cases}$$

Find $E(X)$.

Solution.
By Theorem 29.2, we have

$$\begin{aligned} E(X) &= \int_0^1 100\frac{1}{10}e^{-\frac{t}{10}}dt + \int_1^3 50\frac{1}{10}e^{-\frac{t}{10}}dt \\ &= 100(1-e^{-\frac{1}{10}}) + 50(e^{-\frac{1}{10}} - e^{-\frac{3}{10}}) \\ &= 100 - 50e^{-\frac{1}{10}} - 50e^{-\frac{3}{10}} \blacksquare \end{aligned}$$

Corollary 29.1
For any constants a and b

$$E(aX+b) = aE(X) + b.$$

Proof.
Let $g(x) = ax + b$ in Theorem 29.2 to obtain

$$\begin{aligned} E(aX+b) &= \int_{-\infty}^{\infty} (ax+b)f(x)dx \\ &= a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx \\ &= aE(X) + b \blacksquare \end{aligned}$$

The **variance** of a random variable is a measure of the "spread" of the random variable about its expected value. In essence, it tells us how much variation there is in the values of the random variable from its mean value. The variance of the random variable X, is determined by calculating the expectation of the function $g(X) = (X - E(X))^2$. That is,

$$\mathrm{Var}(X) = E\left[(X - E(X))^2\right].$$

Theorem 29.3
(a) An alternative formula for the variance is given by

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2.$$

(b) For any constants a and b, $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.

Proof.
(a) By Theorem 29.2 we have

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2xE(X) + (E(X))^2) f(x)dx \\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - 2E(X)\int_{-\infty}^{\infty} xf(x)dx + (E(X))^2 \int_{-\infty}^{\infty} f(x)dx \\ &= E(X^2) - (E(X))^2.\end{aligned}$$

(b) We have

$$\mathrm{Var}(aX+b) = E[(aX+b-E(aX+b))^2] = E[a^2(X-E(X))^2] = a^2\mathrm{Var}(X)$$ ■

Example 29.3
Let X be a random variable with probability density function

$$f(x) = \begin{cases} 2 - 4|x| & -\frac{1}{2} < x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the variance of X.
(b) Find the c.d.f. $F(x)$ of X.

Solution.
(a) Since the function $xf(x)$ is odd in $-\frac{1}{2} < x < \frac{1}{2}$, we have $E(X) = 0$. Thus,

$$\begin{aligned}\mathrm{Var}(X) = E(X^2) &= \int_{-\frac{1}{2}}^{0} x^2(2+4x)dx + \int_{0}^{\frac{1}{2}} x^2(2-4x)dx \\ &= \frac{1}{24}.\end{aligned}$$

(b) Since the range of f is the interval $(-\frac{1}{2}, \frac{1}{2})$, we have $F(x) = 0$ for $x \le -\frac{1}{2}$ and $F(x) = 1$ for $x \ge \frac{1}{2}$. Thus it remains to consider the case when $-\frac{1}{2} < x < \frac{1}{2}$. For $-\frac{1}{2} < x \le 0$,

$$F(x) = \int_{-\frac{1}{2}}^{x} (2+4t)dt = 2x^2 + 2x + \frac{1}{2}.$$

For $0 \leq x < \frac{1}{2}$, we have

$$F(x) = \int_{-\frac{1}{2}}^{0} (2+4t)dt + \int_{0}^{x} (2-4t)dt = -2x^2 + 2x + \frac{1}{2}.$$

Combining these cases, we get

$$F(x) = \begin{cases} 0 & x < -\frac{1}{2} \\ 2x^2 + 2x + \frac{1}{2} & -\frac{1}{2} \leq x < 0 \\ -2x^2 + 2x + \frac{1}{2} & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} \blacksquare$$

Example 29.4
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 4xe^{-2x}, & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For this example, you might find the identity $\int_0^\infty t^n e^{-t} dt = n!$ useful.
(a) Find $E(X)$.
(b) Find the variance of X.
(c) Find the probability that $X < 1$.

Solution.
(a) Using the substitution $t = 2x$ we find

$$E(X) = \int_0^\infty 4x^2 e^{-2x} dx = \frac{1}{2} \int_0^\infty t^2 e^{-t} dt = \frac{2!}{2} = 1.$$

(b) First, we find $E(X^2)$. Again, letting $t = 2x$ we find

$$E(X^2) = \int_0^\infty 4x^3 e^{-2x} dx = \frac{1}{4} \int_0^\infty t^3 e^{-t} dt = \frac{3!}{4} = \frac{3}{2}.$$

Hence,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{3}{2} - 1 = \frac{1}{2}.$$

(c) We have

$$\begin{aligned} P(X < 1) = & P(X \leq 1) = \int_0^1 4xe^{-2x} dx = \int_0^2 te^{-t} dt \\ & = -(t+1)e^{-t}\Big|_0^2 = 1 - 3e^{-2} \blacksquare \end{aligned}$$

Example 29.5
Let X be the random variable representing the cost of maintaining a car. Suppose that $E(X) = 200$ and $\text{Var}(X) = 260$. If a tax of 20% is introduced on all items associated with the maintenance of the car, what will the variance of the cost of maintaining a car be?

Solution.
The new cost is $1.2X$, so its variance is $\text{Var}(1.2X) = 1.2^2\text{Var}(X) = (1.44)(260) = 374$. ■

Finally, we define the **standard deviation** X to be the square root of the variance.

Example 29.6
A random variable has a **Pareto** distribution with parameters $\alpha > 0$ and $x_0 > 0$ if its density function has the form

$$f(x) = \begin{cases} \frac{\alpha x_0^\alpha}{x^{\alpha+1}} & x > x_0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f(x)$ is indeed a density function.
(b) Find $E(X)$ and $\text{Var}(X)$.

Solution.
(a) By definition $f(x) > 0$. Also,

$$\int_{x_0}^{\infty} f(x)dx = \int_{x_0}^{\infty} \frac{\alpha x_0^\alpha}{x^{\alpha+1}}dx = -\left(\frac{x_0}{x}\right)\Big|_{x_0}^{\infty} = 1$$

(b) We have

$$E(X) = \int_{x_0}^{\infty} xf(x)dx = \int_{x_0}^{\infty} \frac{\alpha x_0^\alpha}{x^\alpha}dx = \frac{\alpha}{1-\alpha}\left(\frac{x_0^\alpha}{x^{\alpha-1}}\right)\Big|_{x_0}^{\infty} = \frac{\alpha x_0}{\alpha - 1}$$

provided $\alpha > 1$. Similarly,

$$E(X^2) = \int_{x_0}^{\infty} x^2 f(x)dx = \int_{x_0}^{\infty} \frac{\alpha x_0^\alpha}{x^{\alpha-1}}dx = \frac{\alpha}{2-\alpha}\left(\frac{x_0^\alpha}{x^{\alpha-2}}\right)\Big|_{x_0}^{\infty} = \frac{\alpha x_0^2}{\alpha - 2}$$

provided $\alpha > 2$. Hence,

$$\text{Var}(X) = \frac{\alpha x_0^2}{\alpha - 2} - \frac{\alpha^2 x_0^2}{(\alpha-1)^2} = \frac{\alpha x_0^2}{(\alpha-2)(\alpha-1)^2}$$ ■

Practice Problems

Problem 29.1
Let X have the density function given by

$$f(x) = \begin{cases} 0.2 & -1 < x \leq 0 \\ 0.2 + cx & 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.
(b) Find $F(x)$.
(c) Find $P(0 \leq x \leq 0.5)$.
(d) Find $E(X)$.

Problem 29.2
The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $E(X) = \frac{3}{5}$.
(a) Find a and b.
(b) Determine the cdf, $F(x)$, explicitly.

Problem 29.3
Compute $E(X)$ if X has the density function given by
(a)

$$f(x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 29.4
A continuous random variable has a pdf

$$f(x) = \begin{cases} 1 - \frac{x}{2} & 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value and the variance.

Problem 29.5
Let X denote the lifetime (in years) of a computer chip. Let the probability density function be given by

$$f(x) = \begin{cases} 4(1+x)^{-5} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the mean and the standard deviation.
(b) What is the probability that a randomly chosen computer chip expires in less than a year?

Problem 29.6
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{x} & 1 < x < e \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(\ln X)$.

Problem 29.7
Let X have a cdf

$$F(x) = \begin{cases} 1 - \frac{1}{x^6} & x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Var}(X)$.

Problem 29.8
Let X have a pdf

$$f(x) = \begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value and variance of $Y = X^2$.

Problem 29.9 ‡
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & -2 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

Problem 29.10 ‡
An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$ and 0 otherwise.
Determine the company's expected monthly claims.

Problem 29.11 ‡
A random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2-2x+2}{2} & 1 \le x < 2 \\ 1 & x \ge 2. \end{cases}$$

Calculate the variance of X. Hint: See Remark 30.3.

Problem 29.12
Let X have the density function

$$f(x) = \begin{cases} \frac{2x}{k^2} & 0 \le x \le k \\ 0 & \text{otherwise.} \end{cases}$$

For what value of k is the variance of X equal to 2?

Problem 29.13 ‡
A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device has the density function

$$f(t) = \begin{cases} \frac{1}{3}e^{-\frac{t}{3}} & 0 \le x \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$.
Calculate $E(X)$.

Problem 29.14
Find $E(X)$ when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 29.15 ‡
An insurance policy reimburses a loss up to a benefit limit of 10 . The policyholder's loss, X, follows a distribution with density function:

$$f(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

Problem 29.16 ‡
A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \frac{2.5(0.6)^{2.5}}{x^{3.5}} & x > 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2.
What is the mean of the manufacturer's annual losses not paid by the insurance policy?

Problem 29.17 ‡
A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment has density function

$$f(t) = \begin{cases} \frac{1}{10}e^{-\frac{t}{10}} & 0 < t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made.
Calculate x such that the expected payment made under this insurance is 1000.

Problem 29.18 ‡
An insurance policy is written to cover a loss, X, where X has the density function

$$f(x) = \begin{cases} \frac{1}{1000} & 0 \leq x \leq 1000 \\ 0 & \text{otherwise.} \end{cases}$$

The policy has a deductible, d, and the expected payment under the policy is 25% of what it would be with no deductible.
Calculate d.

Problem 29.19 ‡
An actuary determines that the claim size for a certain class of accidents is a continuous random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1-2500t)^4}.$$

Calculate the standard deviation of the claim size for this class of accidents.

Problem 29.20 ‡
An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003e^{-\frac{x}{2}}, & 0 < x < 15 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected claim payment.

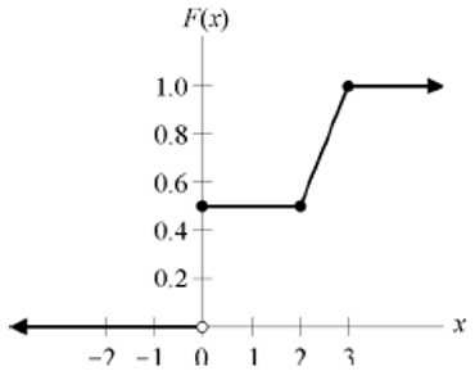
Problem 29.21 ‡
A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260. A tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive).
Calculate the variance of the annual cost of maintaining and repairing a car after the tax is introduced.

Problem 29.22 ‡
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{p-1}{x^p}, & x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the value of p such that $E(X) = 2$.

Problem 29.23 ‡
The figure below shows the cumulative distribution function of a random variable, X.



Calculate $E(X)$.

Problem 29.24 ‡
The proportion X of yearly dental claims that exceed 200 is a random variable with probability density function

$$f(x) = \begin{cases} 60x^3(1-x)^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate Var $\left[\frac{X}{1-X}\right]$.

30 Median, Mode, and Percentiles

Recall that the mean of a random variable is the measure of the center of the data (i.e. a measure of central tendency) whereas the standard deviation is a measure of how data is scattered around the mean (i.e. measure of dispersion). In this section, we consider other measures of central tendency such as the median, the mode and the percentile.

Median of a Random Variable

In statistics, the median of a set of data is the number where half of the data should fall below it. In the context of a discrete random variable X, the **median** is a number M such that $F(M) = P(X \leq M) > 0.5$. If there is a number M such that $F(M) = P(X \leq M) = 0.5$ then the median is the average of this value and the next value of X.

Example 30.1

Given the pmf of a discrete random variable X.

x	0	1	2	3	4	5
$p(x)$	0.35	0.20	0.15	0.15	0.10	0.05

Find the median of X.

Solution.

Since $P(X \leq 1) = 0.55 > 0.5$ so 1 is the median of X ■

Example 30.2

Let X be the discrete random variable with pmf given by $p(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, \cdots$ and 0 otherwise. Find the median of X.

Solution.

Since $P(X \leq 1) = 0.5$, the median of X is $\frac{1}{2}(1 + 2) = 1.5$ ■

In the case of a continuous random variable X, the median is the number M such that $P(X \leq M) = P(X \geq M) = 0.5$. That is, $F(M) = 0.5$, where $F(x)$ is the cumulative distribution function of X.

Example 30.3

Let X be a continuous random variable with pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$ and 0 otherwise. Find the median of X.

Solution.
We must find a number M such that $\int_a^M \frac{dx}{b-a} = 0.5$. This leads to the equation $\frac{M-a}{b-a} = 0.5$. Solving this equation we find $M = \frac{a+b}{2}$ ■

Mode of a Random Variable

The **mode** is defined as the value that maximizes the probability mass function $p(x)$ (discrete case) or the probability density function $f(x)$ (continuous case.) In the discrete case, the mode is the value that is most likely to be sampled. In the continuous case, the mode is where $f(x)$ is at its peak.

Example 30.4
Let X be the discrete random variable with pmf given by $p(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, \cdots$ and 0 otherwise. Find the mode of X.

Solution.
The value of x that maximizes $p(x)$ is $x = 1$. Thus, the mode of X is 1 ■

Example 30.5
Let X be the continuous random variable with pdf given by $f(x) = 0.75(1 - x^2)$ for $-1 \leq x \leq 1$ and 0 otherwise. Find the mode of X.

Solution.
The pdf is maximum for $x = 0$. Thus, the mode of X is 0 ■

Percentiles and Quantiles

In statistics, a percentile is the value of a variable below which a certain percent of observations fall. For example, if a score is in the 85^{th} percentile, it is higher than 85% of the other scores. For a random variable X and $0 < p < 1$, the 100p^{th} **percentile** (or the p^{th} **quantile**) is the number x such

$$P(X < x) \leq p \leq P(X \leq x).$$

For a continuous random variable, this is the solution to the equation $F(x) = p$. The 25^{th} percentile is also known as the **first quartile**, the 50^{th} percentile as the median or second quartile, and the 75^{th} percentile as the third quartile.

Example 30.6
A loss random variable X has the density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}} & x > 200 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 25^{th} and 75^{th} percentiles of X.

Solution.
First, the cdf is given by

$$F(x) = \int_{200}^{x} \frac{2.5(200)^{2.5}}{t^{3.5}} dt.$$

If Q_1 is the 25^{th} percentile then it satisfies the equation

$$F(Q_1) = \frac{1}{4}$$

or equivalently

$$1 - F(Q_1) = \frac{3}{4}.$$

This leads to

$$\frac{3}{4} = \int_{Q_1}^{\infty} \frac{2.5(200)^{2.5}}{t^{3.5}} dt = -\left(\frac{200}{t}\right)^{2.5}\Bigg|_{Q_1}^{\infty} = \left(\frac{200}{Q_1}\right)^{2.5}.$$

Solving for Q_1 we find $Q_1 = 200(4/3)^{0.4} \approx 224.4$. Similarly, the third quartile (i.e. 75^{th} percentile) is given by $Q_3 = 348.2$, The **interquartile range** (i.e., the difference between the 25^{th} and 75^{th} percentiles) is $Q_3 - Q_1 = 348.2 - 224.4 = 123.8$ ■

Example 30.7
Let X be the random variable with pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$ and 0 otherwise. Find the p^{th} quantile of X.

Solution.
We have

$$p = P(X \leq x) = \int_{a}^{x} \frac{dt}{b-a} = \frac{x-a}{b-a}.$$

Solving this equation for x, we find $x = a + (b-a)p$ ■

Example 30.8
What percentile is 0.63 quantile?

Solution.
0.63 quantile is 63^{rd} percentile ■

Practice Problems

Problem 30.1
Suppose the random variable X has pmf

$$p(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n, \quad n = 0, 1, 2, \cdots$$

Find the median and the 70^{th} percentile.

Problem 30.2
Suppose the random variable X has pdf

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the 50^{th} percentile.

Problem 30.3
Let Y be a continuous random variable with cumulative distribution function

$$F(y) = \begin{cases} 0 & y \leq a \\ 1 - e^{-\frac{1}{2}(y-a)^2} & \text{otherwise} \end{cases}$$

where a is a constant. Find the 75th percentile of Y.

Problem 30.4
Let X be a random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find λ if the median of X is $\frac{1}{3}$.

Problem 30.5
People are dispersed on a linear beach with a density function $f(y) = 4y^3$, $0 < y < 1$, and 0 elsewhere. An ice cream vendor wishes to locate her cart at the median of the locations (where half of the people will be on each side of her). Where will she locate her cart?

Problem 30.6
Let X be a discrete random variable with pmf $p(n) = (n-1)(0.4)^2(0.6)^{n-2}$, $n \geq 2$ and 0 otherwise. Find the mode of X.

Problem 30.7
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \lambda \frac{1}{9} x(4-x) & 0 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mode of X.

Problem 30.8
Find the pth quantile of the exponential distribution defined by the cumulative distribution function $F(x) = 1 - e^{-x}$ for $x \geq 0$ and 0 otherwise.

Problem 30.9
A continuous random variable has the pdf $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$. Find the pth quantile of X.

Problem 30.10
Let X be a loss random variable with cdf

$$F(x) = \begin{cases} 1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The 10th percentile is $\theta - k$. The 90th percentile is $5\theta - 3k$. Determine the value of α.

Problem 30.11
Let X be a random variable with density function $f(x) = \frac{4x}{(1+x^2)^3}$ for $x > 0$ and 0 otherwise. Calculate the mode of X.

Problem 30.12
Let X be a random variable with pdf $f(x) = \left(\frac{3}{5000}\right)\left(\frac{5000}{x}\right)^4$ for $x > 5000$ and 0 otherwise. Determine the median of X.

Problem 30.13
Let X be a random variable with cdf

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{27}, & 0 \leq x \leq 3 \\ 1, & x > 3. \end{cases}$$

Find the median of X.

Problem 30.14
A distribution has a pdf $f(x) = \frac{3}{x^4}$ for $x > 1$ and 0 otherwise. Calculate the 0.95th quantile of this distribution.

Problem 30.15 ‡
An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100 . The cumulative distribution function for the incurred losses is given by

$$F(x) = 1 - e^{-\frac{1}{300}x}, \ x > 0$$

and 0 otherwise. What is the 95th percentile of actual losses that exceed the deductible?

Problem 30.16 ‡
Losses under an insurance policy have the density function

$$f(x) = \begin{cases} 0.25e^{-0.25x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The deductible is 1 for each loss.
Calculate the median amount that the insurer pays a policyholder for a loss under the policy.

Problem 30.17 ‡
Losses covered by an insurance policy have the density function

$$f(x) = \begin{cases} 0.001 & 0 \leq x \leq 1000 \\ 0 & \text{otherwise.} \end{cases}$$

An insurance company reimburses losses in excess of a deductible of 250.
Calculate the difference between the median and the 20^{th} percentile of the insurance company reimbursement, over all losses.

Problem 30.18 ‡
The time to failure T of a component in an electronic device has the density function

$$f(x) = \begin{cases} ce^{-cx} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$. The median of X is 4. Calculate the probability that the component will work without failing for at least five hours.

Problem 30.19 ‡
An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is:

$$f(x) = \begin{cases} ce^{-0.004x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

Problem 30.20 ‡
Each time a hurricane arrives, a new home has a 0.4 probability of experiencing damage. The occurrences of damage in different hurricanes are mutually independent.
Calculate the mode of the number of hurricanes it takes for the home to experience damage from two hurricanes.

Problem 30.21 ‡
The number of policies that an agent sells has a Poisson distribution with modes at 2 and 3. K is the smallest number such that the probability of selling more than K policies is less than 25%. Calculate K.

Problem 30.22 ‡
The lifetime of a light bulb has density function

$$f(x) = \begin{cases} \frac{Kx^2}{1+x^3} & 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mode of X.

Problem 30.23 ‡
An insurer's medical reimbursements have density function

$$f(x) = \begin{cases} Kxe^{-x^2} & 0 < x < 1,\ K > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mode of X.

Problem 30.24 ‡
An automobile insurance company issues a one-year policy with a deductible of 500. The probability is 0.8 that the insured automobile has no accident and 0.0 that the automobile has more than one accident. If there is an accident, the loss before application of the deductible is exponentially distributed with mean 3000.
Calculate the 95th percentile of the insurance company payout on this policy.

Problem 30.25 ‡

An insurance company insures a good driver and a bad driver on the same policy. The table below gives the probability of a given number of claims occurring for each of these drivers in the next ten years.

Number of claims	Probability for the good driver	Probability for for the bad driver
0	0.5	0.2
1	0.3	0.3
2	0.2	0.4
3	0.0	0.1

The number of claims occurring for the two drivers are independent. Calculate the mode of the distribution of the total number of claims occurring on this policy in the next ten years.

31 The Continuous Uniform Distribution Function

The simplest continuous distribution is the uniform distribution. A continuous random variable X is said to be **uniformly** distributed over the interval $a \leq x \leq b$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Since $F(x) = \int_{-\infty}^{x} f(t)dt$, the cdf is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

Figure 31.1 presents a graph of $f(x)$ and $F(x)$.

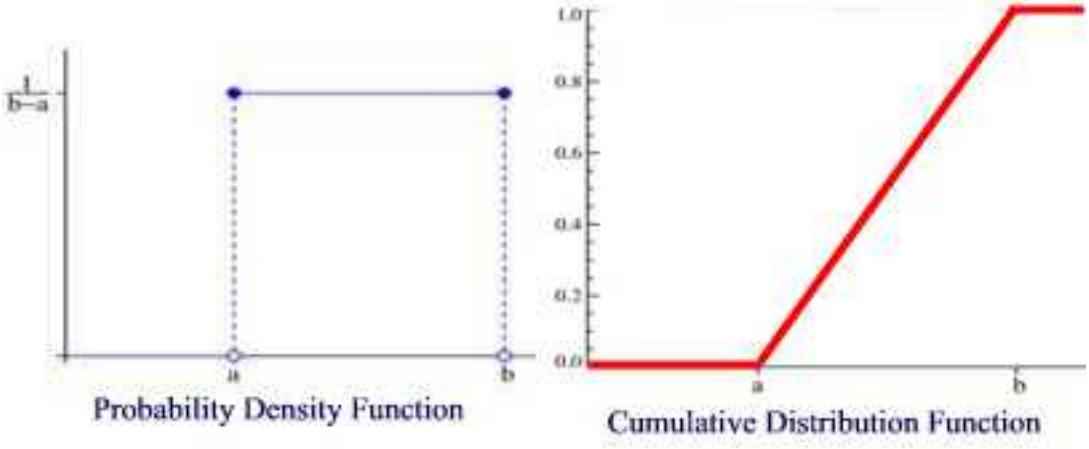


Figure 31.1

If $a = 0$ and $b = 1$ then X is called the **standard uniform** random variable.

Remark 31.1
The values at the two boundaries a and b are usually unimportant because they do not alter the value of the integral of $f(x)$ over any interval. Sometimes they are chosen to be zero, and sometimes chosen to be $\frac{1}{b-a}$. Our

definition above assumes that $f(a) = f(b) = f(x) = \frac{1}{b-a}$. In the case $f(a) = f(b) = 0$ then the pdf becomes

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Because the pdf of a uniform random variable is constant, if X is uniform, then the probability X lies in any interval contained in (a, b) depends only on the length of the interval-not location. That is, for any x and d such that $[x, x + d] \subseteq [a, b]$ we have

$$\int_x^{x+d} f(x)dx = \frac{d}{b-a}.$$

Example 31.1
Find the survival function of a uniform distribution X on the interval $[a, b]$.

Solution.
The survival function is given by

$$S(x) = \begin{cases} 1 & \text{if } x \leq a \\ \frac{b-x}{b-a} & \text{if } a < x < b \\ 0 & \text{if } x \geq b \end{cases} \blacksquare$$

Example 31.2
Let X be a continuous uniform random variable on $[0, 25]$. Find the pdf and cdf of X.

Solution.
The pdf is

$$f(x) = \begin{cases} \frac{1}{25} & 0 \leq x \leq 25 \\ 0 & \text{otherwise} \end{cases}$$

and the cdf is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{25} & \text{if } 0 \leq x \leq 25 \\ 1 & \text{if } x > 25 \end{cases} \blacksquare$$

Example 31.3
Suppose that X has a uniform distribution on the interval $(0, a)$, where $a > 0$. Find $P(X > X^2)$.

Solution.
If $a \leq 1$ then $P(X > X^2) = \int_0^a \frac{1}{a} dx = 1$. If $a > 1$ then $P(X > X^2) = \int_0^1 \frac{1}{a} dx = \frac{1}{a}$. Thus, $P(X > X^2) = \min\{1, \frac{1}{a}\}$ ■

The expected value of X is

$$\begin{aligned} E(X) &= \int_a^b x f(x) = \int_a^b \frac{x}{b-a} dx \\ &= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{a+b}{2} \end{aligned}$$

and so the expected value of a uniform random variable is halfway between a and b.
The second moment about the origin is

$$\begin{aligned} E(X^2) &= \int_a^b \frac{x^2}{b-a} dx = \frac{x^3}{3(b-a)}\Big|_a^b \\ &= \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + b^2 + ab}{3}. \end{aligned}$$

The variance of X is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{a^2 + b^2 + ab}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

Practice Problems

Problem 31.1
Let X be the total time to process a passport application by the state department. It is known that X is uniformly distributed between 3 and 7 weeks.
(a) Find $f(x)$.
(b) What is the probability that an application will be processed in fewer than 3 weeks ?
(c) What is the probability that an application will be processed in 5 weeks or less ?

Problem 31.2
In a sushi bar, customers are charged for the amount of sushi they consume. Suppose that the amount of sushi consumed is uniformly distributed between 5 ounces and 15 ounces. Let X be the random variable representing a plate filling weight.
(a) Find the probability density function of X.
(b) What is the probability that a customer will take between 12 and 15 ounces of sushi?
(c) Find $E(X)$ and Var(X).

Problem 31.3
Suppose that X has a uniform distribution over the interval [0,1]. Find
(a) $F(x)$.
(b) Show that $P(a \leq X \leq a+b)$ for $a, b \geq 0,\ a+b \leq 1$ depends only on b.

Problem 31.4
Let X be uniform on [0,1]. Compute $E(X^n)$ where n is a positive integer.

Problem 31.5
Let X be a uniform random variable on the interval [1,2] and let $Y = \frac{1}{X}$. Find $E[Y]$.

Problem 31.6
A commuter train arrives at a station at some time that is uniformly distributes between 10:00 AM and 10:30 AM. Let X be the waiting time (in minutes) for the train. What is the probability that you will have to wait longer than 10 minutes?

Problem 31.7 ‡
An insurance policy is written to cover a loss, X, where X has a uniform distribution on [0, 1000].
At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

Problem 31.8 ‡
The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

Problem 31.9 ‡
The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250 . In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $[0, 1500]$.
Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

Problem 31.10
Let X be a random variable distributed uniformly over the interval $[-1, 1]$.
(a) Compute $E(e^{-X})$.
(b) Compute $\text{Var}(e^{-X})$.

Problem 31.11
Let X be a random variable with a continuous uniform distribution on the interval $[1, a], a > 1$. If $E(X) = 6\text{Var}(X)$, what is the value of a?

Problem 31.12
Let X be a random variable with a continuous uniform distribution on the interval $[0, 10]$. What is $P(X + \frac{10}{X} > 7)$?

Problem 31.13 ‡
An investment account earns an annual interest rate R that follows a uniform

distribution on the interval $[0.04, 0.08]$. The value of a 10,000 initial investment in this account after one year is given by $V = 10,000e^R$..
Let F be the cumulative distribution function of V. Determine $F(v)$ for values of v that satisfy $0 < F(v) < 1$.

Problem 31.14 ‡
Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes.
Let R denote the average rate, in customers per minute, at which the representative responds to inquiries, and let $f(r)$ be the density function for R. Determine $f(r)$, for $\frac{10}{12} \leq r \leq \frac{10}{8}$.

Problem 31.15 ‡
An insurer offers a travelers insurance policy. Losses under the policy are uniformly distributed on the interval [0,5]. The insurer reimburses a policyholder for a loss up to a maximum of 4.
Determine the cumulative distribution function, F, of the benefit that the insurer pays a policyholder who experiences exactly one loss under the policy.

Problem 31.16 ‡
Losses covered by a flood insurance policy are uniformly distributed on the interval [0,2]. The insurer pays the amount of the loss in excess of a deductible d. The probability that the insurer pays at least 1.20 on a random loss is 0.30. Calculate the probability that the insurer pays at least 1.44 on a random loss.

Problem 31.17 ‡
An insurance company issues policies covering damage to automobiles. The amount of damage is modeled by a uniform distribution on $[0, b]$. The policy payout is subject to a deductible of $0.1b$. A policyholder experiences automobile damage.
Calculate the ratio of the standard deviation of the policy payout to the standard deviation of the amount of the damage.

Problem 31.18
The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval [0, 25]. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$.

Problem 31.19
Let X be a uniform distribution on the interval $[a, b]$. Find an expression for $M_X(t) = E(e^{tX})$.

Problem 31.20
The weight X , in pounds, of a package is uniformly distributed on the interval $[14, 20]$. Find the 40^{th} percentile of X.

Problem 31.21 ‡
Automobile claim amounts are modeled by a uniform distribution on the interval $[0, 10,000]$. Actuary A reports X, the claim amount divided by 1000. Actuary B reports Y, which is X rounded to the nearest integer from 0 to 10.
Calculate the absolute value of the difference between the 4^{th} moment of X and the 4^{th} moment of Y.

Problem 31.22 ‡
For a certain health insurance policy, losses are uniformly distributed on the interval $[0, b]$. The policy has a deductible of 180 and the expected value of the un-reimbursed portion of a loss is 144. Calculate b.

Problem 31.23 ‡
For a certain health insurance policy, losses are uniformly distributed on the interval [0,450]. The policy has a deductible of d and the expected value of the unreimbursed portion of a loss is 56. Calculate d.

Problem 31.24 ‡
Under a liability insurance policy, losses are uniformly distributed on $[0, b]$, where b is a positive constant. There is a deductible of $0.5b$. Calculate the ratio of the variance of the claim payment (greater than or equal to zero) from a given loss to the variance of the loss.

Problem 31.25 ‡
A government employee's yearly dental expense follows a uniform distribution on the interval from 200 to 1200. The government's primary dental plan reimburses an employee for up to 400 of dental expense incurred in a year, while a supplemental plan pays up to 500 of any remaining dental expense. Let Y represent the yearly benefit paid by the supplemental plan to a government employee. Calculate $\text{Var}(Y)$.

Problem 31.26 ‡

A car and a bus arrive at a railroad crossing at times independently and uniformly distributed between 7:15 and 7:30. A train arrives at the crossing at 7:20 and halts traffic at the crossing for five minutes. Calculate the probability that the waiting time of the car or the bus at the crossing exceeds three minutes.

32 Normal Random Variables

A **normal** random variable with parameters μ and σ^2 has a pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

This density function is a bell-shaped curve that is symmetric about μ (See Figure 32.1).

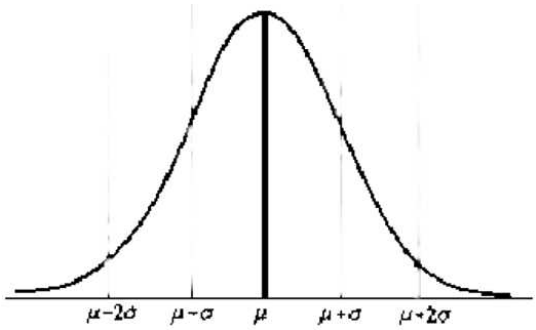


Figure 32.1

The normal distribution is used to model phenomenon such as a person's height at a certain age or the measurement error in an experiment. Observe that the distribution is symmetric about the point μ−hence the experiment outcome being modeled should be equally likely to assume points above μ as points below μ. The normal distribution is probably the most important distribution because of a result known as the **central limit theorem** to be discussed in Section 54.

To prove that the given $f(x)$ is indeed a pdf we must show that the area under the normal curve is 1. That is,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

First note that using the substitution $y = \frac{x-\mu}{\sigma}$ we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy.$$

Toward this end, let $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$. Then

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$
$$= \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} rd\theta dr = 2\pi \int_0^{\infty} re^{-\frac{r^2}{2}} dr = 2\pi.$$

Thus, $I = \sqrt{2\pi}$ and the result is proved. Note that in the process above, we used the polar substitution $x = r\cos\theta, y = r\sin\theta$, and $dydx = rdrd\theta$.

Example 32.1
Let X be a normal random variable with mean 950 and standard deviation 10. Find $P(947 \leq X \leq 950)$.

Solution.
We have

$$P(947 \leq X \leq 950) = \frac{1}{10\sqrt{2\pi}} \int_{947}^{950} e^{-\frac{(x-950)^2}{200}} dx \approx 0.118$$

where the value of the integral is found by using a calculator ■

Theorem 32.1
If X is a normal distribution with parameters (μ, σ^2) then $Y = aX + b$ is a normal distribution with parameters $(a\mu + b, a^2\sigma^2)$.

Proof.
We prove the result when $a > 0$. The proof is similar for $a < 0$. Let F_Y denote the cdf of Y. Then

$$F_Y(x) = P(Y \leq x) = P(aX + b \leq x)$$
$$= P\left(X \leq \frac{x-b}{a}\right) = F_X\left(\frac{x-b}{a}\right).$$

Differentiating both sides to obtain

$$f_Y(x) = \frac{1}{a} f_X\left(\frac{x-b}{a}\right) = \frac{1}{\sqrt{2\pi}a\sigma} exp\left[-(\frac{x-b}{a} - \mu)^2/(2\sigma^2)\right]$$
$$= \frac{1}{\sqrt{2\pi}a\sigma} exp\left[-(x - (a\mu + b))^2/2(a\sigma)^2\right]$$

which shows that Y is normal with parameters $(a\mu + b, a^2\sigma^2)$ ■
Note that if $Z = \frac{X-\mu}{\sigma}$ then this is a normal distribution with parameters (0,1). Such a random variable is called the **standard normal** random variable.

Theorem 32.2
If X is a normal random variable with parameters (μ, σ^2) then
(a) $E(X) = \mu$
(b) $\text{Var}(X) = \sigma^2$.

Proof.
(a) Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal distribution. Then

$$E(Z) = \int_{-\infty}^{\infty} x f_Z(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} = 0.$$

Thus,

$$E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu.$$

(b)

$$\text{Var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx.$$

Using integration by parts with $u = x$ and $dv = xe^{-\frac{x^2}{2}}$ we find

$$\text{Var(Z)} = \frac{1}{\sqrt{2\pi}} \left[-xe^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1.$$

Thus,

$$\text{Var}(X) = \text{Var}(\sigma Z + \mu) = \sigma^2 \text{Var}(Z) = \sigma^2 \ \blacksquare$$

Figure 32.2 shows different normal curves with the same μ and different σ.

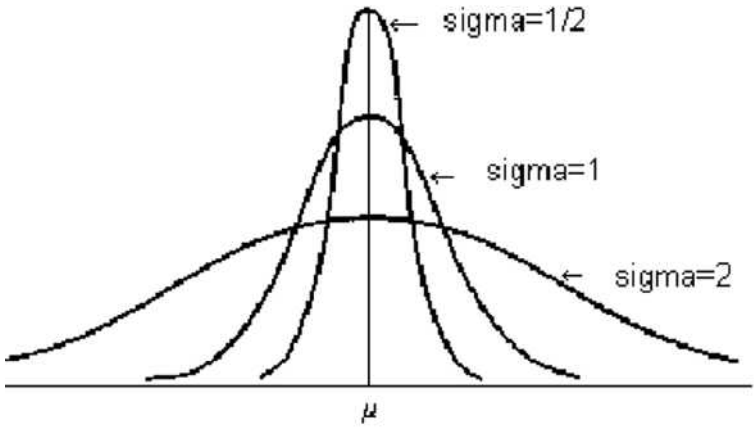


Figure 32.2

It is traditional to denote the cdf of Z by $\Phi(x)$. That is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

Now, since $f_Z(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $f_Z(x)$ is an even function. This implies that $\Phi'(-x) = \Phi'(x)$. Integrating we find that $\Phi(x) = -\Phi(-x) + C$. Letting $x = 0$ we find that $C = 2\Phi(0) = 2(0.5) = 1$. Thus,

$$\Phi(x) = 1 - \Phi(-x), \quad -\infty < x < \infty. \tag{32.1}$$

This implies that

$$P(Z \leq -x) = P(Z > x).$$

Now, $\Phi(x)$ is the area under the standard curve to the left of x. The values of $\Phi(x)$ for $x \geq 0$ are given in Table 34.1 below. Equation 32.1 is used for $x < 0$.

Example 32.2
Let X be a normal random variable with parameters $\mu = 24$ and $\sigma_X^2 = 9$.
(a) Find $P(X > 27)$ using Table 34.1.
(b) Solve $S(x) = 0.05$ where $S(x)$ is the survival function of X.

Solution.
(a) The desired probability is given by

$$\begin{aligned} P(X > 27) =& P\left(\frac{X - 24}{3} > \frac{27 - 24}{3}\right) = P(Z > 1) \\ =& 1 - P(Z \leq 1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587. \end{aligned}$$

(b) The equation $P(X > x) = 0.05$ is equivalent to $P(X \leq x) = 0.95$. Note that

$$P(X \leq x) = P\left(\frac{X - 24}{3} < \frac{x - 24}{3}\right) = P\left(Z < \frac{x - 24}{3}\right) = 0.95$$

From Table 32.1 we find $P(Z \leq 1.65) = 0.95$. Thus, we set $\frac{x-24}{3} = 1.65$ and solve for x we find $x = 28.95$ ■

From the above example, we see that probabilities involving normal random variables can be reduced to the ones involving standard normal variable. For example

$$P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Example 32.3
Let X be a normal random variable with parameters μ and σ^2. Find
(a)$P(\mu - \sigma \leq X \leq \mu + \sigma)$.
(b)$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.
(c)$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$.

Solution.
(a) We have

$$\begin{aligned} P(\mu - \sigma \leq X \leq \mu + \sigma) =& P(-1 \leq Z \leq 1) \\ =& \Phi(1) - \Phi(-1) \\ =& 2(0.8413) - 1 = 0.6826. \end{aligned}$$

Thus, 68.26% of all possible observations lie within one standard deviation to either side of the mean.
(b) We have

$$\begin{aligned} P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) =& P(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) \\ =& 2(0.9772) - 1 = 0.9544. \end{aligned}$$

Thus, 95.44% of all possible observations lie within two standard deviations to either side of the mean.
(c) We have

$$\begin{aligned} P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) =& P(-3 \leq Z \leq 3) = \Phi(3) - \Phi(-3) \\ =& 2(0.9987) - 1 = 0.9974. \end{aligned}$$

Thus, 99.74% of all possible observations lie within three standard deviations to either side of the mean. See Figure 32.3 ■

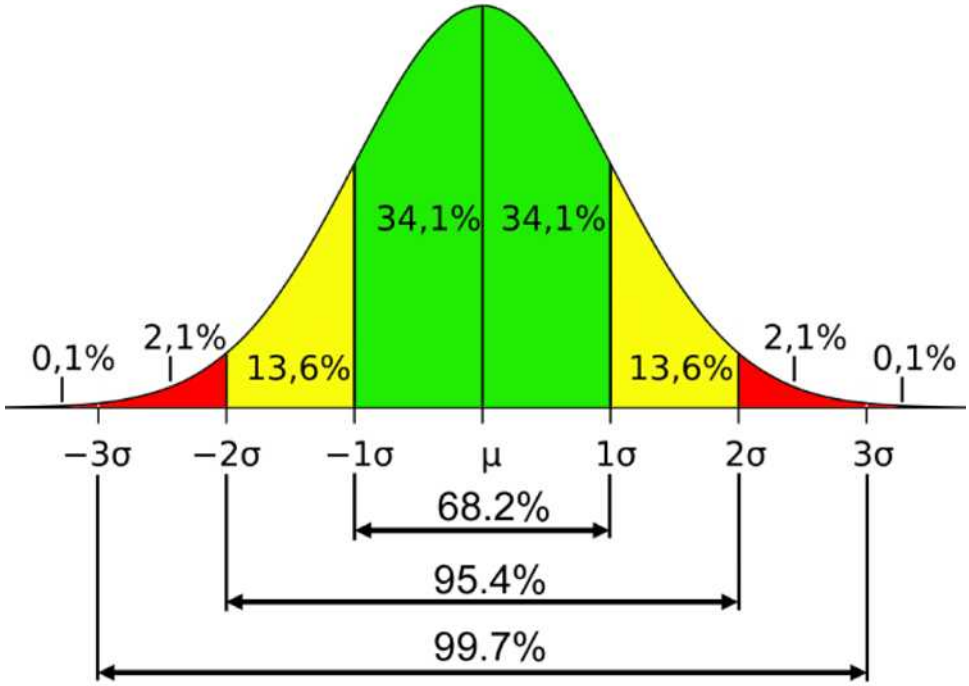


Figure 32.3

Table 32.1: Area under the Standard Normal Curve from $-\infty$ to x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Practice Problems

Problem 32.1
The scores on a statistics test are Normally distributed with parameters $\mu = 80$ and $\sigma^2 = 196$. Find the probability that a randomly chosen score is
(a) no greater than 70
(b) at least 95
(c) between 70 and 95.
(d) Approximately, what is the raw score corresponding to a percentile score of 72%?

Problem 32.2
Let X be a normal random variable with parameters $\mu = 0.381$ and $\sigma^2 = 0.031^2$. Compute the following:
(a) $P(X > 0.36)$.
(b) $P(0.331 < X < 0.431)$.
(c) $P(|X - .381| > 0.07)$.

Problem 32.3
Assume the time required for a cyclist to travel a distance d follows a normal distribution with mean 4 minutes and variance 4 seconds.
(a) What is the probability that this cyclist with travel the distance in less than 4 minutes?
(b) What is the probability that this cyclist will travel the distance in between 3min55sec and 4min5sec?

Problem 32.4
It has been determined that the lifetime of a certain light bulb has a normal distribution with $\mu = 2000$ hours and $\sigma = 200$ hours.
(a) Find the probability that a bulb will last between 2000 and 2400 hours.
(b) What is the probability that a light bulb will last less than 1470 hours?

Problem 32.5
Let X be a normal random variable with mean 100 and standard deviation 15. Find $P(X > 130)$ given that $\Phi(2) = .9772$.

Problem 32.6
The lifetime X of a randomly chosen battery is normally distributed with mean 50 and standard deviation 5.
(a) Find the probability that the battery lasts at least 42 hours.
(b) Find the probability that the battery will last between 45 to 60 hours.

Problem 32.7 ‡
An insurance company's annual profit is normally distributed with mean 100 and variance 400. Let Z be normally distributed with mean 0 and variance 1 and let Φ be the cumulative distribution function of Z.
Determine, in terms of $\Phi(x)$, the probability that the company's profit in a year is at most 60, given that the profit in the year is positive.

Problem 32.8
Let X be a normal random variable with $P(X < 500) = 0.5$ and $P(X > 650) = 0.0227$. Find the standard deviation of X.

Problem 32.9
Suppose that X is a normal random variable with parameters $\mu = 5, \sigma^2 = 49$. Using the table of the normal distribution , compute: (a) $P(X > 5.5)$, (b) $P(4 < X < 6.5)$, (c) $P(X < 8)$, (d) $P(|X - 7| \geq 4)$.

Problem 32.10
Let X be a normal random variable with mean 1 and variance 4. Find $P(X^2 - 2X \leq 8)$.

Problem 32.11
Let X be a normal random variable with mean 360 and variance 16.
(a) Calculate $P(X < 355)$.
(b) Suppose the variance is kept at 16 but the mean is to be adjusted so that $P(X < 355) = 0.025$. Find the adjusted mean.

Problem 32.12
The length of time X (in minutes) it takes to go from your home to downtown is normally distributed with $\mu = 30$ minutes and $\sigma_X = 5$ minutes. What is the latest time that you should leave home if you want to be over 99% sure of arriving in time for a job interview taking place in downtown at 2pm?

Problem 32.13
The minimum force required to break a particular type of cable is normally distributed with mean 12,432 and standard deviation 25. Determine the probability that a randomly selected cable will not break under a force of 12,400.

Problem 32.14 ‡
In 1982 Abby's mother scored at the 93rd percentile in the math SAT exam. In 1982 the mean score was 503 and the variance of the scores was 9604.
In 2008 Abby took the math SAT and got the same numerical score as her mother had received 26 years before. In 2008 the mean score was 521 and the variance of the scores was 10,201.
Math SAT scores are normally distributed and stated in multiples of ten. Calculate the percentile for Abby's score.

Problem 32.15 ‡
The working lifetime, in years, of a particular model of bread maker is normally distributed with mean 10 and variance 4.
Calculate the 12th percentile of the working lifetime, in years.

Problem 32.16 ‡
The profits of life insurance companies A and B are normally distributed with the same mean. The variance of company B's profit is 2.25 times the variance of company A's profit. The 14th percentile of company A's profit is the same as the pth percentile of company B's profit. Calculate p.

Problem 32.17 ‡
Insurance companies A and B each earn an annual profit that is normally distributed with the same positive mean. The standard deviation of company A's annual profit is one half of its mean.
In a given year, the probability that company B has a loss (negative profit) is 0.9 times the probability that company A has a loss.
Calculate the ratio of the standard deviation of company B's annual profit to the standard deviation of company A's annual profit.

Problem 32.18 ‡
An insurance policy covers losses incurred by a policyholder, subject to a deductible of 10,000. Incurred losses follow a normal distribution with mean 12,000 and standard deviation c. The probability that a loss is less than k is 0.9582, where k is a constant. Given that the loss exceeds the deductible, there is a probability of 0.9500 that it is less than k.
Calculate c.

Problem 32.19 ‡
The annual profit of a life insurance company is normally distributed. The

probability that the annual profit does not exceed 2000 is 0.7642. The probability that the annual profit does not exceed 3000 is 0.9066.
Calculate the probability that the annual profit does not exceed 1000.

Problem 32.20 ‡
A gun shop sells gunpowder. Monthly demand for gunpowder is normally distributed, averages 20 pounds, and has a standard deviation of 2 pounds. The shop manager wishes to stock gunpowder inventory at the beginning of each month so that there is only a 2% chance that the shop will run out of gunpowder (i.e., that demand will exceed inventory) in any given month.
Calculate the amount of gunpowder to stock in inventory, in pounds.

Problem 32.21 ‡
A company's annual profit, in billions, has a normal distribution with variance equal to the cube of its mean. The probability of an annual loss is 5%.
Calculate the company's expected annual profit.

Problem 32.22 ‡
Losses incurred by a policyholder follow a normal distribution with mean 20,000 and standard deviation 4,500. The policy covers losses, subject to a deductible of 15,000. Calculate the 95^{th} percentile of losses that exceed the deductible. Round your answer to the nearest hundreds.

Problem 32.23 ‡
An insurance policy will reimburse only one claim per year. For a random policyholder, there is a 20% probability of no loss in the next year, in which case the claim amount is 0. If a loss occurs in the next year, the claim amount is normally distributed with mean 1000 and standard deviation 400. Calculate the median claim amount in the next year for a random policyholder.

33 The Normal Approximation to the Binomial Distribution

When the number of trials in a binomial distribution is very large, the use of the probability distribution formula $p(x) =_n C_x p^x q^{n-x}$ becomes tedious. An attempt was made to approximate this distribution for large values of n. The approximating distribution is the normal distribution.
Historically, the normal distribution was discovered by De Moivre as an approximation to the binomial distribution. The result is the so-called *De Moivre-Laplace theorem.*

Theorem 33.1
Let X denote the number of successes that occur with n independent Bernoulli trials, each with probability p of success. Then, X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$. Then

$$\lim_{n\to\infty}\left(\frac{X-np}{\sqrt{np(1-p)}}\right) = \frac{N-np}{\sqrt{np(1-p)}}$$

where N is the normal random variable with $\mu = np$ and $\sigma^2 = np(1-p)$.

Proof.
This result is a special case of the central limit theorem. See Section 54 ■

Remark 33.1
How large should n be so that a normal approximation to the binomial distribution is adequate? A rule-of-thumb for the normal distribution to be a good approximation to the binomial distribution is to have $np > 5$ and $nq > 5$.

Remark 33.2 *(continuity correction)*
Suppose we are approximating a binomial random variable X with a normal random variable N. Say we want to find $P(8 \leq X \leq 10)$ where X is a binomial distribution. Then

$$P(8 \leq X \leq 10) = P(X = 8) + P(X = 9) + P(X = 10).$$

According to Figure 33.1, the probability in question is the area of the three rectangles centered at 8,9 and 10. When using the standard normal distribution (a continuous random variable) to approximate the binomial distribution (a discrete random variable), the area under the pdf N from 7.5 to 10.5 must be found. That is,

$$P(8 \leq X \leq 10) \approx P(7.5 \leq N \leq 10.5).$$

In practice, then, we apply a *continuity correction,* when approximating a discrete random variable with a continuous random variable.

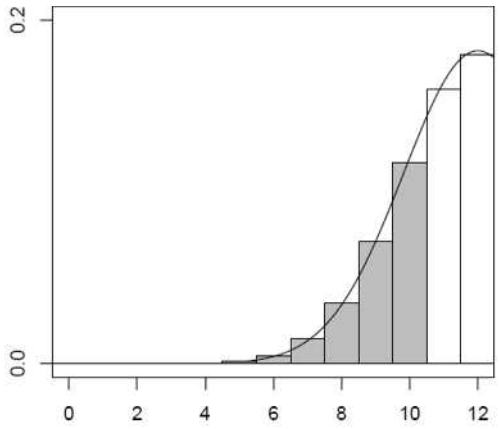


Figure 33.1

Example 33.1
In a box of 100 light bulbs, 10 are found to be defective. What is the probability that the number of defectives exceeds 13?

Solution.
Let X be the number of defective items. Then X is binomial with $n = 100$ and $p = 0.1$. Since $np = 10 > 5$ and $nq = 90 > 5$ we can use the normal approximation to the binomial with $\mu = np = 10$ and $\sigma^2 = np(1-p) = 9$. We want $P(X > 13)$. Using continuity correction we find

$$\begin{aligned} P(X > 13) =& P(X \geq 14) \approx P(N \geq 13.5) \\ =& P(\frac{N-10}{\sqrt{9}} \geq \frac{13.5-10}{\sqrt{9}}) \\ \approx& 1 - \Phi(1.17) = 1 - 0.8790 = 0.121 \blacksquare \end{aligned}$$

Example 33.2
In a small town, it was found that out of every 6 people 1 is left-handed. Consider a random sample of 612 persons from the town, estimate the probability that the number of left-handed persons is strictly between 90 and 150.

Solution.
Let X be the number of left-handed people in the sample. Then X is a binomial random variable with $n = 612$ and $p = \frac{1}{6}$. Since $np = 102 > 5$ and $n(1-p) = 510 > 5$ we can use the normal approximation to the binomial with $\mu = np = 102$ and $\sigma^2 = np(1-p) = 85$. Using continuity correction we find

$$\begin{aligned} P(90 < X < 150) =& P(91 \leq X \leq 149) \approx P(90.5 \leq N \leq 149.5) \\ =& P\left(\frac{90.5 - 102}{\sqrt{85}} \leq \frac{N - 102}{\sqrt{85}} \leq \frac{149.5 - 102}{\sqrt{85}}\right) \\ =& P(-1.25 \leq Z \leq 5.15) \approx 0.8943 \blacksquare \end{aligned}$$

Example 33.3
There are 90 students in a statistics class. Suppose each student has a standard deck of 52 cards of his/her own, and each of them selects 13 cards at random without replacement from his/her own deck independent of the others. What is the chance that there are more than 50 students who got at least 2 aces ?

Solution.
Let X be the number of students who got at least 2 aces or more, then clearly X is a binomial random variable with $n = 90$ and

$$p = \frac{{}_4C_2 \cdot {}_{48}C_{11}}{{}_{52}C_{13}} + \frac{{}_4C_3 \cdot {}_{48}C_{10}}{{}_{52}C_{13}} + \frac{{}_4C_4 \cdot {}_{48}C_9}{{}_{52}C_{13}} \approx 0.2573$$

Since $np \approx 23.157 > 5$ and $n(1-p) \approx 66.843 > 5$, X can be approximated by a normal random variable with $\mu = 23.157$ and $\sigma = \sqrt{np(1-p)} \approx 4.1473$. Thus,

$$\begin{aligned} P(X > 50) =& 1 - P(X \leq 50) \approx 1 - P(N \leq 50.5) \\ =& 1 - \Phi\left(\frac{50.5 - 23.157}{4.1473}\right) \\ \approx& 1 - \Phi(6.59) \blacksquare \end{aligned}$$

Practice Problems

Problem 33.1
Suppose that 25% of all the students who took a given test fail. Let X be the number of students who failed the test in a random sample of 50.
(a) What is the probability that the number of students who failed the test is at most 10?
(b) What is the probability that the number of students who failed the test is between 5 and 15 inclusive?

Problem 33.2
A vote on whether to allow the use of medical marijuana is being held. A polling company will survey 200 individuals to measure support for the new law. If in fact 53% of the population oppose the new law, use the normal approximation to the binomial, with a continuity correction, to approximate the probability that the poll will show a majority in favor?

Problem 33.3
A company manufactures 50,000 light bulbs a day. For every 1,000 bulbs produced there are 50 bulbs defective. Consider testing a random sample of 400 bulbs from today's production. Find the probability that the sample contains
(a) At least 14 and no more than 25 defective bulbs.
(b) At least 33 defective bulbs.

Problem 33.4
Suppose that the probability of a family with two children is 0.25 that the children are boys. Consider a random sample of 1,000 families with two children. Find the probability that at most 220 families have two boys.

Problem 33.5
A survey shows that 10% of the students in a college are left-handed. In a random sample of 818, what is the probability that at most 100 students are left-handed?

Problem 33.6 ‡
The minimum force required to break a particular type of cable is normally distributed with mean 12,432 and standard deviation 25. A random sample of 400 cables of this type is selected.
Calculate the probability that at least 349 of the selected cables will not break under a force of 12,400.

Problem 33.7 ‡

A student takes a multiple-choice test with 40 questions. The probability that the student answers a given question correctly is 0.5, independent of all other questions. The probability that the student answers more than N questions correctly is greater than 0.10. The probability that the student answers more than $N + 1$ questions correctly is less than 0.10. Calculate N using a normal approximation with the continuity correction.

34 Exponential Random Variables

An **exponential** random variable with parameter $\lambda > 0$ is a random variable with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The parameter λ is called the **rate parameter**. Note that

$$\int_0^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^\infty = 1.$$

The graph of the probability density function is shown in Figure 34.1

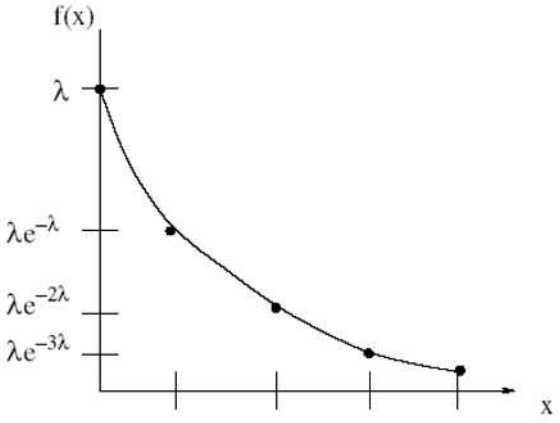


Figure 34.1

Exponential random variables are often used to model arrival times, waiting times, and equipment failure times. For example, in physics the exponential distribution is used to represent the lifetime of a particle, the parameter λ representing the rate at which the particle ages.
The expected value of X can be found using integration by parts with $u = x$ and $dv = \lambda e^{-\lambda x} dx$:

$$\begin{aligned} E(X) &= \int_0^\infty x\lambda e^{-\lambda x} dx = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= \left[-xe^{-\lambda x}\right]_0^\infty + \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}. \end{aligned}$$

Likewise, using integration by parts twice, we obtain

$$E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = \int_0^\infty x^2 d(-e^{-\lambda x}) = \left[-x^2 e^{-\lambda x}\right]_0^\infty + 2\int_o^\infty x e^{-\lambda x} dx$$
$$= 2\left[-\frac{x}{\lambda} e^{-\lambda x}\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x} dx = \frac{2}{\lambda}\left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{2}{\lambda^2}.$$

Thus,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Example 34.1
The time between calls received by a 911 operator has an exponential distribution with an average of 3 calls per hour.
(a) Find the expected time between calls.
(b) Find the probability that the next call is received within 5 minutes.

Solution.
Let X denote the time (in hours) between calls. We are told that $\lambda = 3$.
(a) We have $E(X) = \frac{1}{\lambda} = \frac{1}{3}$.
(b) $P(X < \frac{1}{12}) = \int_0^{\frac{1}{12}} 3e^{-3x} dx \approx 0.2212$ ■

The cumulative distribution function of an exponential random variable X is given by

$$F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda u} du = -e^{-\lambda u}\,|_0^x = 1 - e^{-\lambda x}$$

for $x \geq 0$, and 0 otherwise.

Example 34.2
Suppose that the waiting time (in minutes) at a post office is an exponential random variable with mean 10 minutes. If someone arrives immediately ahead of you at the post office, find the probability that you have to wait
(a) more than 10 minutes
(b) between 10 and 20 minutes.

Solution.
Let X be the time you must wait in line at the post office. Then X is an exponential random variable with parameter $\lambda = 0.1$.

(a) We have $P(X > 10) = 1 - F(10) = 1 - (1 - e^{-1}) = e^{-1} \approx 0.3679$.
(b) We have $P(10 \leq X \leq 20) = F(20) - F(10) = e^{-1} - e^{-2} \approx 0.2325$ ■

The most important property of the exponential distribution is known as the **memoryless** property:

$$P(X > s + t | X > s) = P(X > t), \quad s, t \geq 0.$$

This says that the probability that we have to wait for an additional time t (and therefore a total time of $s + t$) given that we have already waited for time s is the same as the probability at the start that we would have had to wait for time t. So the exponential distribution "forgets" that it is larger than s.
To see why the memoryless property holds, note that for all $t \geq 0$, we have

$$P(X > t) = \int_t^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x} |_t^\infty = e^{-\lambda t}.$$

It follows that

$$\begin{aligned} P(X > s + t | X > s) =& \frac{P(X > s + t \text{ and } X > s)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)} \\ =& \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t). \end{aligned}$$

Example 34.3
Suppose that the time X (in hours) required to repair a car has an exponential distribution with parameter $\lambda = 0.25$. Find
(a) the cumulative distribution function of X.
(b) $P(X > 4)$.
(c) $P(X > 10 | X > 8)$.

Solution.
(a) It is easy to see that the cumulative distribution function is

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{4}} & x \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

(b) $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - e^{-\frac{4}{4}}) = e^{-1} \approx 0.368$.
(c) By the memoryless property, we find

$$\begin{aligned} P(X > 10|X > 8) =& P(X > 8 + 2|X > 8) = P(X > 2) \\ =& 1 - P(X \leq 2) = 1 - F(2) \\ =& 1 - (1 - e^{-\frac{1}{2}}) = e^{-\frac{1}{2}} \approx 0.6065 \end{aligned}$$ ■

Example 34.4
The time between hits to my website is an exponential distribution with an average of 5 minutes between hits.
(a) What is the probability that there are no hits in a 20-minute period?
(b) What is the probability that the first observed hit occurs between 15 and 20 minutes?
(c) Given that there are no hits in the first 5 minutes observed, what is the probability that there are no hits in the next 15 minutes?

Solution.
Let X denote the time between two hits. Then, X is an exponential random variable with $\mu = \frac{1}{E(X)} = \frac{1}{5} = 0.2$ hit/minute.
(a)

$$P(X > 20) = \int_{20}^{\infty} 0.2e^{-0.2x}dx = -e^{-0.2x}\big|_{20}^{\infty} = e^{-4} \approx 0.01831.$$

(b)

$$P(15 < X < 20) = \int_{15}^{20} 0.2e^{-0.2x}dx = -e^{-0.2x}\big|_{15}^{20} \approx 0.03147.$$

(c) By the memoryless property, we have

$$P(X > 15+5|X > 5) = P(X > 15) = \int_{15}^{\infty} 0.2e^{-0.2x}dx = -e^{-0.2x}\big|_{15}^{\infty} \approx 0.04979$$ ■

The exponential random variable is the only named continuous random variable with range $[0, \infty)$ (and differentiable cdf) that possesses the memoryless property. To see this, suppose that X is a memoryless continuous random variable with range $[0, \infty)$ and differentiable cdf $F(x)$. Let $g(x) = P(X > x)$. Since X is memoryless, we have for $h > 0$

$$P(X > h) = P(X > x+h|X > x) = \frac{P(X > x + h \text{ and } X > x)}{P(X > h)} = \frac{P(X > x + h)}{P(X > x)}$$

and this implies

$$P(X > x + h) = P(X > x)P(X > h)$$

Hence, g satisfies the equation

$$g(x + h) = g(x)g(h).$$

Theorem 34.1
The only solution to the functional equation $g(x + h) = g(x)g(h)$ is $g(x) = e^{-\lambda x}$ for some $\lambda > 0$.

Proof.
We have

$$\frac{g(x + h) - g(x)}{h} = g(x)\frac{g(h) - 1}{h}.$$

Thus,

$$g'(x) = \lim_{h\to 0^+} \frac{g(x + h) - g(x)}{h} = g(x) \lim_{h\to 0^+} \frac{g(h) - 1}{h} = g(x)g'(0).$$

This is the familiar differential equation for exponential decay which can be solved by using the method of separation of variables. The general solution is

$$g(x) = g(0)e^{g'(0)x}.$$

But $g(0) = 1 - F(0) = 1$ and $\lambda = -g'(0) = F'(0) > 0$ since F is strictly increasing. Hence,

$$g(x) = e^{-\lambda x},\ \lambda > 0.$$

It follows that $F(x) = P(X \leq x) = 1 - g(x) = 1 - e^{-\lambda x}$ and hence $f(x) = F'(x) = \lambda e^{-\lambda x}$ which shows that X is exponentially distributed ■

Example 34.5
Very often, credit card customers are placed on hold when they call for inquiries. Suppose the amount of time until a service agent assists a customer has an exponential distribution with mean 5 minutes. Given that a customer has already been on hold for 2 minutes, what is the probability that he/she will remain on hold for a total of more than 5 minutes?

Solution.
Let X represent the total time on hold. Then X is an exponential random variable with $\lambda = \frac{1}{5}$. Thus,

$$P(X > 3 + 2|X > 2) = P(X > 3) = 1 - F(3) = e^{-\frac{3}{5}}$$ ■

Practice Problems

Problem 34.1
Let X have an exponential distribution with a mean of 40. Compute $P(X < 36)$.

Problem 34.2
Let X be an exponential function with mean equals to 5. Graph $f(x)$ and $F(x)$.

Problem 34.3
A continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \frac{1}{100}e^{-\frac{x}{100}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(0 \leq X \leq 50)$.

Problem 34.4
Let X be an exponential random variable with mean equals to 4. Find $P(X \leq 0.5)$.

Problem 34.5
The life length X (in years) of a DVD player is exponentially distributed with mean 5 years. What is the probability that a more than 5-year old DVD would still work for more than 3 years?

Problem 34.6
Suppose that the spending time X (in minutes) of a customer at a bank has an exponential distribution with mean 3 minutes.
(a) What is the probability that a customer spends more than 5 minutes in the bank?
(b) Under the same conditions, what is the probability of spending between 2 and 4 minutes?

Problem 34.7
The waiting time X (in minutes) of a train arrival to a station has an exponential distribution with mean 3 minutes.
(a) What is the probability of having to wait 6 or more minutes for a train?
(b) What is the probability of waiting between 4 and 7 minutes for a train?
(c) What is the probability of having to wait at least 9 more minutes for the train given that you have already waited 3 minutes?

Problem 34.8 ‡
Ten years ago at a certain insurance company, the size of claims under homeowner insurance policies had an exponential distribution. Furthermore, 25% of claims were less than \$1000. Today, the size of claims still has an exponential distribution but, owing to inflation, every claim made today is twice the size of a similar claim made 10 years ago. Determine the probability that a claim made today is less than \$1000.

Problem 34.9
The lifetime (in hours) of a battery installed in a radio is an exponentially distributed random variable with parameter $\lambda = 0.01$. What is the probability that the battery is still in use one week after it is installed?

Problem 34.10 ‡
The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.
What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

Problem 34.11 ‡
The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year.
If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

Problem 34.12 ‡
A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$.
Determine $E[X]$.

Problem 34.13 ‡
A piece of equipment is being insured against early failure. The time from

purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000 ?

Problem 34.14 ‡
An insurance policy reimburses dental expense, X, up to a maximum benefit of 250 . The probability density function for X is:

$$f(x) = \begin{cases} ce^{-0.004x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

Problem 34.15 ‡
The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.
Calculate the probability that the component will work without failing for at least five hours.

Problem 34.16
Let X be an exponential random variable such that $P(X \leq 2) = 2P(X > 4)$. Find the variance of X.

Problem 34.17 ‡
The cumulative distribution function for health care costs experienced by a policyholder is modeled by the function

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{100}}, & \text{for } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The policy has a deductible of 20. An insurer reimburses the policyholder for 100% of health care costs between 20 and 120 less the deductible. Health care costs above 120 are reimbursed at 50%. Let G be the cumulative distribution function of reimbursements given that the reimbursement is positive. Calculate $G(115)$.

Problem 34.18 ‡
The lifespan, in years, of a certain computer is exponentially distributed. The probability that its lifespan exceeds four years is 0.30.
Let $f(x)$ represent the density function of the computer's lifespan, in years, for $x > 0$. Determine the formula for $f(x)$.

Problem 34.19 ‡
A car is new at the beginning of a calendar year. The time, in years, before the car experiences its first failure is exponentially distributed with mean 2. Calculate the probability that the car experiences its first failure in the last quarter of some calendar year.

Problem 34.20 ‡
Losses due to accidents at an amusement park are exponentially distributed. An insurance company offers the park owner two different policies, with different premiums, to insure against losses due to accidents at the park.
Policy A has a deductible of 1.44. For a random loss, the probability is 0.640 that under this policy, the insurer will pay some money to the park owner.
Policy B has a deductible of d. For a random loss, the probability is 0.512 that under this policy, the insurer will pay some money to the park owner.
Calculate d.

Problem 34.21 ‡
Losses due to burglary are exponentially distributed with mean 100. The probability that a loss is between 40 and 50 equals the probability that a loss is between 60 and r, with $r > 60$.
Calculate r.

Problem 34.22 ‡
The time until the next car accident for a particular driver is exponentially distributed with a mean of 200 days.
Calculate the probability that the driver has no accidents in the next 365 days, but then has at least one accident in the 365-day period that follows this initial 365-day period.

Problem 34.23 ‡
Losses under an insurance policy are exponentially distributed with mean 4. The deductible is 1 for each loss.
Calculate the median amount that the insurer pays a policyholder for a loss under the policy.

Problem 34.24 ‡
The loss L due to a boat accident is exponentially distributed. Boat insurance policy A covers up to 1 unit for each loss. Boat insurance policy B covers up to 2 units for each loss.
The probability that a loss is fully covered under policy B is 1.9 times the probability that it is fully covered under policy A.
Calculate the variance of L.

Problem 34.25 ‡
Losses, X, under an insurance policy are exponentially distributed with mean 10. For each loss, the claim payment Y is equal to the amount of the loss in excess of a deductible $d > 0$.
Calculate $\text{Var}(Y)$.

Problem 34.26 ‡
An auto insurance policy has a deductible of 1 and a maximum claim payment of 5. Auto loss amounts follow an exponential distribution with mean 2. Calculate the expected claim payment made for an auto loss.

Problem 34.27 ‡
Let X be a random variable with density function

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $P(X \leq 0.5 | X \leq 1.0)$.

Problem 34.28 ‡
The distribution of values of the retirement package offered by a company to new employees is modeled by the probability density function

$$f_X(x) = \begin{cases} \frac{1}{5}e^{-\frac{(x-5)}{5}}, & x \geq 5, \\ 0 & \text{otherwise} \end{cases}$$

Calculate the variance of the retirement package value for a new employee, given that the value is at least 10.

Problem 34.29 ‡
A certain town experiences an average of 5 tornadoes in any four year period. The number of years from now until the town experiences its next tornado as well as the number of years between tornadoes have identical exponential distributions and all such times are mutually independent Calculate the median number of years from now until the town experiences its next tornado.

35 Gamma Distribution

We start this section by introducing the **Gamma function** defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy, \quad \alpha > 0.$$

For example,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = -e^{-y} \,|_0^\infty = 1.$$

For $\alpha > 1$ we can use integration by parts with $u = y^{\alpha-1}$ and $dv = e^{-y}dy$ to obtain

$$\begin{aligned}
\Gamma(\alpha) =& - e^{-y} y^{\alpha-1} \,|_0^\infty + \int_0^\infty e^{-y}(\alpha-1) y^{\alpha-2} dy \\
=& (\alpha-1) \int_0^\infty e^{-y} y^{\alpha-2} dy \\
=& (\alpha-1)\Gamma(\alpha-1).
\end{aligned}$$

If n is a positive integer greater than 1 then by applying the previous relation repeatedly we find

$$\begin{aligned}
\Gamma(n) =& (n-1)\Gamma(n-1) \\
=& (n-1)(n-2)\Gamma(n-2) \\
& \vdots \\
=& (n-1)(n-2)\cdots 3 \cdot 2 \cdot \Gamma(1) = (n-1)!
\end{aligned}$$

Example 35.1
Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Solution.
Using the substitution $y = \frac{z^2}{2}$, we find

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) =& \int_0^\infty y^{-\frac{1}{2}} e^{-y} dy = \sqrt{2} \int_0^\infty e^{-\frac{z^2}{2}} dz \\
=& \frac{\sqrt{2}}{2}\sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{z^2}{2}} dz\right] = \sqrt{\pi}
\end{aligned}$$

where we used the fact that Z is the standard normal distribution with

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz = 1 \blacksquare$$

A **Gamma** random variable with parameters $\alpha > 0$ and $\lambda > 0$ has a pdf

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We call α the **shape parameter** because changing α changes the shape of the density function. We call λ the **scale parameter** because if X is a Gamma distribution with parameters (α, λ) then cX is also a Gamma distribution with parameters $(\alpha, \frac{\lambda}{c})$ where $c > 0$ is a constant. See Problem 35.1. The parameter λ rescales the density function without changing its shape.

To see that $f(x)$ is indeed a probability density function we have

$$\begin{aligned} \Gamma(\alpha) &= \int_0^{\infty} e^{-x}x^{\alpha-1}dx \\ 1 &= \int_0^{\infty} \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}dx \\ 1 &= \int_0^{\infty} \frac{\lambda e^{-\lambda y}(\lambda y)^{\alpha-1}}{\Gamma(\alpha)}dy \end{aligned}$$

where we used the substitution $x = \lambda y$.
The Gamma distribution is skewed right as shown in Figure 35.1

Figure 35.1

Note that the above computation involves a $\Gamma(\alpha)$ integral. Thus, the origin of the name of the random variable.

The cdf of the Gamma distribution is

$$F(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-\lambda y} dy.$$

The following reduction formula is useful when computing $F(x)$:

$$\int x^n e^{-\lambda x} dx = -\frac{1}{\lambda} x^n e^{-\lambda x} + \frac{n}{\lambda} \int x^{n-1} e^{-\lambda x} dx. \tag{35.1}$$

The Gamma distribution can be used to model a number of physical quantities such as service times, lifetimes of objects, and repair times. Gamma distribution plays a prominent role in actuarial science since most total insurance claim distributions have roughly the same shape as Gamma distributions: skewed to the right, non-negatively supported and uni-modal. See Figure 37.1.

Example 35.2
Let X be a Gamma random variable with $\alpha = 4$ and $\lambda = \frac{1}{2}$. Compute $P(2 < X < 4)$.

Solution.
We have

$$P(2 < X < 4) = \int_2^4 \frac{1}{2^4 \Gamma(4)} x^3 e^{-\frac{x}{2}} dx = \frac{1}{96} \int_2^4 x^3 e^{-\frac{x}{2}} dx \approx 0.124$$

where we used the reduction formula (35.1) ■

The next result provides formulas for the expected value and the variance of a Gamma distribution.

Theorem 35.1
If X is a Gamma random variable with parameters (λ, α) then
(a) $E(X) = \frac{\alpha}{\lambda}$
(b) $\text{Var}(X) = \frac{\alpha}{\lambda^2}$.

Solution.
(a) We have

$$\begin{aligned} E(X) =& \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda x e^{-\lambda x} (\lambda x)^{\alpha-1} dx = \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^\alpha dx \\ =& \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda}. \end{aligned}$$

(b) We have

$$E(X^2) = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^2 e^{-\lambda x} \lambda^\alpha x^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} \lambda^\alpha e^{-\lambda x} dx$$
$$= \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha+1} \lambda^{\alpha+2} e^{-\lambda x}}{\Gamma(\alpha+2)} dx = \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)}$$

where the last integral is the integral of the pdf of a Gamma random variable with parameters $(\alpha+2, \lambda)$. Thus,

$$E(X^2) = \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^2 \Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\lambda^2}.$$

Finally,

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2} \blacksquare$$

Example 35.3
In a certain city, the daily consumption of water (in millions of liters) can be treated as a random variable having a Gamma distribution with $\alpha = 3$ and $\lambda = 0.5$.
(a) What is the random variable? What is the expected daily consumption?
(b) If the daily capacity of the city is 12 million liters, what is the probability that this water supply will be inadequate on a given day? Set up the appropriate integral but do not evaluate.
(c) What is the variance of the daily consumption of water?

Solution.
(a) The random variable is the daily consumption of water in millions of liters. The expected daily consumption is the expected value of a Gamma distributed variable with parameters $\alpha = 3$ and $\lambda = \frac{1}{2}$ which is $E(X) = \frac{\alpha}{\lambda} = 6$.
(b) The probability is $\frac{1}{2^3\Gamma(3)} \int_{12}^\infty x^2 e^{-\frac{x}{2}} dx = \frac{1}{16} \int_{12}^\infty x^2 e^{-\frac{x}{2}} dx$.
(c) The variance is

$$\text{Var}(X) = \frac{3}{0.5^2} = 12 \blacksquare$$

Remark 35.1
It is easy to see that when the parameter set is restricted to $(\alpha, \lambda) = (1, \lambda)$ the Gamma distribution becomes the exponential distribution.

Practice Problems

Problem 35.1
Let X be a Gamma distribution with parameters (α, λ). Let $Y = cX$ with $c > 0$. Show that

$$F_Y(y) = \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)} \int_0^y z^{\alpha-1} e^{-\lambda \frac{z}{c}} dz.$$

Hence, Y is a Gamma distribution with parameters $\left(\alpha, \frac{\lambda}{c}\right)$.

Problem 35.2
If X has a probability density function given by

$$f(x) = \begin{cases} 4x^2 e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance.

Problem 35.3
Let X be a Gamma random variable with $\lambda = 1.8$ and $\alpha = 3$. Compute $P(X > 3)$.

Problem 35.4
Suppose the time (in hours) taken by a technician to fix a computer is a random variable X having a Gamma distribution with parameters $\alpha = 3$ and $\lambda = 0.5$. What is the probability that it takes at most 1 hour to fix a computer?

Problem 35.5
Suppose the continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \frac{1}{16} x^2 e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X^3)$.

Problem 35.6
Let X be the standard normal distribution. Show that X^2 is a Gamma distribution with $\alpha = \lambda = \frac{1}{2}$.

Problem 35.7
Let X be a Gamma random variable with parameter (α, λ). Find $E(e^{tX})$.

Problem 35.8
Show that the Gamma density function with parameters $\alpha > 1$ and $\lambda > 0$ has a relative maximum at $x = \frac{1}{\lambda}(\alpha - 1)$.

Problem 35.9
Let X be a Gamma distribution with parameters $\alpha = 3$, and $\lambda = \frac{1}{6}$.
(a) Give the density function, as well as the mean and standard deviation of X.
(b) Find $E(3X^2 + X - 1)$.

Problem 35.10 ‡
An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1 - 2500t)^4}, \ t < 0.0004.$$

Calculate the standard deviation of the claim size for this class of accidents.

Problem 35.11 ‡
A claimant places calls to an insurer's claims call center. Let X be the time elapsed before the claimant gets to speak with call center representatives. The moment generating function of X is

$$M_X(t) = \frac{1}{(1 - 1.5t)^2}, \ t < \frac{2}{3}.$$

Calculate the standard deviation of X.

Problem 35.12
Suppose that the time it takes to get service in a restaurant follows a Gamma distribution with mean 8 minutes and standard deviation 4 minutes. Find the parameters α and λ.

Problem 35.13
An interesting special case of the Gamma distribution is when the parameter set is $(\alpha, \lambda) = \left(\frac{n}{2}, \frac{1}{2}\right)$ where n is a positive integer. This distribution is called the **chi-squared** distribution with **degrees of freedom** n.. The chi-squared random variable is usually denoted by χ^2_n.
Find the pdf, mean and variance of the chi-squared distribution with degrees of freedom n.

Problem 35.14
Show that the random variable X with $E(e^{tX}) = (1-2t)^{-6},\ t < \frac{1}{2}$ is a χ^2 distribution. What is the degrees of freedom of this distribution?

Problem 35.15
Let X be a Gamma distribution with $\alpha = 2$ and λ. Express $\text{Var}(X^3)$ in terms of λ.

Problem 35.16
Let X be a Gamma random variable with shape parameter 2 and scale parameter 1. Determine the density function and cumulative distribution function the random variable $Y = e^X$.

36 The Distribution of a Function of a Continuous Random Variable

Let X be a continuous random variable. Let $g(x)$ be a function. Then $g(X)$ is also a random variable. In this section we are interested in finding the probability density function of $g(X)$.
The following example illustrates the method of finding the probability density function by finding first its cdf.

Example 36.1
If the probability density of X is given by

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the probability density of $Y = X^3$.

Solution.
We have

$$\begin{aligned} F(y) =& P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{\frac{1}{3}}) \\ =& \int_0^{y^{\frac{1}{3}}} 6x(1-x)dx = 3y^{\frac{2}{3}} - 2y \end{aligned}$$

Hence, $f(y) = F'(y) = 2(y^{-\frac{1}{3}} - 1)$, for $0 < y < 1$ and 0 otherwise ■

Example 36.2
Let X be a random variable with probability density $f(x)$. Find the probability density function of $Y = |X|$.

Solution.
Clearly, $F_Y(y) = 0$ for $y \leq 0$. So assume that $y > 0$. Then

$$\begin{aligned} F_Y(y) =& P(Y \leq y) = P(|X| \leq y) \\ =& P(-y \leq X \leq y) = F_X(y) - F_X(-y) \end{aligned}$$

Thus, $f_Y(y) = F_Y'(y) = f_X(y) + f_X(-y)$ for $y > 0$ and 0 otherwise ■

The following theorem provides a formula for finding the probability density of $g(X)$ for monotone g without the need for finding the distribution function.

Theorem 36.1
Let X be a continuous random variable with pdf f_X. Let $g(x)$ be a monotone and differentiable function of x. Suppose that $g^{-1}(Y) = X$. Then the random variable $Y = g(X)$ has a pdf given by

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|.$$

Proof.
Suppose first that $g(\cdot)$ is increasing. Then

$$\begin{aligned} F_Y(y) =& P(Y \leq y) = P(g(X) \leq y) \\ =& P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) \end{aligned}$$

Differentiating and using the chain rule, we find

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X[g^{-1}(y)] \frac{d}{dy} g^{-1}(y).$$

Now, suppose that $g(\cdot)$ is decreasing. Then

$$\begin{aligned} F_Y(y) =& P(Y \leq y) = P(g(X) \leq y) \\ =& P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)) \end{aligned}$$

Differentiating we find

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -f_X[g^{-1}(y)] \frac{d}{dy} g^{-1}(y) \blacksquare$$

Example 36.3
Let X be a continuous random variable with pdf f_X. Find the pdf of $Y = -X$.

Solution.
By the previous theorem we have

$$f_Y(y) = f_X(-y) \blacksquare$$

Example 36.4
Let X be a continuous random variable with pdf f_X. Find the pdf of $Y = aX + b,\ a > 0$.

Solution.
Let $g(x) = ax + b$. Then $g^{-1}(y) = \frac{y-b}{a}$. By the previous theorem, we have

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \blacksquare$$

Example 36.5
Suppose X is a random variable with the following density :

$$f(x) = \frac{1}{\pi(x^2+1)}, \quad -\infty < x < \infty.$$

(a) Find the cdf of $|X|$.
(b) Find the pdf of X^2.

Solution.
(a) $|X|$ takes values in $[0, \infty)$. Thus, $F_{|X|}(x) = 0$ for $x \leq 0$. Now, for $x > 0$ we have

$$F_{|X|}(x) = P(|X| \leq x) = \int_{-x}^{x} \frac{1}{\pi(y^2+1)} dy = \frac{2}{\pi} \tan^{-1} x.$$

Hence,

$$F_{|X|}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{2}{\pi} \tan^{-1} x & x > 0. \end{cases}$$

(b) X^2 also takes only nonnegative values, so the density $f_{X^2}(x) = 0$ for $x \leq 0$. Furthermore, $F_{X^2}(x) = P(X^2 \leq x) = P(|X| \leq \sqrt{x}) = \frac{2}{\pi} \tan^{-1} \sqrt{x}$. So by differentiating we get

$$f_{X^2}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{\pi\sqrt{x}(1+x)} & x > 0 \blacksquare \end{cases}$$

Remark 36.1
In general, if a function does not have a unique inverse, we must sum over all possible inverse values.

Practice Problems

Problem 36.1
Suppose $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$ and let $Y = aX + b$. Find $f_Y(y)$.

Problem 36.2
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find probability density function for $Y = 3X - 1$.

Problem 36.3
Let X be a random variable with density function

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of $Y = 8X^3$.

Problem 36.4
Suppose X is an exponential random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the density function of $Y = e^X$?

Problem 36.5
Gas molecules move about with varying velocity which has, according to the Maxwell- Boltzmann law, a probability density given by

$$f(v) = cv^2 e^{-\beta v^2}, \quad v \geq 0$$

The kinetic energy is given by $Y = E = \frac{1}{2}mv^2$ where m is the mass. What is the density function of Y?

Problem 36.6
Let X be a random variable that is uniformly distributed in (0,1). Find the probability density function of $Y = -\ln X$.

Problem 36.7
Let X be a uniformly distributed function over $[-\pi, \pi]$. That is

$$f(x) = \begin{cases} \frac{1}{2\pi} & -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = \cos X$.

Problem 36.8
Suppose X has the uniform distribution on (0, 1). Compute the probability density function and expected value of:
(a) X^α, $\alpha > 0$ (b) $\ln X$ (c) e^X (d) $\sin \pi X$

Problem 36.9 ‡
The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1 - \left(\frac{2}{t}\right)^2 & t > 2 \\ 0 & \text{otherwise} \end{cases}$$

The resulting cost to the company is $Y = T^2$. Determine the density function of Y, for $y > 4$.

Problem 36.10 ‡
An investment account earns an annual interest rate R that follows a uniform distribution on the interval $(0.04, 0.08)$. The value of a 10,000 initial investment in this account after one year is given by $V = 10,000e^R$.
Determine the cumulative distribution function, $F_V(v)$ of V.

Problem 36.11 ‡
An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1 year.
Determine the probability density function $f_Y(y)$, for $y > 0$, of the random variable Y.

Problem 36.12 ‡
Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries.
Find the density function $f_R(r)$ of R.

Problem 36.13 ‡
The monthly profit of Company A can be modeled by a continuous random variable with density function f_A. Company B has a monthly profit that is twice that of Company A.
Determine the probability density function of the monthly profit of Company B.

Problem 36.14
Let X have normal distribution with mean 1 and standard deviation 2.
(a) Find $P(|X| \leq 1)$.
(b) Let $Y = e^X$. Find the probability density function $f_Y(y)$ of Y.

Problem 36.15
Let X be a uniformly distributed random variable on the interval $(-1, 1)$. Find the pdf of $Y = X^2$

Problem 36.16
Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the pdf of $Y = 3X$.
(b) Find the pdf of $Z = 3 - X$.

Problem 36.17
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 1 - |x| & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the density function of $Y = X^2$.

Problem 36.18
If $f(x) = xe^{-\frac{x^2}{2}}$, for $x > 0$ and $Y = \ln X$, find the density function for Y.

Problem 36.19
Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} 2(1 - x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the pdf of $Y = 10X - 2$.
(b) Find the expected value of Y.
(c) Find $P(Y < 0)$.

Problem 36.20
Let X be a continuous random variable with pdf f_X. Find the pdf of $Y = X^2$.

Problem 36.21 ‡
X is a random variable with probability density function

$$f_X(x) = \begin{cases} e^{-2x}, & x \geq 0 \\ 2e^{4x}, & x < 0. \end{cases}$$

Determine the probability density function for $T = X^2$ for positive values of t.

Problem 36.22 ‡
The time until failure, T, of a product is modeled by a uniform distribution on [0,10]. An extended warranty pays a benefit of 100 if failure occurs between time $t = 1.5$ and $t = 8$.
The present value, W, of this benefit is

$$W = \begin{cases} 0, & 0 \leq T < 1.5, \\ 100e^{-0.04T}, & 1.5 \leq T < 8, \\ 0, & 8 \leq T \leq 10. \end{cases}$$

Calculate $P(W < 79)$.

Calculus Prerequisite II

In this chapter, we collect concepts from calculus deemed necessary in the understanding of the topics that follow later in this book.

37 Graphing Systems of Inequalities in Two Variables

When evaluating double integrals over a certain region, the region under consideration is the solution to a system of inequalities in two variables. The purpose of this section is to represent the solution graphically. We will consider systems of linear inequalities. However, the discussion applies to any system of inequalities.
By a **linear inequality** in the variables x and y we mean anyone of the following

$$ax + by \leq c,\ ax + by \geq c,\ ax + by < c,\ ax + by > c.$$

A pair of numbers (x, y) that satisfies a linear inequality is called a **solution.** A solution set of a linear inequality is a half-plane in the Cartesian coordinates system. The **boundary** of the region is the graph of the line $ax + by = c$. The boundary is represented by a dashed line in the case of inequalities involving either $<$ or $>$. Otherwise, the boundary is represented by a solid line to show that the points on the line are included in the solution set.
To solve a linear inequality of the type above, one starts by drawing the boundary line. This boundary line partition the Cartesian coordinates system into two half-planes. One of them is the solution set. To determine which of the two half-planes is the solution set, one picks a point, called a **test point**, in one of the half-plane. If the chosen point is a solution to the

linear inequality then the half-plane containing the point is the solution set. Otherwise, the half-plane not containing the point is the solution set.

Example 37.1
Solve graphically each of the following inequalities:
(a) $y \leq x - 2$
(b) $y < x - 2$
(c) $y \geq x - 2$
(d) $y > x - 2$.

Solution.
(a) First, we graph the line $y = x-2$ as a solid line. The test point $(0, 0)$ does not satisfy the inequality so that the lower-half plane including the boundary line is the solution set. See Figure 37.1(a).
(b) The boundary line is a dashed-line. As in (a), the solution set is the lower-half plane. See Figure 37.1(b).
(c) The solution set is the upper-half plane together with the boundary line. See Figure 37.1(c).
(d) The solution set is the upper-half plane. See Figure 37.1(d) ■

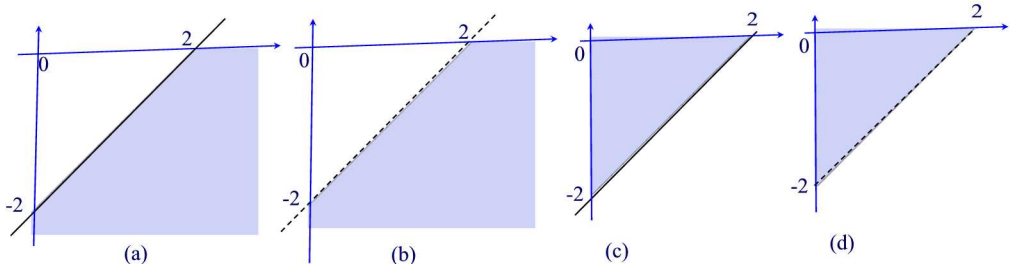


Figure 37.1

We next consider systems of linear inequalities. The solution set is the region in the Cartesian coordinate system consisting of all pairs that simultaneously satisfy all the inequalities in the system. The solution region is known as the **feasible region**.
To find the feasible region, we solve graphically each linear inequality in the system. The feasible region is the region where all the solution sets overlap. The intersection of two boundary lines is called a **corner point**.

Example 37.2
Determine if the point $(x, y) = (1, 2)$ is in the feasible region of the system

$$\begin{cases} 2x + 3y \geq 6 \\ 2x - 3y \geq 15. \end{cases}$$

Solution.
The give point satisfies the first inequality. However, it does not satisfy the second inequality since $2x - 3y = 2(1) - 3(2) = -4 < 15$. Hence, the point $(1, 2)$ is not in the feasible region of the given system ■

Example 37.3
Solve the linear system

$$\begin{cases} -3x + 4y \leq 12 \\ x + 2y < 6 \\ -x + 5y \geq -5. \end{cases}$$

Solution.
We graph the solution set to each linear inequality on the same set of axes. The overlapping region is the triangle with corners $(0, 3)$, $\left(-\frac{80}{11}, -\frac{27}{11}\right)$, $\left(\frac{40}{7}, \frac{1}{7}\right)$ as shown in Figure 37.2 ■

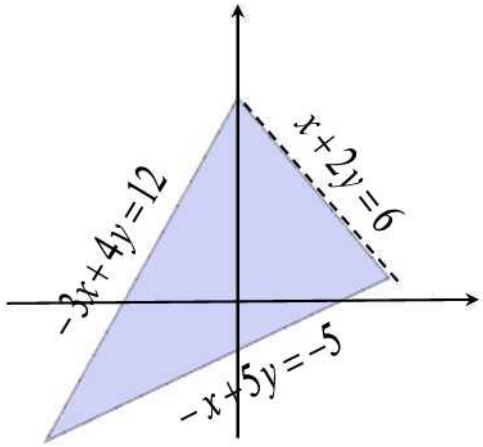


Figure 37.2

Practice Problems

Problem 37.1
Solve graphically $2x - 3y \leq 6$.

Problem 37.2
Solve graphically $x > 3$.

Problem 37.3
Solve graphically $y \leq 2$.

Problem 37.4
Solve graphically $2x + 5y > 20$.

Problem 37.5
Solve the system of inequalities:

$$\begin{cases} x - y < 1 \\ 2x + 3y \leq 12 \\ x \geq 0. \end{cases}$$

Problem 37.6
Solve the system of inequalities:

$$\begin{cases} x + 2y \leq 3 \\ -3x + y < 5 \\ -3x + 8y \geq -23. \end{cases}$$

Problem 37.7
Solve the system of inequalities:

$$\begin{cases} y < 2x + 1 \\ x + y \geq -5. \end{cases}$$

Problem 37.8
Solve the system of inequalities:

$$\begin{cases} x > -2 \\ y \leq 4 \\ 3x + 4y \leq 24. \end{cases}$$

Problem 37.9
Find the feasible region of the system

$$\begin{cases} x+y>1 \\ 0<x<1 \\ 0<y<2. \end{cases}$$

Problem 37.10
Find the feasible region of the system

$$\begin{cases} x+y<1 \\ 0<x<0.2 \\ y>0. \end{cases}$$

Problem 37.11
Graph the region $x^2 \leq y \leq x$.

Problem 37.12
Find the feasible region of the system

$$\begin{cases} x<50-y \\ 20<x,y<30. \end{cases}$$

Problem 37.13
Find the feasible region of the system

$$\begin{cases} x^2>y \\ 0<x,y<1. \end{cases}$$

Problem 37.14
Find the feasible region of the system

$$\begin{cases} \frac{x}{2} \leq y \leq x \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 1. \end{cases}$$

Problem 37.15
Find the feasible region of the system

$$\begin{cases} x-\frac{1}{2} \leq y \leq x+\frac{1}{2} \\ 0 \leq x,y \leq 1. \end{cases}$$

Problem 37.16
Find the feasible region of the system

$$\begin{cases} x + 2y < 3 \\ 0 \leq x, y \leq 2. \end{cases}$$

Problem 37.17
Find the feasible region of the system

$$\begin{cases} y \geq 2x \\ 0 \leq y \leq 3 - x \\ x \geq 0 \end{cases}$$

Problem 37.18
Find the feasible region of the system

$$\begin{cases} x - 20 \leq y \leq x + 20 \\ 2000 \leq x, y \leq 2200. \end{cases}$$

38 Iterated Double Integrals

In this section, we see how to compute double integrals exactly using one-variable integrals.
Going back to the definition of the integral over a region as the limit of a double Riemann sum:

$$\begin{aligned}\int_R f(x,y)dxdy &= \lim_{m,n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*,y_j^*)\Delta x\Delta y \\ &= \lim_{m,n\to\infty}\sum_{j=1}^{m}\left(\sum_{i=1}^{n} f(x_i^*,y_j^*)\Delta x\right)\Delta y \\ &= \lim_{m,n\to\infty}\sum_{j=1}^{m}\Delta y\left(\sum_{i=1}^{n} f(x_i^*,y_j^*)\Delta x\right) \\ &= \lim_{m\to\infty}\sum_{j=1}^{m}\Delta y\int_a^b f(x,y_j^*)dx.\end{aligned}$$

We now let

$$F(y_j^*) = \int_a^b f(x,y_j^*)dx$$

and, substituting into the expression above, we obtain

$$\int_R f(x,y)dxdy = \lim_{m\to\infty}\sum_{j=1}^{m} F(y_j^*)\Delta y = \int_c^d F(y)dy = \int_c^d\int_a^b f(x,y)dxdy.$$

Thus, if f is continuous over a rectangle R then the integral of f over R can be expressed as an **iterated integral**. To evaluate this iterated integral, first perform the inside integral with respect to x, holding y constant, then integrate the result with respect to y.

Example 38.1
Compute $\int_0^{16}\int_0^8 \left(12-\frac{x}{4}-\frac{y}{8}\right)dxdy$.

Solution.
We have

$$\begin{aligned}\int_0^{16}\int_0^8 \left(12-\frac{x}{4}-\frac{y}{8}\right)dxdy &= \int_0^{16}\left(\int_0^8 \left(12-\frac{x}{4}-\frac{y}{8}\right)dx\right)dy\\ &= \int_0^{16}\left[12x-\frac{x^2}{8}-\frac{xy}{8}\right]_0^8 dy\\ &= \int_0^{16}(88-y)dy = 88y-\frac{y^2}{2}\Big|_0^{16} = 1280 \blacksquare\end{aligned}$$

We note, that we can repeat the argument above for establishing the iterated integral, reversing the order of the summation so that we sum over j first and i second (i.e. integrate over y first and x second) so the result has the order of integration reversed. That is we can show that

$$\int_R f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dydx.$$

Example 38.2
Compute $\int_0^8\int_0^{16}\left(12-\frac{x}{4}-\frac{y}{8}\right)dydx$.

Solution.
We have

$$\begin{aligned}\int_0^8\int_0^{16} \left(12-\frac{x}{4}-\frac{y}{8}\right)dydx &= \int_0^8\left(\int_0^{16} \left(12-\frac{x}{4}-\frac{y}{8}\right)dy\right)dx\\ &= \int_0^8\left[12y-\frac{xy}{4}-\frac{y^2}{16}\right]_0^{16} dx\\ &= \int_0^8(176-4x)dx = 176x-2x^2\Big|_0^8 = 1280 \blacksquare\end{aligned}$$

Iterated Integrals Over Non-Rectangular Regions
So far we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\int_R f(x,y)dxdy$$

where R is any region. We consider the two types of regions shown in Figure 38.1.

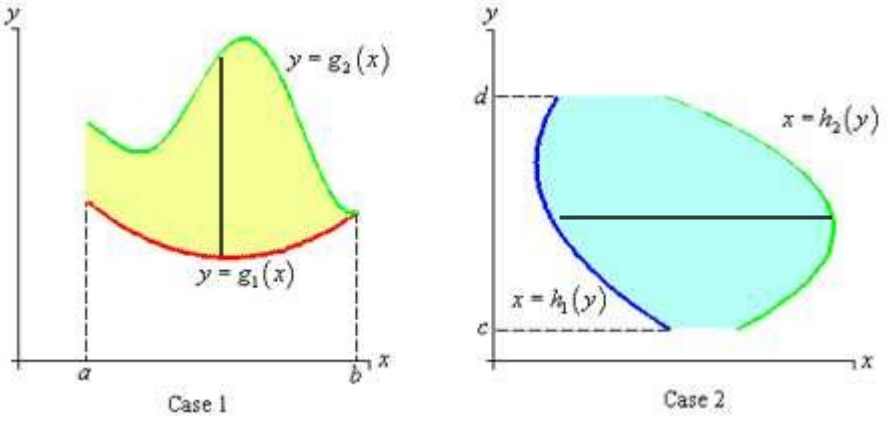


Figure 38.1

In Case 1, the iterated integral of f over R is defined by

$$\int_R f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

This means, that we are integrating using vertical strips from $g_1(x)$ to $g_2(x)$ and moving these strips from $x = a$ to $x = b$.
In case 2, we have

$$\int_R f(x,y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

so we use horizontal strips from $h_1(y)$ to $h_2(y)$. Note that in both cases, the limits on the outer integral must always be constants.

Remark 38.1
Chosing the order of integration will depend on the problem and is usually determined by the function being integrated and the shape of the region R. The order of integration which results in the "simplest" evaluation of the integrals is the one that is preferred.

Example 38.3
Let $f(x,y) = xy$. Integrate $f(x,y)$ for the triangular region bounded by the $x-$axis, the $y-$axis, and the line $y = 2 - 2x$.

Solution.
Figure 38.2 shows the region of integration for this example.

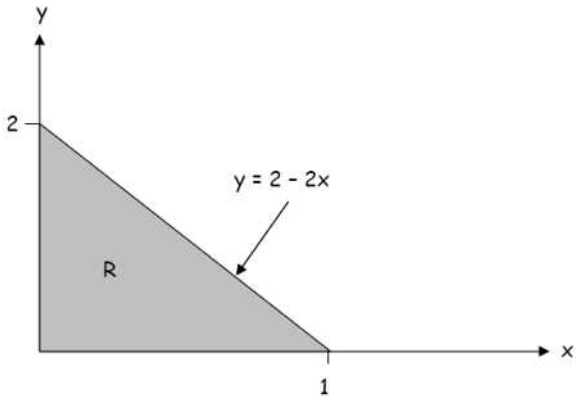


Figure 38.2

Graphically integrating over y first is equivalent to moving along the x axis from 0 to 1 and integrating from $y = 0$ to $y = 2 - 2x$. That is, summing up the vertical strips as shown in Figure 38.3(I).

$$\begin{aligned}\int_R xydxdy &= \int_0^1 \int_0^{2-2x} xydydx \\ &= \int_0^1 \frac{xy^2}{2}\Big|_0^{2-2x} dx = \frac{1}{2}\int_0^1 x(2-2x)^2 dx \\ &= 2\int_0^1 (x - 2x^2 + x^3)dx = 2\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{6}.\end{aligned}$$

If we choose to do the integral in the opposite order, then we need to invert the $y = 2 - 2x$ i.e. express x as function of y. In this case we get $x = 1 - \frac{1}{2}y$. Integrating in this order corresponds to integrating from $y = 0$ to $y = 2$ along horizontal strips ranging from $x = 0$ to $x = 1 - \frac{1}{2}y$, as shown in Figure

38.3(II)

$$\begin{aligned}\int_R xydxdy &= \int_0^2 \int_0^{1-\frac{1}{2}y} xydxdy \\ &= \int_0^2 \frac{x^2y}{2}\Big|_0^{1-\frac{1}{2}y} dy = \frac{1}{2}\int_0^2 y(1-\frac{1}{2}y)^2 dy \\ &= \frac{1}{2}\int_0^2 (y - y^2 + \frac{y^3}{4})dy = \frac{y^2}{4} - \frac{y^3}{6} + \frac{y^4}{32}\Big|_0^2 = \frac{1}{6} \blacksquare\end{aligned}$$

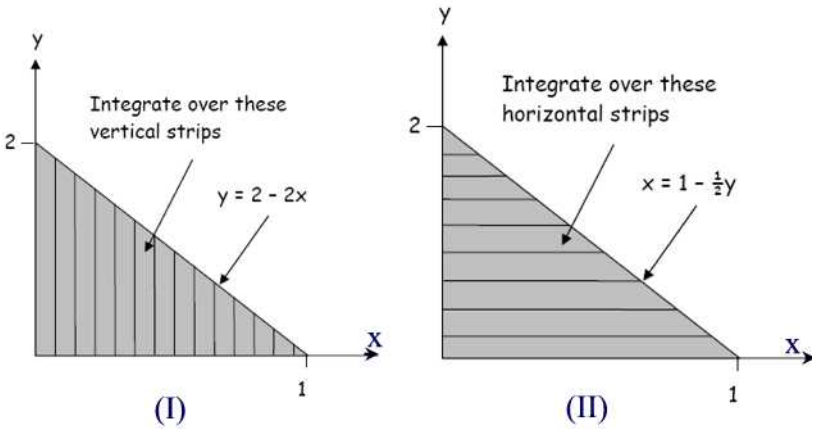


Figure 38.3

Example 38.4
Find $\int_R (4xy - y^3)dxdy$ where R is the region bounded by the curves $y = \sqrt{x}$ and $y = x^3$.

Solution.
A sketch of R is given in Figure 38.4. Using horizontal strips we can write

$$\begin{aligned}\int_R (4xy - y^3)dxdy &= \int_0^1 \int_{y^2}^{\sqrt[3]{y}} (4xy - y^3)dxdy \\ &= \int_0^1 2x^2y - xy^3\Big|_{y^2}^{\sqrt[3]{y}} dy = \int_0^1 \left(2y^{\frac{5}{3}} - y^{\frac{10}{3}} - y^5\right) dy \\ &= \frac{3}{4}y^{\frac{8}{3}} - \frac{3}{13}y^{\frac{13}{3}} - \frac{1}{6}y^6\Big|_0^1 = \frac{55}{156} \blacksquare\end{aligned}$$

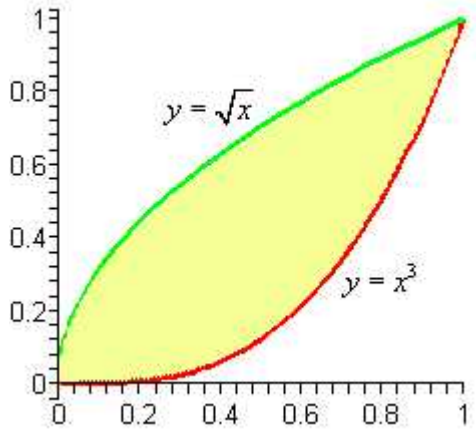


Figure 38.4

Example 38.5
Sketch the region of integration of $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xydydx$

Solution.
A sketch of the region is given in Figure 38.5.

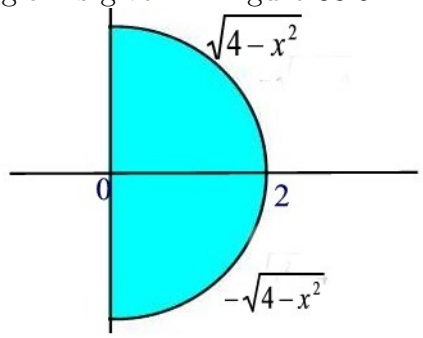


Figure 38.5

Practice Problems

Problem 38.1
Set up a double integral of $f(x, y)$ over the region given by $0 < x < 1; x < y < x + 1$.

Problem 38.2
Set up a double integral of $f(x, y)$ over the part of the unit square $0 \leq x \leq 1; 0 \leq y \leq 1$, on which $y \leq \frac{x}{2}$.

Problem 38.3
Set up a double integral of $f(x, y)$ over the part of the unit square on which both x and y are greater than 0.5.

Problem 38.4
Set up a double integral of $f(x, y)$ over the part of the unit square on which at least one of x and y is greater than 0.5.

Problem 38.5
Set up a double integral of $f(x, y)$ over the part of the region given by $0 < x < 50 - y < 50$ on which both x and y are greater than 20.

Problem 38.6
Set up a double integral of $f(x, y)$ over the set of all points (x, y) in the first quadrant with $|x - y| \leq 1$.

Problem 38.7
Evaluate $\int_R e^{-x-y} dxdy$, where R is the region in the first quadrant in which $x + y \leq 1$.

Problem 38.8
Evaluate $\int_R e^{-x-2y} dxdy$, where R is the region in the first quadrant in which $x \leq y$.

Problem 38.9
Evaluate $\int_R (x^2 + y^2) dxdy$, where R is the region $0 \leq x \leq y \leq L$.

Problem 38.10
Write as an iterated integral $\int_R f(x, y) dxdy$, where R is the region inside the unit square in which both coordinates x and y are greater than 0.5.

Problem 38.11
Evaluate $\int_R (x - y + 1)dxdy$, where R is the region inside the unit square in which $x + y \geq 0.5$.

Problem 38.12
Evaluate $\int_0^1 \int_0^1 x\max(x, y)dydx$.

Problem 38.13
Evaluate $\int_R \frac{1}{375}(20 - x - y)dxdy$ where $R = \{(x, y) : 1 \leq x \leq 2,\ 2 \leq y \leq 3\}$.

Problem 38.14
Evaluate $\int_R x^\alpha y^{1-\alpha}dxdy$ where $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

Problem 38.15
Evaluate $\int_R \frac{x+y}{8}dxdy$ where $R = \{(x, y) : 1 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Problem 38.16
Evaluate $\int_R \frac{2x+2-y}{4}dxdy$ where $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 2, x + y \geq 1\}$.

Problem 38.17
Evaluate $\int_R 6[1-(x+y)]dxdy$ where $R = \{(x, y) : 0 \leq x \leq 0.2,\ y \geq 0, x+y \leq 1\}$.

Problem 38.18
Evaluate $\int_R \frac{6}{125000}(50 - x - y)dxdy$ where $R = \{(x, y) : x, y \geq 20,\ 0 \leq x \leq 50 - y\}$.

Problem 38.19
Evaluate $\int_R (x + y)dxdy$ where $R = \{(x, y) : 0 \leq x, y \leq 1, x \geq \sqrt{y}\}$.

Problem 38.20
Evaluate $\int_R xydxdy$ where $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1,\ \frac{x}{2} \leq y \leq x\}$.

Problem 38.21
Evaluate $\int_R \frac{1}{800}e^{-\frac{x}{40}-\frac{y}{20}}dxdy$ where $R = \{(x, y) : x, y \geq 0,\ x + y \leq 60\}$.

Problem 38.22
Evaluate $\int_R e^{-(x+y)}dxdy$ where $R = \{(x, y) : x, y \geq 0, x \leq y\}$.

Problem 38.23
Evaluate $\int_R 6y(1-y)dxdy$ where $R = \{(x,y) : 0 \leq x \leq y \leq 1, x \leq \frac{3}{4}, y \geq \frac{1}{2}\}$.

Problem 38.24
Evaluate $\int_R 3xy^3dxdy$ where $R = \{(x,y) : 0 \leq x, y \leq 1, y \geq 2x\}$.

Problem 38.25
Evaluate $\int_R 6xy^2dxdy$ where $R = \{(x,y) : 0 \leq x, y \leq 1, x - \frac{1}{2} \leq y \leq x + \frac{1}{2}\}$.

Problem 38.26
Evaluate $\int_R \frac{8}{7}x^2y^{-3}dxdy$ where $R = \{(x,y) : 1 \leq x, y \leq 2, x \geq y\}$.

Problem 38.27
Evaluate $\int_R \frac{3x^2+2y}{24}dxdy$ where $R = \{(x,y) : 0 \leq x, y \leq 2, x + 2y \leq 3\}$.

Problem 38.28
Evaluate $\int_R \frac{4}{9}dxdy$ where $R = \{(x,y) : 0 \leq x \leq y \leq 3 - x, y \geq 2x\}$.

Problem 38.29
Evaluate $\int_R \frac{1}{6}e^{-(\frac{x}{2}+\frac{y}{3})}dxdy$ where $R = \{(x,y) : x, y \geq 0, y \leq x\}$.

Joint Distributions

There are many situations which involve the presence of several random variables and we are interested in their joint behavior. This chapter is concerned with the joint probability structure of two or more random variables defined on the same sample space.

39 Jointly Distributed Random Variables

Suppose that X and Y are two random variables defined on the same sample space S. The **joint cumulative distribution function** of X and Y is the function

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = P(\{e \in S : X(e) \leq x \text{ and } Y(e) \leq y\}).$$

Example 39.1
Consider the experiment of throwing a fair coin and a fair die simultaneously. The sample space is

$$S = \{(H,1),(H,2),\cdots,(H,6),(T,1),(T,2),\cdots,(T,6)\}.$$

Let X be the number of heads showing on the coin, $X \in \{0,1\}$. Let Y be the number showing on the die, $Y \in \{1,2,3,4,5,6\}$. Thus, if $e = (H,1)$ then $X(e) = 1$ and $Y(e) = 1$. Find $F_{XY}(1,2)$.

Solution.

$$\begin{aligned} F_{XY}(1,2) =& P(X \leq 1, Y \leq 2) \\ =& P(\{(H,1),(H,2),(T,1),(T,2)\}) \\ =& \frac{4}{12} = \frac{1}{3} \blacksquare \end{aligned}$$

In what follows, individual cdfs will be referred to as **marginal distributions**. These cdfs are obtained from the joint cumulative distribution as follows

$$\begin{aligned} F_X(x) =& P(X \le x) \\ =& P(X \le x, Y < \infty) \\ =& P\left(\bigcup_{-\infty<y<\infty} \{X \le x, Y \le y\} \right) \\ =& \lim_{y\to\infty} P(X \le x, Y \le y) \\ =& \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,\infty) \end{aligned}$$

where we used Proposition 25.1(a). In a similar way, one can show that

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y) = F_{XY}(\infty, y).$$

Next, we have

$$F_{XY}(\infty,\infty) = P(X < \infty, Y < \infty) = P(\{e \in S : X(e) < \infty, Y(e) < \infty\}) = P(S) = 1.$$

Moreover,

$$F_{XY}(-\infty, y) = 0.$$

This follows from

$$0 \le F_{XY}(-\infty, y) = P(X < -\infty, Y \le y) \le P(X < -\infty) = F_X(-\infty) = 0.$$

Likewise ,

$$F_{XY}(x, -\infty) = 0.$$

All joint probability statements about X and Y can be answered in terms of their joint distribution functions. For example,

$$\begin{aligned} P(X > x, Y > y) =& 1 - P(\{X > x, Y > y\}^c) \\ =& 1 - P(\{X > x\}^c \cup \{Y > y\}^c) \\ =& 1 - [P(\{X \le x\} \cup \{Y \le y\}) \\ =& 1 - [P(X \le x) + P(Y \le y) - P(X \le x, Y \le y)] \\ =& 1 - F_X(x) - F_Y(y) + F_{XY}(x,y). \end{aligned}$$

Also, if $a_1 < a_2$ and $b_1 < b_2$ then

$$\begin{aligned} P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =& P(X \leq a_2, Y \leq b_2) - P(X \leq a_2, Y \leq b_1) \\ & - P(X \leq a_1, Y \leq b_2) + P(X \leq a_1, Y \leq b_1) \\ =& F_{XY}(a_2, b_2) - F_{XY}(a_1, b_2) - F_{XY}(a_2, b_1) + F_{XY}(a_1, b_1). \end{aligned}$$

This is clear if you use the concept of area shown in Figure 39.1

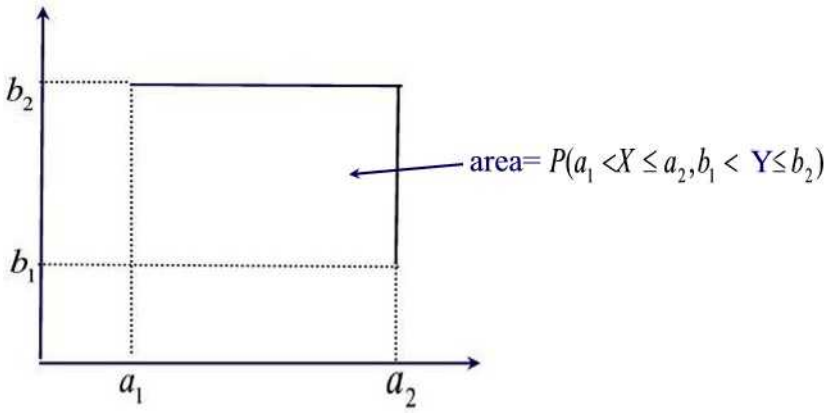


Figure 39.1

If X and Y are both discrete random variables, we define the **joint probability mass function** of X and Y by

$$p_{XY}(x, y) = P(X = x, Y = y).$$

The marginal probability mass function of X can be obtained from $p_{XY}(x, y)$ by

$$p_X(x) = P(X = x) = \sum_{y: p_{XY}(x,y) > 0} p_{XY}(x, y).$$

Similarly, we can obtain the marginal pmf of Y by

$$p_Y(y) = P(Y = y) = \sum_{x: p_{XY}(x,y) > 0} p_{XY}(x, y).$$

This simply means that to find the probability that X takes on a specific value we sum across the row associated with that value. To find the probability that Y takes on a specific value we sum the column associated with that value as illustrated in the next example.

Example 39.2
A fair coin is tossed 4 times. Let the random variable X denote the number of heads in the first 3 tosses, and let the random variable Y denote the number of heads in the last 3 tosses.
(a) What is the joint pmf of X and Y?
(b) What is the probability 2 or 3 heads appear in the first 3 tosses and 1 or 2 heads appear in the last three tosses?
(c) What is the joint cdf of X and Y?
(d) What is the probability less than 3 heads occur in both the first and last 3 tosses?
(e) Find the probability that one head appears in the first three tosses.

Solution.
(a) The joint pmf is given by the following table

$X\backslash Y$	0	1	2	3	$p_X(.)$
0	1/16	1/16	0	0	2/16
1	1/16	3/16	2/16	0	6/16
2	0	2/16	3/16	1/16	6/16
3	0	0	1/16	1/16	2/16
$p_Y(.)$	2/16	6/16	6/16	2/16	1

(b) $P((X,Y) \in \{(2,1),(2,2),(3,1),(3,2)\}) = P(2,1) + P(2,2) + P(3,1) + P(3,2) = \frac{3}{8}$.
(c) The joint cdf is given by the following table

$X\backslash Y$	0	1	2	3
0	1/16	2/16	2/16	2/16
1	2/16	6/16	8/16	8/16
2	2/16	8/16	13/16	14/16
3	2/16	8/16	14/16	1

(d) $P(X < 3, Y < 3) = F(2,2) = \frac{13}{16}$.
(e) $P(X = 1) = P((X,Y) \in \{(1,0),(1,1),(1,2),(1,3)\}) = 1/16 + 3/16 + 2/16 = \frac{3}{8}$ ■

Example 39.3
Suppose two balls are chosen from a box containing 3 white, 2 red and 5 blue balls. Let X = the number of white balls chosen and Y = the number of blue balls chosen. Find the joint pmf of X and Y.

Solution.

$$\begin{aligned}
p_{XY}(0,0) =& \frac{{}_2C_2}{{}_{10}C_2} = \frac{1}{45}\\
p_{XY}(0,1) =& \frac{{}_2C_1 \cdot {}_5C_1}{{}_{10}C_2} = \frac{10}{45}\\
p_{XY}(0,2) =& \frac{{}_5C_2}{{}_{10}C_2} = \frac{10}{45}\\
p_{XY}(1,0) =& \frac{{}_2C_1 \cdot {}_3C_1}{{}_{10}C_2} = \frac{6}{45}\\
p_{XY}(1,1) =& \frac{{}_5C_1 \cdot {}_3C_1}{{}_{10}C_2} = \frac{15}{45}\\
p_{XY}(1,2) =& 0\\
p_{XY}(2,0) =& \frac{{}_3C_2}{{}_{10}C_2} = \frac{3}{45}\\
p_{XY}(2,1) =& 0\\
p_{XY}(2,2) =& 0.
\end{aligned}$$

The pmf of X is

$$\begin{aligned}
p_X(0) =& P(X=0) = \sum_{y:p_{XY}(0,y)>0} p_{XY}(0,y) = \frac{1+10+10}{45} = \frac{21}{45}\\
p_X(1) =& P(X=1) = \sum_{y:p_{XY}(1,y)>0} p_{XY}(1,y) = \frac{6+15}{45} = \frac{21}{45}\\
p_X(2) =& P(X=2) = \sum_{y:p_{XY}(2,y)>0} p_{XY}(2,y) = \frac{3}{45} = \frac{3}{45}.
\end{aligned}$$

The pmf of Y is

$$\begin{aligned}
p_Y(0) =& P(Y=0) = \sum_{x:p_{XY}(x,0)>0} p_{XY}(x,0) = \frac{1+6+3}{45} = \frac{10}{45}\\
p_Y(1) =& P(Y=1) = \sum_{x:p_{XY}(x,1)>0} p_{XY}(x,1) = \frac{10+15}{45} = \frac{25}{45}\\
p_Y(2) =& P(Y=2) = \sum_{x:p_{XY}(x,2)>0} p_{XY}(x,2) = \frac{10}{45} = \frac{10}{45}
\end{aligned}$$ ■

Two random variables X and Y are said to be **jointly continuous** if there exists a function $f_{XY}(x,y) \geq 0$ with the property that for every subset C of $\mathbb{R}^2$ we have

$$P((X,Y) \in C) = \iint_{(x,y)\in C} f_{XY}(x,y)dA.$$

The function $f_{XY}(x,y)$ is called the **joint probability density function** of X and Y.
If A and B are any sets of real numbers then by letting $C = \{(x,y) : x \in A, y \in B\}$ we have

$$P(X \in A, Y \in B) = \int_B \int_A f_{XY}(x,y)dxdy.$$

As a result of this last equation we can write

$$\begin{aligned} F_{XY}(x,y) =& P(X \in (-\infty, x], Y \in (-\infty, y]) \\ =& \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u,v)dudv. \end{aligned}$$

It follows upon differentiation that

$$f_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y)$$

whenever the partial derivatives exist.

Example 39.4
The cumulative distribution function for the joint distribution of the continuous random variables X and Y is $F_{XY}(x,y) = 0.2(3x^3y + 2x^2y^2),\ 0 \leq x \leq 1, 0 \leq y \leq 1$. Find $f_{XY}(\frac{1}{2}, \frac{1}{2})$.

Solution.
Since

$$f_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = 0.2(9x^2 + 8xy)$$

we find $f_{XY}(\frac{1}{2}, \frac{1}{2}) = \frac{17}{20}$ ■

Now, if X and Y are jointly continuous then they are individually continuous, and their probability density functions can be obtained as follows:

$$\begin{aligned} P(X \in A) =& P(X \in A, Y \in (-\infty, \infty)) \\ &= \int_A \int_{-\infty}^{\infty} f_{XY}(x,y) dy dx \\ &= \int_A f_X(x) dx \end{aligned}$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy$$

is thus the probability density function of X. Similarly, the probability density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) dx.$$

Example 39.5
Let X and Y be random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4} & -1 \leq x, y \leq 1 \\ 0 & \text{Otherwise.} \end{cases}$$

Determine
(a) $P(X^2 + Y^2 < 1)$,
(b) $P(2X - Y > 0)$,
(c) $P(|X + Y| < 2)$.

Solution.
(a)

$$P(X^2 + Y^2 < 1) = \int_0^{2\pi} \int_0^1 \frac{1}{4} r dr d\theta = \frac{\pi}{4}.$$

(b)

$$P(2X - Y > 0) = \int_{-1}^1 \int_{\frac{y}{2}}^1 \frac{1}{4} dx dy = \frac{1}{2}.$$

Note that $P(2X - Y > 0)$ is the area of the region bounded by the lines $y = 2x, x = 1, y = -1$ and $y = 1$. A graph of this region will help you

understand the integration process used above.
(c) Since the square with vertices $(1,1),(1,-1),(-1,1),(-1,-1)$ is completely contained in the region $-2 < x+y < 2$, we have

$$P(|X+Y| < 2) = 1 \blacksquare$$

Remark 39.1
An important remark to make in here. Note that the area of the region $-1 \leq x \leq 1, \ -1 \leq y \leq 1$ is 4 and $f_{XY}(x,y) = \frac{1}{4}$. In this case, we say that X and Y have a **uniformly joint distribution** in the given region.

Remark 39.2
Joint pdfs and joint cdfs for three or more random variables are obtained as straightforward generalizations of the above definitions and conditions.

Practice Problems

Problem 39.1
A security check at an airport has two express lines. Let X and Y denote the number of customers in the first and second line at any given time. The joint probability function of X and $Y, p_{XY}(x, y)$, is summarized by the following table

$X\backslash Y$	0	1	2	3	$p_X(.)$
0	0.1	0.2	0	0	0.3
1	0.2	0.25	0.05	0	0.5
2	0	0.05	0.05	0.025	0.125
3	0	0	0.025	0.05	0.075
$p_Y(.)$	0.3	0.5	0.125	0.075	1

(a) Show that $p_{XY}(x, y)$ is a joint probability mass function.
(b) Find the probability that more than two customers are in line.
(c) Find $P(|X - Y| = 1)$.
(d) Find $p_X(x)$.

Problem 39.2
Given:

$X\backslash Y$	1	2	3	$p_X(.)$
1	0.1	0.05	0.02	0.17
2	0.1	0.35	0.05	0.50
3	0.03	0.1	0.2	0.33
$p_Y(.)$	0.23	0.50	0.27	1

Find $P(X \geq 2, Y \geq 3)$.

Problem 39.3
Given:

$X\backslash Y$	0	1	2	$p_X(.)$
0	0.4	0.12	0.08	0.6
1	0.15	0.08	0.03	0.26
2	0.1	0.03	0.01	0.14
$p_Y(.)$	0.65	0.23	0.12	1

Find the following: (a) $P(X = 0, Y = 2)$.
(b) $P(X > 0, Y \leq 1)$.
(c) $P(X \leq 1)$.
(d) $P(Y > 0)$.
(e) $P(X = 0)$.
(f) $P(Y = 0)$.
(g) $P(X = 0, Y = 0)$.

Problem 39.4
Given:

$X \backslash Y$	15	16	$p_X(.)$
129	0.12	0.08	0.2
130	0.4	0.30	0.7
131	0.06	0.04	0.1
$p_Y(.)$	0.58	0.42	1

(a) Find $P(X = 130, Y = 15)$.
(b) Find $P(X \geq 130, Y \geq 15)$.

Problem 39.5
Suppose the random variables X and Y have a joint pdf

$$f_{XY}(x, y) = \begin{cases} \frac{20-x-y}{375} & 0 \leq x, y \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(1 \leq X \leq 2, 2 \leq Y \leq 3)$.

Problem 39.6
Assume the joint pdf of X and Y is

$$f_{XY}(x, y) = \begin{cases} xye^{-\frac{x^2+y^2}{2}} & 0 < x, y \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $F_{XY}(x, y)$.
(b) Find $f_X(x)$ and $f_Y(y)$.

Problem 39.7
Show that the following function is not a joint probability density function?

$$f_{XY}(x, y) = \begin{cases} x^a y^{1-a} & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < a < 1$. What factor should you multiply $f_{XY}(x, y)$ to make it a joint probability density function?

Problem 39.8 ‡
A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f_{XY}(x,y) = \begin{cases} \frac{x+y}{8} & 0 < x, y < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the device fails during its first hour of operation?

Problem 39.9 ‡
An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{2x+2-y}{4} & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the total loss is at least 1 ?

Problem 39.10 ‡
A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold. Given the following information

$$\begin{aligned} P(X = 0, Y = 0) &= \frac{1}{6} \\ P(X = 1, Y = 0) &= \frac{1}{12} \\ P(X = 1, Y = 1) &= \frac{1}{6} \\ P(X = 2, Y = 0) &= \frac{1}{12} \\ P(X = 2, Y = 1) &= \frac{1}{3} \\ P(X = 2, Y = 2) &= \frac{1}{6} \end{aligned}$$

What is the variance of X?

Problem 39.11 ‡
A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f_{XY}(x,y) = \begin{cases} 6[1-(x+y)] & x>0, y>0, x+y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

Problem 39.12 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 15y & x^2 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal density function of Y.

Problem 39.13 ‡
Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{64}(10-xy^2) & 2 \le x \le 10, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

Problem 39.14 ‡
A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails.
Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f_{XY}(x,y) = \begin{cases} 6e^{-x}e^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time at which the device fails?

Problem 39.15 ‡
The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f_{XY}(x,y) = \begin{cases} \frac{6}{125000}(50-x-y) & 0 < x < 50-y < 50 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both components are still functioning 20 months from now?

Problem 39.16
Suppose the random variables X and Y have a joint pdf

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X > \sqrt{Y})$.

Problem 39.17 ‡
Let X and Y be random losses with joint density function

$$f_{XY}(x,y) = e^{-(x+y)}, \quad x > 0, y > 0$$

and 0 otherwise. An insurance policy is written to reimburse $X + Y$. Calculate the probability that the reimbursement is less than 1.

Problem 39.18
Let X and Y be continuous random variables with joint cumulative distribution $F_{XY}(x,y) = \frac{1}{250}(20xy - x^2y - xy^2)$ for $0 \le x \le 5$ and $0 \le y \le 5$. Compute $P(X > 2)$.

Problem 39.19
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} xy & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(\frac{X}{2} \le Y \le X)$.

Problem 39.20
Let X and Y be random variables with common range $\{1, 2\}$ and such that $P(X = 1) = 0.7, P(X = 2) = 0.3, P(Y = 1) = 0.4, P(Y = 2) = 0.6$, and $P(X = 1, Y = 1) = 0.2$.
(a) Find the joint probability mass function $p_{XY}(x, y)$.
(b) Find the joint cumulative distribution function $F_{XY}(x, y)$.

Problem 39.21 ‡
A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s, t)$, where $0 < s < 1$ and $0 < t < 1$.
What is the probability that the device fails during the first half hour of operation?

Problem 39.22 ‡
A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent. The joint density function of X and Y can be modeled by

$$f(x, y) = \begin{cases} \frac{1}{800} e^{-\frac{x}{40} - \frac{y}{20}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that a client spends less than 60 minutes at the agent's office. You do NOT have to evaluate the integrals.

Problem 39.23 *(Trinomial Distribution)*‡
The trinomial distribution arises from an extension of the binomial experiment to situations where each trial has three different outcomes O_1, O_2, and O_3 of probability p_1, p_2, and p_3 respectively, where $p_1 + p_2 + p_3 = 1$. The trials are assumed to be independent. Let X_i be the number of trials with outcome O_i for $i = 1, 2, 3$. The trinomial mass function of X_1, X_2, X_3 is defined by

$$f(x_1, x_2, x_3) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where n is the number of trials.
A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool.

This month, the insurance company writes four new policies for adults earning their first driver's license.
Calculate the probability that these four will contain at least two more high-risk drivers than low-risk drivers.

Problem 39.24 ‡
The random variables X and Y have joint probability function $p(x, y)$ for $x = 0, 1$ and $y = 0, 1, 2$. Suppose $3p(1, 1) = p(1, 2)$, and $p(1, 1)$ maximizes the variance of XY. Calculate the probability that X or Y is 0.

Problem 39.25 ‡
Let X denote the proportion of employees at a large firm who will choose to be covered under the firm's medical plan, and let Y denote the proportion who will choose to be covered under both the firm's medical and dental plans.
Suppose that for $0 \leq y \leq x \leq 1$, X and Y have the joint cumulative distribution function

$$F_{XY}(x, y) = y(x^2 + xy - y^2).$$

Calculate the expected proportion of employees who will choose to be covered under both plans.

Problem 39.26 ‡
An insurance company sells automobile liability and collision insurance. Let X denote the percentage of liability policies that will be renewed at the end of their terms and Y the percentage of collision policies that will be renewed at the end of their terms. X and Y have the joint cumulative distribution function

$$F_{XY}(x, y) = \frac{xy(x + y)}{2,000,000}, \quad 0 \leq x \leq 100, \ 0 \leq y \leq 100.$$

Calculate $\text{Var}(X)$.

Problem 39.27 ‡
A hurricane policy covers both water damage, X, and wind damage, Y, where X and Y have joint density function

$$f_{XY}(x, y) = \begin{cases} 0.13e^{-0.5x-0.2y} - 0.06e^{-x-0.2y} - 0.06e^{-0.5x-0.4y} + 0.12e^{-x-0.4y}, & x > 0, y > 0, \\ 0, \text{otherwise.} \end{cases}$$

Calculate the standard deviation of X.

Problem 39.28 ‡
Random variables X and Y have joint distribution

	$X=0$	$X=1$	$X=2$
$Y=0$	1/15	a	2/15
$Y=1$	a	b	a
$Y=2$	2/15	a	1/15

Let a be the value that minimizes the variance of X. Calculate the variance of Y.

Problem 39.29 ‡
Batteries A and B have lifetimes that are independent and exponentially distributed with a common mean of m years. The probability that battery B outlasts battery A by more than one year is 0.33. Calculate m.

Problem 39.30 ‡
The number of minor surgeries, X, and the number of major surgeries, Y, for a policyholder, this decade, has joint cumulative distribution function

$$F(x,y) = [1-(0.5)^{x+1}][1-(0.2)^{y+1}]$$

for non-negative integers x and y. Calculate the probability that the policyholder experiences exactly three minor surgeries and exactly three major surgeries this decade.

Problem 39.31 ‡
The table below shows the joint probability function of a sailor's number of boating accidents X and number of hospitalizations from these accidents Y for this year.

$Y\backslash X$	0	1	2	3
0	0.70	0	0	0
1	0.15	0.05	0	0
2	0.06	0.02	0.01	0
3	0.005	0.002	0.002	0.001

Calculate the sailor's expected number of hospitalizations from boating accidents this year.

40 Independent Random Variables

Let X and Y be two random variables defined on the same sample space S. We say that X and Y are **independent** random variables if and only if for any two sets of real numbers A and B we have

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B). \tag{40.1}$$

That is, the events $E = \{X \in A\}$ and $F = \{Y \in B\}$ are independent.
The following theorem expresses independence in terms of pdfs.

Theorem 40.1
If X and Y are discrete random variables, then X and Y are independent if and only if

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

where $p_X(x)$ and $p_Y(y)$ are the marginal pmfs of X and Y respectively. Similar result holds for continuous random variables where sums are replaced by integrals and pmfs are replaced by pdfs.

Proof.
Suppose that X and Y are independent. Then by letting $A = \{x\}$ and $B = \{y\}$ in Equation 40.1 we obtain

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

that is

$$p_{XY}(x, y) = p_X(x)p_Y(y).$$

Conversely, suppose that $p_{XY}(x, y) = p_X(x)p_Y(y)$. Let A and B be any sets of real numbers. Then

$$\begin{aligned} P(X \in A, Y \in B) &= \sum_{y \in B} \sum_{x \in A} p_{XY}(x, y) \\ &= \sum_{y \in B} \sum_{x \in A} p_X(x)p_Y(y) \\ &= \sum_{y \in B} p_Y(y) \sum_{x \in A} p_X(x) \\ &= P(Y \in B)P(X \in A) \end{aligned}$$

and thus equation 40.1 is satisfied. That is, X and Y are independent ■

Example 40.1
A month of the year is chosen at random (each with probability $\frac{1}{12}$). Let X be the number of letters in the month's name, and let Y be the number of days in the month (ignoring leap year).
(a) Write down the joint pdf of X and Y. From this, compute the pdf of X and the pdf of Y.
(b) Find $E(Y)$.
(c) Are the events "$X \leq 6$" and "$Y = 30$" independent?
(d) Are X and Y independent random variables?

Solution.
(a) The joint pdf is given by the following table

Y\ X	3	4	5	6	7	8	9	$p_Y(y)$
28	0	0	0	0	0	$\frac{1}{12}$	0	$\frac{1}{12}$
30	0	$\frac{1}{12}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$
31	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	0	$\frac{7}{12}$
$p_X(x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	1

(b) $E(Y) = \left(\frac{1}{12}\right) \times 28 + \left(\frac{4}{12}\right) \times 30 + \left(\frac{7}{12}\right) \times 31 = \frac{365}{12}$
(c) We have $P(X \leq 6) = \frac{6}{12} = \frac{1}{2}$, $P(Y = 30) = \frac{4}{12} = \frac{1}{3}$, $P(X \leq 6, Y = 30) = \frac{2}{12} = \frac{1}{6}$. Since, $P(X \leq 6, Y = 30) = P(X \leq 6)P(Y = 30)$, the two events are independent.
(d) Since $p_{XY}(5, 28) = 0 \neq p_X(5)p_Y(28) = \frac{1}{6} \times \frac{1}{12}$, X and Y are dependent ■

In the jointly continuous case the condition of independence is equivalent to

$$f_{XY}(x, y) = f_X(x)f_Y(y).$$

It follows from the previous theorem, that if you are given the joint pdf of the random variables X and Y, you can determine whether or not they are independent by calculating the marginal pdfs of X and Y and determining whether or not the relationship $f_{XY}(x, y) = f_X(x)f_Y(y)$ holds.

Example 40.2
The joint pdf of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} 4e^{-2(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{Otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
Marginal density $f_X(x)$ is given by

$$f_X(x) = \int_0^\infty 4e^{-2(x+y)}dy = 2e^{-2x}\int_0^\infty 2e^{-2y}dy = 2e^{-2x},\ x > 0$$

and 0 otherwise. Similarly, the marginal density $f_Y(y)$ is given by

$$f_Y(y) = \int_0^\infty 4e^{-2(x+y)}dx = 2e^{-2y}\int_0^\infty 2e^{-2x}dx = 2e^{-2y},\ y > 0.$$

and 0 otherwise. Now since

$$f_{XY}(x,y) = 4e^{-2(x+y)} = [2e^{-2x}][2e^{-2y}] = f_X(x)f_Y(y)$$

X and Y are independent ■

Example 40.3
The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 3(x+y) & 0 \le x+y \le 1,\ 0 \le x,y < \infty \\ 0 & \text{Otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
For the limit of integration see Figure 40.1 below.

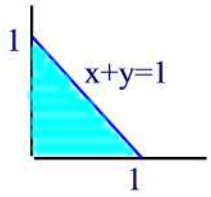


Figure 40.1

The marginal pdf of X is

$$f_X(x) = \int_0^{1-x} 3(x+y)dy = 3xy + \frac{3}{2}y^2\Big|_0^{1-x} = \frac{3}{2}(1-x^2),\ 0 \le x \le 1$$

and 0 otherwise. The marginal pdf of Y is

$$f_Y(y) = \int_0^{1-y} 3(x+y)dx = \frac{3}{2}x^2 + 3xy\Big|_0^{1-y} = \frac{3}{2}(1-y^2),\ 0 \le y \le 1$$

and 0 otherwise. Since

$$f_{XY}(x,y) = 3(x+y) \neq \frac{3}{2}(1-x^2)\frac{3}{2}(1-y^2) = f_X(x)f_Y(y)$$

X and Y are dependent ■

The following theorem provides a necessary and sufficient condition for two random variables to be independent.

Theorem 40.2
Two continuous random variables X and Y are independent if and only if their joint probability density function can be expressed as

$$f_{XY}(x,y) = h(x)g(y), \quad -\infty < x < \infty, -\infty < y < \infty.$$

The same result holds for discrete random variables.

Proof.
Suppose first that X and Y are independent. Then $f_{XY}(x,y) = f_X(x)f_Y(y)$. Let $h(x) = f_X(x)$ and $g(y) = f_Y(y)$.
Conversely, suppose that $f_{XY}(x,y) = h(x)g(y)$. Let $C = \int_{-\infty}^{\infty} h(x)dx$ and $D = \int_{-\infty}^{\infty} g(y)dy$. Then

$$\begin{aligned} CD &= \left(\int_{-\infty}^{\infty} h(x)dx\right)\left(\int_{-\infty}^{\infty} g(y)dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1. \end{aligned}$$

Furthermore,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} h(x)g(y)dy = h(x)D$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} h(x)g(y)dx = g(y)C.$$

Hence,

$$f_X(x)f_Y(y) = h(x)g(y)CD = h(x)g(y) = f_{XY}(x,y).$$

This proves that X and Y are independent ■

Example 40.4
The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} xye^{-\frac{(x^2+y^2)}{2}} & 0 \le x, y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
We have

$$f_{XY}(x,y) = xye^{-\frac{(x^2+y^2)}{2}} = xe^{-\frac{x^2}{2}}ye^{-\frac{y^2}{2}}.$$

By the previous theorem, X and Y are independent ■

Example 40.5
The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

Solution.
Let

$$I(x,y) = \begin{cases} 1 & 0 \le x < 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f_{XY}(x,y) = (x+y)I(x,y)$$

which clearly does not factor into a part depending only on x and another depending only on y. Thus, by the previous theorem X and Y are dependent ■

Example 40.6 *(Order statistics)*
Let X and Y be two independent random variables with X having a normal distribution with mean μ and variance 2 and Y being the standard normal distribution.
(a) Find the density of $Z = \min\{X, Y\}$.
(b) Assume that $X - Y$ is normal with mean μ and variance 1. Calculate $P(\max(X,Y) - \min(X,Y) > t)$ for $t \in \mathbb{R}$.

Solution.
(a) Fix a real number z. Then

$$\begin{aligned} F_Z(z) =& P(Z \leq z) = 1 - P(\min(X, Y) > z) \\ =& 1 - P(X > z)P(Y > z) = 1 - (1 - \Phi(z - \mu))(1 - \Phi(z)). \end{aligned}$$

Hence,

$$f_Z(z) = (1 - \Phi(z - \mu))\phi(z) + (1 - \Phi(z))\phi(z - \mu)$$

where $\phi(z)$ is the pdf of the standard normal distribution.
(b) First, note that $\max(X, Y) - \min(X, Y)$. Thus, for $t \leq 0$ then $P(\max(X, Y) - \min(X, Y) > t) = 1$. If $t > 0$ then

$$\begin{aligned} P(\max(X, Y) - \min(X, Y) > t) =& P(|X - Y| > t) \\ =& 1 - \Phi\left(\frac{t - \mu}{\sqrt{2}}\right) + \Phi\left(\frac{-t - \mu}{\sqrt{2}}\right) \blacksquare \end{aligned}$$

Example 40.7 *(Order statistics)*
Suppose $X_1, \cdots, X_n$ are independent and identically distributed[9] random variables with cdf $F_X(x)$. Define U and L as

$$\begin{aligned} U =& \max\{X_1, X_2, \cdots, X_n\} \\ L =& \min\{X_1, X_2, \cdots, X_n\}. \end{aligned}$$

(a) Find the cdf of U.
(b) Find the cdf of L.
(c) Are U and L independent?

Solution.
(a) First note the following equivalence of events

$$\{U \leq u\} \Leftrightarrow \{X_1 \leq u, X_2 \leq u, \cdots, X_n \leq u\}.$$

Thus,

$$\begin{aligned} F_U(u) =& P(U \leq u) = P(X_1 \leq u, X_2 \leq u, \cdots, X_n \leq u) \\ =& P(X_1 \leq u)P(X_2 \leq u) \cdots P(X_n \leq u) = (F_X(u))^n. \end{aligned}$$

[9] $f_{X_1}(x) = f_{X_2}(x) = \cdots = f_{X_n}(x) = f_X(x)$.

(b) Note the following equivalence of events

$$\{L > \ell\} \Leftrightarrow \{X_1 > \ell, X_2 > \ell, \cdots, X_n > \ell\}.$$

Thus,

$$\begin{aligned} F_L(\ell) =& P(L \leq \ell) = 1 - P(L > \ell) \\ =& 1 - P(X_1 > \ell, X_2 > \ell, \cdots, X_n > \ell) \\ =& 1 - P(X_1 > \ell)P(X_2 > \ell) \cdots P(X_n > \ell) \\ =& 1 - (1 - F_X(\ell))^n. \end{aligned}$$

(c) No. First note that $P(L > \ell) = 1 - F_L(\ell)$. From the definition of cdf there must be a number ℓ_0 such that $F_L(\ell_0) \neq 1$. Thus, $P(L > \ell_0) \neq 0$. But $P(L > \ell_0 | U \leq u) = 0$ for any $u < \ell_0$. This shows that $P(L > \ell_0 | U \leq u) \neq P(L > \ell_0)$ ■

Remark 40.1
L defined in the previous example is referred to as the **first order statistics**. U is referred to as the n^{th} **order statistics**.

Practice Problems

Problem 40.1
Let X and Y be random variables with joint pdf given by

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & 0 \le x, y \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?
(b) Find $P(X < Y)$.
(c) Find $P(X < a)$.

Problem 40.2
The random vector (X, Y) is said to be **uniformly distributed** over a region R in the plane if, for some constant c, its joint pdf is

$$f_{XY}(x,y) = \begin{cases} c & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $c = \frac{1}{A(R)}$ where $A(R)$ is the area of the region R.
(b) Suppose that $R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$. Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.
(c) Find $P(X^2 + Y^2 \le 1)$.

Problem 40.3
Let X and Y be random variables with joint pdf given by

$$f_{XY}(x,y) = \begin{cases} 6(1-y) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \le \frac{3}{4}, Y \ge \frac{1}{2})$.
(b) Find $f_X(x)$ and $f_Y(y)$.
(c) Are X and Y independent?

Problem 40.4
Let X and Y have the joint pdf given by

$$f_{XY}(x,y) = \begin{cases} kxy & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find k.
(b) Find $f_X(x)$ and $f_Y(y)$.
(c) Are X and Y independent?

Problem 40.5
Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} kxy^2 & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find k.
(b) Compute the marginal densities of X and of Y.
(c) Compute $P(Y > 2X)$.
(d) Compute $P(|X - Y| < 0.5)$.
(e) Are X and Y independent?

Problem 40.6
Suppose the joint density of random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} kx^2y^{-3} & 1 \le x, y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find k.
(b) Are X and Y independent?
(c) Find $P(X > Y)$.

Problem 40.7
Let X and Y be continuous random variables, with the joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{3x^2+2y}{24} & 0 \le x, y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $f_X(x)$ and $f_Y(y)$.
(b) Are X and Y independent?
(c) Find $P(X + 2Y < 3)$.

Problem 40.8
Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} \frac{4}{9} & x \le y \le 3 - x, 0 \le x \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the marginal densities of X and Y.
(b) Compute $P(Y > 2X)$.
(c) Are X and Y independent?

Problem 40.9 ‡
A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants).
What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

Problem 40.10 ‡
The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively.
What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

Problem 40.11 ‡
An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days.
What is the probability that the next claim will be a Deluxe Policy claim?

Problem 40.12 ‡
Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200 . The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200.
Determine the probability that the company considers the two bids further.

Problem 40.13 ‡
A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy has a deductible of 1 and the other has a deductible of 2. The family experiences exactly one loss under each policy.
Calculate the probability that the total benefit paid to the family does not exceed 5.

Problem 40.14 ‡
In a small metropolitan area, annual losses due to storm, fire, and theft are assumed to be independent, exponentially distributed random variables with respective means 1.0, 1.5, and 2.4 .
Determine the probability that the maximum of these losses exceeds 3.

Problem 40.15 ‡
A device containing two key components fails when, and only when, both components fail. The lifetimes, X and Y, of these components are independent with common density function $f(t) = e^{-t}, t > 0$. The cost, Z, of operating the device until failure is $2X + Y$.
Find the probability density function of Z.

Problem 40.16 ‡
A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums.
What is the density function of X?

Problem 40.17
Let X and Y be independent continuous random variables with common density function

$$f_X(x) = f_Y(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is $P(X^2 \geq Y^3)$?

Problem 40.18
Let X and Y be two discrete random variables with joint distribution given by the following table.

Y\ X	1	5
2	$\theta_1 + \theta_2$	$\theta_1 + 2\theta_2$
4	$\theta_1 + 2\theta_2$	$\theta_1 + \theta_2$

We assume that $-0.25 \leq \theta_1 \leq 2.5$ and $0 \leq \theta_2 \leq 0.35$. Find θ_1 and θ_2 when X and Y are independent.

Problem 40.19
Let X_1, X_2, X_3 be three independent, identically distributed random variables each with density function

$$f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \max\{X_1, X_2, X_3\}$. Find $P(Y > \frac{1}{2})$.

Problem 40.20 ‡
Automobile policies are separated into two groups: low-risk and high-risk. Actuary Rahul examines low-risk policies, continuing until a policy with a claim is found and then stopping. Actuary Toby follows the same procedure with high-risk policies. Each low-risk policy has a 10% probability of having a claim. Each high-risk policy has a 20% probability of having a claim. The claim statuses of policies are mutually independent.
Calculate the probability that Actuary Rahul examines fewer policies than Actuary Toby.

Problem 40.21 ‡
Losses follow an exponential distribution with mean 1. Two independent losses are observed.
Calculate the expected value of the smaller loss.

Problem 40.22 ‡
Claim amounts for wind damage to insured homes are mutually independent random variables with common density function

$$f(x) = \begin{cases} \frac{3}{x^4} & x > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where x is the amount of a claim in thousands. Suppose 3 such claims will be made. Calculate the expected value of the largest of the three claims.

Problem 40.23 ‡
Skateboarders A and B practice one difficult stunt until becoming injured while attempting the stunt. On each attempt, the probability of becoming injured is p, independent of the outcomes of all previous attempts.
Let $F(x, y)$ represent the probability that skateboarders A and B make no more than x and y attempts, respectively, where x and y are positive integers. It is given that $F(2, 2) = 0.0441$. Calculate $F(1, 5)$.

Problem 40.24 ‡

A couple takes out a medical insurance policy that reimburses them for days of work missed due to illness. Let X and Y denote the number of days missed during a given month by the wife and husband, respectively. The policy pays a monthly benefit of 50 times the maximum of X and Y, subject to a benefit limit of 100. X and Y are independent, each with a discrete uniform distribution on the set $\{0, 1, 2, 3, 4\}$. Calculate the expected monthly benefit for missed days of work that is paid to the couple.

41 Sum of Two Independent Random Variables: Discrete Case

In this section we turn to the important question of determining the distribution of a sum of independent random variables in terms of the distributions of the individual constituents. In this section, we consider only sums of discrete random variables, reserving the case of continuous random variables for the next section. We consider here only discrete random variables whose values are non-negative integers. Their distribution mass functions are then defined on these integers.

Suppose X and Y are two independent discrete random variables with pmf $p_X(x)$ and $p_Y(y)$ respectively. We would like to determine the pmf of the random variable $X + Y$. To do this, we note first that for any non-negative integer n we have

$$\{X + Y = n\} = \bigcup_{k=0}^{n} A_k$$

where $A_k = \{X = k\} \cap \{Y = n - k\}$. Note that $A_i \cap A_j = \emptyset$ for $i \neq j$. Since the A_i's are pairwise disjoint and X and Y are independent, we have

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k).$$

Thus,

$$p_{X+Y}(n) = p_X(n) * p_Y(n)$$

where $p_X(n) * p_Y(n)$ is called the **convolution** of p_X and p_Y.

Example 41.1
A die is rolled twice. Let X and Y be the outcomes, and let $Z = X + Y$ be the sum of these outcomes. Find the probability mass function of Z.

Solution. Note that X and Y have the common pmf :

x	1	2	3	4	5	6
p_X	1/6	1/6	1/6	1/6	1/6	1/6

The probability mass function of Z is then the convolution of p_X with itself. Thus,

$$\begin{aligned}
P(Z=2) =& p_X(1)p_X(1) = \frac{1}{36} \\
P(Z=3) =& p_X(1)p_X(2) + p_X(2)p_X(1) = \frac{2}{36} \\
P(Z=4) =& p_X(1)p_X(3) + p_X(2)p_X(2) + p_X(3)p_X(1) = \frac{3}{36}
\end{aligned}$$

Continuing in this way we would find $P(Z=5) = 4/36, P(Z=6) = 5/36, P(Z=7) = 6/36, P(Z=8) = 5/36, P(Z=9) = 4/36, P(Z=10) = 3/36, P(Z=11) = 2/36$, and $P(Z=12) = 1/36$ ■

Example 41.2
Let X and Y be two independent Poisson random variables with respective parameters λ_1 and λ_2. Compute the pmf of $X+Y$.

Solution.
We have

$$\begin{aligned}
p_{X+Y}(n) =& P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k) \\
=& \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\
=& \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\
=& e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!} \\
=& \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} \\
=& \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n.
\end{aligned}$$

Thus, $X+Y$ is a Poisson random variable with parameter $\lambda_1+\lambda_2$ ■

Example 41.3
Let X and Y be two independent binomial random variables with respective parameters (n,p) and (m,p). Compute the pmf of $X+Y$.

Solution.
X represents the number of successes in n independent trials, each of which results in a success with probability p; similarly, Y represents the number of successes in m independent trials, each of which results in a success with probability p. Hence, as X and Y are assumed to be independent, it follows that $X+Y$ represents the number of successes in $n+m$ independent trials, each of which results in a success with probability p. So $X+Y$ is a binomial random variable with parameters $(n+m,p)$ ■

Example 41.4
Two biased coins are being flipped repeatedly. The probability that coin 1 comes up heads is $\frac{1}{4}$, while that of coin 2 is $\frac{3}{4}$. Each coin is being flipped until a head comes up. What is the pmf of the total number of flips until both coins come up heads?

Solution.
Let X and Y be the number of flips of coins 1 and 2 to come up heads for the first time. Then, $X+Y$ is the total number of flips until both coins come up heads for the first time. The random variables X and Y are independent geometric random variables with parameters 1/4 and 3/4, respectively. By convolution, we have

$$\begin{aligned} p_{X+Y}(n) =& \sum_{k=1}^{n-1} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \frac{3}{4}\left(\frac{1}{4}\right)^{n-k-1} \\ =& \frac{1}{4^n}\sum_{k=1}^{n-1} 3^k = \frac{3^n-3}{2^{2n+1}} \blacksquare \end{aligned}$$

Practice Problems

Problem 41.1
Let X and Y be two independent discrete random variables with probability mass functions defined in the tables below. Find the probability mass function of $Z = X + Y$.

x	0	1	2	3
$p_X(x)$	0.10	0.20	0.30	0.40

y	0	1	2
$p_Y(y)$	0.25	0.40	0.35

Problem 41.2
Suppose X and Y are two independent binomial random variables with respective parameters $(20, 0.2)$ and $(10, 0.2)$. Find the pmf of $X + Y$.

Problem 41.3
Let X and Y be independent random variables each geometrically distributed with parameter p, i.e.

$$p_X(n) = p_Y(n) = \begin{cases} p(1-p)^{n-1} & n = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability mass function of $X + Y$.

Problem 41.4
Consider the following two experiments: the first has outcome X taking on the values 0, 1, and 2 with equal probabilities; the second results in an (independent) outcome Y taking on the value 3 with probability 1/4 and 4 with probability 3/4. Find the probability mass function of $X + Y$.

Problem 41.5 ‡
An insurance company determines that N, the number of claims received in a week, is a random variable with $P[N = n] = \frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week.
Determine the probability that exactly seven claims will be received during a given two-week period.

Problem 41.6
Suppose X and Y are independent, each having Poisson distribution with means 2 and 3, respectively. Let $Z = X + Y$. Find $P(X + Y = 1)$.

Problem 41.7
Suppose that X has Poisson distribution with parameter λ and that Y has geometric distribution with parameter p and is independent of X. Find simple formulas in terms of λ and p for the following probabilities. (The formulas should not involve an infinite sum.)
(a) $P(X+Y=2)$
(b) $P(Y>X)$

Problem 41.8
Let X and Y be two independent random variables with common pmf given by

x	0	1	2	y
$p_X(x)$	0.5	0.25	0.25	$p_Y(y)$

Find the probability mass function of $X+Y$.

Problem 41.9
Let X and Y be two independent random variables with pmfs given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{3} & y = 1 \\ \frac{1}{6} & y = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability mass function of $X+Y$.

Problem 41.10
Let X and Y be two independent identically distributed geometric distributions with parameter p. Show that $X+Y$ is a negative binomial distribution with parameters $(2,p)$.

Problem 41.11
Let X, Y, Z be independent Poisson random variables with $E(X)=3, E(Y)=1$, and $E(Z)=4$. What is $P(X+Y+Z \leq 1)$?

Problem 41.12
The number of phone calls received by an operator in 5-minute period follows a Poisson distribution with a mean of λ. Find the probability that the total number of phone calls received in 10 randomly selected 5-minute periods is 10.

Problem 41.13 ‡
In each of the months June, July, and August, the number of accidents occurring in that month is modeled by a Poisson random variable with mean 1. In each of the other 9 months of the year, the number of accidents occurring is modeled by a Poisson random variable with mean 0.5. Assume that these 12 random variables are mutually independent.
Calculate the probability that exactly two accidents occur in July through November.

Problem 41.14 ‡
In a given region, the number of tornadoes in a one-week period is modeled by a Poisson distribution with mean 2. The numbers of tornadoes in different weeks are mutually independent.
Calculate the probability that fewer than four tornadoes occur in a three-week period.

Problem 41.15 ‡
The number of days an employee is sick each month is modeled by a Poisson distribution with mean 1. The numbers of sick days in different months are mutually independent.
Calculate the probability that an employee is sick more than two days in a three-month period.

Problem 41.16
Let X and Y be two independent uniform discrete random variables that takes the integer values $0, 1, 2, \cdots, n$ where n is a positive integer. Derive the probability mass function of $Z = X + Y$.

Problem 41.17
Let X be a negative binomial random variable with parameters r and p and Y be a negative binomial random variable with parameters s and p. Suppose that X and Y are independent and for any positive integer n, we have

$$\sum_{k=0}^{n} C(k+r-1,k)C(n-k+s-1,n-k) = C(n+r+s-1,n).$$

Derive the probability mass function of $Z = X + Y$.

Problem 41.18
Let X and Y be two independent geometric random variables with common parameter p. Find $\mathrm{P}(Y = y|X + Y = z)$ where $z \geq 2$ and $y = 1, 2, \cdots, z - 1$.

Problem 41.19
Let X and Y be two independent discrete random variables taking values on S_X and S_Y respectively. The expected value of the product XY is defined by the sum

$$E(XY) = \sum_{x \in S_X} \sum_{y \in S_Y} xy p_{XY}(x, y).$$

(a) Show that $E(XY) = E(X)E(Y)$.
(b) Show that $E(X + Y) = E(X) + E(Y)$.
(c) Show that $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Problem 41.20
Let X and Y be two independent discrete random variables. Show that $E[e^{t(X+Y)}] = E(e^{tX})E(e^{tY})$.

42 Sum of Two Independent Random Variables: Continuous Case

In this section, we consider the continuous version of the problem posed in Section 41: How are sums of independent continuous random variables distributed?

Example 42.1
Let X and Y be two random variables with joint probability density

$$f_{XY}(x,y) = \begin{cases} 6e^{-3x-2y} & x>0, y>0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the probability density of $Z = X + Y$.

Solution.
Integrating the joint probability density over the shaded region of Figure 42.1, we get

$$F_Z(a) = P(Z \leq a) = \int_0^a \int_0^{a-y} 6e^{-3x-2y} dxdy = 1 + 2e^{-3a} - 3e^{-2a}$$

and differentiating with respect to a we find

$$f_Z(a) = 6(e^{-2a} - e^{-3a})$$

for $a > 0$ and 0 elsewhere ■

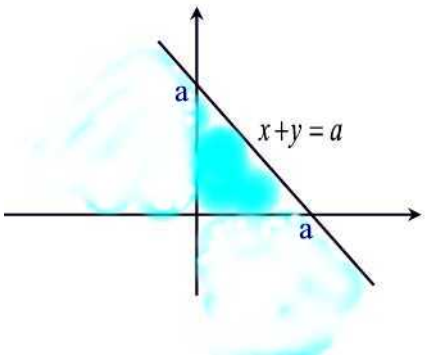


Figure 42.1

The above process can be generalized with the use of convolutions which we define next. Let X and Y be two continuous random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Assume that both $f_X(x)$ and $f_Y(y)$ are defined for all real numbers. Then the **convolution** $f_X * f_Y$ of f_X and f_Y is the function given by

$$(f_X * f_Y)(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy = \int_{-\infty}^{\infty} f_Y(a-x) f_X(x) dx.$$

This definition is analogous to the definition, given for the discrete case, of the convolution of two probability mass functions. Thus, it should not be surprising that if X and Y are independent, then the probability density function of their sum is the convolution of their densities.

Theorem 42.1
Let X and Y be two continuous independent random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all x and y. Then the sum $X+Y$ is a random variable with density function $f_{X+Y}(a)$, where f_{X+Y} is the convolution of f_X and f_Y.

Proof.
The cumulative distribution function is obtained as follows:

$$\begin{aligned} F_{X+Y}(a) =& P(X+Y \leq a) = \iint_{x+y \leq a} f_X(x) f_Y(y) dx dy \\ =& \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_X(x) f_Y(y) dx dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{a-y} f_X(x) dx \right) f_Y(y) dy \\ =& \int_{-\infty}^{\infty} F_X(a-y) f_Y(y) dy. \end{aligned}$$

Differentiating the previous equation with respect to a and assuming that differentiation and integration can be interchanged, we find

$$\begin{aligned} f_{X+Y}(a) =& \frac{d}{da} \int_{-\infty}^{\infty} F_X(a-y) f_Y(y) dy \\ =& \int_{-\infty}^{\infty} \frac{d}{da} F_X(a-y) f_Y(y) dy \\ =& \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy \\ =& (f_X * f_Y)(a) \ \blacksquare \end{aligned}$$

Example 42.2
Let X and Y be two independent random variables uniformly distributed on $[0, 1]$. Compute the probability density function of $X + Y$.

Solution.
Since

$$f_X(a) = f_Y(a) = \begin{cases} 1 & 0 \leq a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

by the previous theorem

$$f_{X+Y}(a) = \int_0^1 f_X(a-y)dy = \int_{a-1}^a f_X(y)dy.$$

If $0 < a \leq 1$ then

$$f_{X+Y}(a) = \int_0^a dy = a.$$

If $1 < a < 2$ then $0 < a - 1 < 1$. In this case,

$$f_{X+Y}(a) = \int_{a-1}^1 dy = 2 - a.$$

Hence,

$$f_{X+Y}(a) = \begin{cases} a & 0 < a \leq 1 \\ 2-a & 1 < a < 2 \\ 0 & \text{otherwise} \end{cases} \blacksquare$$

Example 42.3
Let X and Y be two independent exponential random variables with common parameter λ. Compute $f_{X+Y}(a)$.

Solution.
We have

$$f_X(z) = f_Y(z) = \begin{cases} \lambda e^{-\lambda z} & 0 \leq z \\ 0 & \text{otherwise.} \end{cases}$$

If $a \geq 0$ then

$$\begin{aligned} f_{X+Y}(a) = &\int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy = \int_0^a [\lambda e^{-\lambda(a-y)}][\lambda e^{-\lambda y}]dy \\ =&\lambda^2 \int_0^a e^{-\lambda a}dy = a\lambda^2 e^{-\lambda a}. \end{aligned}$$

If $a < 0$ then $f_{X+Y}(a) = 0$. Hence,

$$f_{X+Y}(a) = \begin{cases} a\lambda^2 e^{-\lambda a} & 0 \leq a \\ 0 & \text{otherwise} \end{cases} \blacksquare$$

Example 42.4
Let X and Y be two independent random variables, each with the standard normal density. Compute $f_{X+Y}(a)$.

Solution.
We have

$$f_X(a) = f_Y(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

By Theorem 42.1 we have

$$\begin{aligned} f_{X+Y}(a) =& \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2}} e^{-\frac{y^2}{2}} dy \\ =& \frac{1}{2\pi} e^{-\frac{a^2}{4}} \int_{-\infty}^{\infty} e^{-(y-\frac{a}{2})^2} dy \\ =& \frac{1}{2\pi} e^{-\frac{a^2}{4}} \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-w^2} dw \right], \quad w = y - \frac{a}{2}. \end{aligned}$$

The expression in the brackets equals 1, since it is the integral of the normal density function with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$. Hence,

$$f_{X+Y}(a) = \frac{1}{\sqrt{4\pi}} e^{-\frac{a^2}{4}} \blacksquare$$

Example 42.5
Let X and Y be two independent gamma random variables with respective parameters (s, λ) and (t, λ). Show that $X + Y$ is a gamma random variable with parameters $(s + t, \lambda)$.

Solution.
We have

$$f_X(a) = \frac{\lambda e^{-\lambda a} (\lambda a)^{s-1}}{\Gamma(s)} \quad \text{and} \quad f_Y(a) = \frac{\lambda e^{-\lambda a} (\lambda a)^{t-1}}{\Gamma(t)}$$

By Theorem 42.1 we have

$$\begin{aligned} f_{X+Y}(a) =& \frac{1}{\Gamma(s)\Gamma(t)} \int_0^a \lambda e^{-\lambda(a-y)} [\lambda(a-y)]^{s-1} \lambda e^{-\lambda y} (\lambda y)^{t-1} dy \\ =& \frac{\lambda^{s+t} e^{-\lambda a}}{\Gamma(s)\Gamma(t)} \int_0^a (a-y)^{s-1} y^{t-1} dy \\ =& \frac{\lambda^{s+t} e^{-\lambda a} a^{s+t-1}}{\Gamma(s)\Gamma(t)} \int_0^1 (1-x)^{s-1} x^{t-1} dx, \quad x = \frac{y}{a}. \end{aligned}$$

Using Problem 42.13, we have

$$\int_0^1 (1-x)^{s-1} x^{t-1} dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Thus,

$$f_{X+Y}(a) = \frac{\lambda e^{-\lambda a} (\lambda a)^{s+t-1}}{\Gamma(s+t)} \blacksquare$$

Example 42.6
The joint distribution function of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{11}(5x+y) & x, y > 0, \quad x + 2y < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the probability density of $Z = X + Y$.

Solution.
Note first that the region of integration is the interior of the triangle with vertices at $(0,0), (0,1)$, and $(2,0)$. From Figure 42.2, we see that $F(a) = 0$ if $a < 0$. If $0 \le a < 1$ then

$$F_Z(a) = P(Z \le a) = \int_0^a \int_0^{a-y} \frac{3}{11}(5x+y) dx dy = \frac{3}{11} a^3.$$

If $1 \le a < 2$ then the two lines $x + y = a$ and $x + 2y = 2$ intersect at $(2a-2, 2-a)$. In this case,

$$\begin{aligned} F_Z(a) =& \int_0^{2-a} \int_0^{a-y} \frac{3}{11}(5x+y) dx dy + \int_{2-a}^1 \int_0^{2-2y} \frac{3}{11}(5x+y) dx dy \\ =& \frac{3}{11} \left(-\frac{7}{3} a^3 + 9a^2 - 8a + \frac{7}{3} \right). \end{aligned}$$

If $a \geq 2$ then $F_Z(a)$ is the area of the shaded triangle which is equal to 1. Differentiating with respect to a we find

$$f_Z(a) = \begin{cases} \frac{9}{11}a^2 & 0 < a \leq 1 \\ \frac{3}{11}(-7a^2 + 18a - 8) & 1 < a < 2 \\ 0 & \text{elsewhere} \end{cases} \blacksquare$$

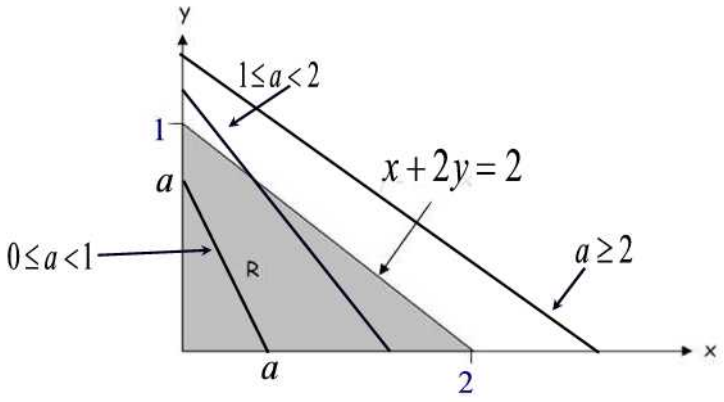


Figure 42.2

Practice Problems

Problem 42.1
Let X be an exponential random variable with parameter λ and Y be an exponential random variable with parameter 2λ independent of X. Find the probability density function of $X + Y$.

Problem 42.2
Let X be an exponential random variable with parameter λ and Y be a uniform random variable on [0,1] independent of X. Find the probability density function of $X + Y$.

Problem 42.3
Let X and Y be two independent random variables with probability density functions (p.d.f.) , f_X and f_Y respectively. Find the pdf of $X + 2Y$.

Problem 42.4
Consider two independent random variables X and Y. Let $f_X(x) = 1 - \frac{x}{2}$ if $0 \leq x \leq 2$ and 0 otherwise. Let $f_Y(y) = 2 - 2y$ for $0 \leq y \leq 1$ and 0 otherwise. Find the probability density function of $X + Y$.

Problem 42.5
Let X and Y be two independent and identically distributed random variables with common density function

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $X + Y$.

Problem 42.6
Let X and Y be independent exponential random variables with pairwise distinct respective parameters α and β. Find the probability density function of $X + Y$.

Problem 42.7
Let X and Y be two random variables with common pdf

$$f_X(t) = f_Y(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of $W = 2X + Y$.

Problem 42.8
Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} & 3 \leq y \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function of $X + Y$.

Problem 42.9
Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{2} & 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function of $X + Y$.

Problem 42.10
Let X have a uniform distribution on the interval $(1, 3)$. What is the probability that the sum of 2 independent observations of X is greater than 5?

Problem 42.11
Let X and Y be two independent exponential random variables each with mean 1. Find the pdf of $Z = X + Y$.

Problem 42.12
X_1 and X_2 are independent exponential random variables each with a mean of 1. Find $P(X_1 + X_2 < 1)$.

Problem 42.13
A **beta function** or **Euler integral of the first kind** is the function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \ x > 0, y > 0.$$

Show that

$$\int_0^\infty \int_0^\infty e^{-u-v}u^{x-1}v^{y-1}dudv = \left(\int_0^\infty e^{-z}z^{x+y-1}dz\right)\left(\int_0^1 t^{x-1}(1-t)^{y-1}dt\right).$$

That is,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Hint: Use the substitutions $u = zt$ and $v = z(1-t)$ and recall the change of formula

$$\int\int_R f(x,y)dxdy = \int\int_T f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|.$$

Problem 42.14 ‡
For any two continuous random variables X and Y and any function $g(x,y)$, we define

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)dxdy.$$

Let X and Y be two independent continuous random variables. Show that $E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY})$.

Problem 42.15 ‡
A company insures homes in three cities, J, K, and L. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are mutually independent.
You are given:

$$E(e^{tJ}) = (1-2t)^{-3} \quad E(e^{tK}) = (1-2t)^{-2.5} \quad E(e^{tL}) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

Problem 42.16
Let X and Y be two continuous random variables.
(a) Show that $E(X+Y) = E(X) + E(Y)$.
(b) Show that if X and Y are independent then $E(XY) = E(X)E(Y)$.
(c) Show that if X and Y are independent then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

Problem 42.17 ‡
A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails.
Calculate the variance of the total time that the generators produce electricity.

Problem 42.18
The life (in days) of a certain machine has an exponential distribution with a mean of 1 day. The machine comes supplied with one spare. Find the density function (t measure in days) of the combined life of the machine and its spare if the life of the spare has the same distribution as the first machine, but is independent of the first machine.

Problem 42.19 ‡
A policyholder has probability 0.7 of having no claims, 0.2 of having exactly one claim, and 0.1 of having exactly two claims. Claim amounts are uniformly distributed on the interval [0,60] and are independent. The insurer covers 100% of each claim. Calculate the probability that the total benefit paid to the policyholder is 48 or less.

43 Conditional Distributions: Discrete Case

Recall that for any two events E and F the conditional probability of E given F is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided that $P(F) > 0$.
In a similar way, if X and Y are discrete random variables then we define the **conditional probability mass function** of X given that $Y = y$ by

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{XY}(x, y)}{p_Y(y)} \qquad (43.1)$$

provided that $p_Y(y) > 0$.

Example 43.1
Two coins are being tossed repeatedly. The tossing of each coin stops when the coin comes up a head. Let X be the number of tosses of the first coin before getting a head, and Y be the number of tosses of the second coin before getting a head.
(a) Find the probability that the two coins come up heads at the same time.
(b) Find the conditional distribution of the number of coin tosses given that the two coins come up heads simultaneously.

Solution.
(a) X and Y are independent identically distributed geometric random variables with parameter $p = \frac{1}{2}$. Thus,

$$\begin{aligned} P(X = Y) &= \sum_{k=1}^{\infty} P(X = k, Y = k) = \sum_{k=1}^{\infty} P(X = k)P(Y = k) \\ &= \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3} \end{aligned}$$

(b) For any $k \geq 1$, we have

$$\mathrm{P}(X = k|Y = k) = \frac{\Pr(X = k, Y = k)}{\Pr(X = Y)} = \frac{\frac{1}{4^k}}{\frac{1}{3}} = \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}.$$

Thus, the conditional distribution follows a geometric distribution with parameter $p = \frac{3}{4}$ ■

Sometimes it is not the joint distribution that is known, but rather, for each y, one knows the conditional distribution of X given $Y = y$. If one also knows the distribution of Y, then one can recover the joint distribution using (43.1). We also mention one more use of (43.1):

$$p_X(x) = \sum_y p_{XY}(x, y) = \sum_y p_{X|Y}(x|y)p_Y(y). \tag{43.2}$$

Thus, given the conditional distribution of X given $Y = y$ for each possible value y, and the (marginal) distribution of Y, one can compute the (marginal) distribution of X, using (43.2).
The **conditional cumulative distribution** of X given that $Y = y$ is defined by

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{a \leq x} p_{X|Y}(a|y).$$

Note that if X and Y are independent, then the conditional mass function and the conditional cumulative distribution function are the same as the unconditional ones. This follows from the next theorem.

Theorem 43.1
If X and Y are independent and $p_Y(y) > 0$ then

$$p_{X|Y}(x|y) = p_X(x).$$

Proof.
We have

$$\begin{aligned} p_{X|Y}(x|y) =& P(X = x|Y = y) \\ =& \frac{P(X = x, Y = y)}{P(Y = y)} \\ =& \frac{P(X = x)P(Y = y)}{P(Y = y)} \\ =& P(X = x) = p_X(x) \end{aligned}$$ ■

Example 43.2
Given the following table.

X\ Y	Y=1	Y=2	Y=3	$p_X(x)$
X=1	.01	.20	.09	.3
X=2	.07	.00	.03	.1
X=3	.09	.05	.06	.2
X=4	.03	.25	.12	.4
$p_Y(y)$	.2	.5	.3	1

Find $p_{X|Y}(x|y)$ where $Y = 2$.

Solution.

$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{.2}{.5} = 0.4$$
$$p_{X|Y}(2|2) = \frac{p_{XY}(2,2)}{p_Y(2)} = \frac{0}{.5} = 0$$
$$p_{X|Y}(3|2) = \frac{p_{XY}(3,2)}{p_Y(2)} = \frac{.05}{.5} = 0.1$$
$$p_{X|Y}(4|2) = \frac{p_{XY}(4,2)}{p_Y(2)} = \frac{.25}{.5} = 0.5$$
$$p_{X|Y}(x|2) = \frac{p_{XY}(x,2)}{p_Y(2)} = \frac{0}{.5} = 0, \quad \text{otherwise } \blacksquare$$

Example 43.3
If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X, given that $X + Y = n$.

Solution.
We have

$$\begin{aligned}
P(X=k|X+Y=n) =& \frac{P(X=k, X+Y=n)}{P(X+Y=n)} \\
=& \frac{P(X=k, Y=n-k)}{P(X+Y=n)} \\
=& \frac{P(X=k)P(Y=n-k))}{P(X+Y=n)} \\
=& \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} \\
=& \frac{n!}{k!(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
=& \left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}.
\end{aligned}$$

In other words, the conditional mass distribution function of X given that $X+Y=n$, is the binomial distribution with parameters n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$ ■

Practice Problems

Problem 43.1
Given the following table.

X\ Y	Y=0	Y=1	$p_X(x)$
X=0	.4	.1	.5
X=1	.2	.3	.5
$p_Y(y)$	.6	.4	1

Find $p_{X|Y}(x|y)$ where $Y = 1$.

Problem 43.2
Let X be a random variable with range the set $\{1, 2, 3, 4, 5\}$ and Y be a random variable with range the set $\{1, 2, \cdots, X\}$.
(a) Find $p_{XY}(x, y)$.
(b) Find $p_{X|Y}(x|y)$.
(c) Are X and Y independent?

Problem 43.3
The following is the joint distribution function of X and Y.

$X\backslash Y$	4	3	2	$p_X(x)$
5	0.1	0.05	0	0.15
4	0.15	0.15	0	0.3
3	0.10	0.15	0.10	0.35
2	0	0.05	0.10	0.15
1	0	0	0.05	0.05
$p_Y(y)$	0.35	0.40	0.25	1

Find $P(X|Y = 4)$ for $X = 3, 4, 5$.

Problem 43.4
A fair coin is tossed 4 times. Let the random variable X denote the number of heads in the first 3 tosses, and let the random variable Y denote the number of heads in the last 3 tosses. The joint pmf is given by the following table

$X\backslash Y$	0	1	2	3	$p_X(x)$
0	1/16	1/16	0	0	2/16
1	1/16	3/16	2/16	0	6/16
2	0	2/16	3/16	1/16	6/16
3	0	0	1/16	1/16	2/16
$p_Y(y)$	2/16	6/16	6/16	2/16	1

What is the conditional pmf of the number of heads in the first 3 coin tosses given exactly 1 head was observed in the last 3 tosses?

Problem 43.5
Two dice are rolled. Let X and Y denote, respectively, the largest and smallest values obtained. Compute the conditional mass function of Y given $X = x$, for $x = 1, 2, \cdots, 6$. Are X and Y independent?

Problem 43.6
Let X and Y be discrete random variables with joint probability function

$$p_{XY}(x,y) = \begin{cases} \frac{n!y^x(pe^{-1})^y(1-p)^{n-y}}{y!(n-y)!x!} & y = 0, 1, \cdots, n \ ; x = 0, 1, \cdots \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $p_Y(y)$.
(b) Find the conditional probability distribution of X, given $Y = y$. Are X and Y independent? Justify your answer.

Problem 43.7
Let X and Y have the joint probability function $p_{XY}(x,y)$ described as follows:

X\ Y	0	1	$p_X(x)$
0	1/18	3/18	4/18
1	4/18	3/18	7/18
2	6/18	1/18	7/18
$p_Y(y)$	11/18	7/18	1

Find $p_{X|Y}(x|y)$ and $p_{Y|X}(y|x)$.

Problem 43.8
Let X and Y be random variables with joint probability mass function

$$p_{XY}(x,y) = c(1 - 2^{-x})^y$$

where $x = 0, 1, \cdots, N-1$ and $y = 0, 1, 2, \cdots$
(a) Find c.
(b) Find $p_X(x)$.
(c) Find $p_{Y|X}(y|x)$, the conditional probability mass function of Y given $X = x$.

Problem 43.9
Let X and Y be identically independent Poisson random variables with parameter λ. Find $P(X = k|X + Y = n)$.

Problem 43.10
If two cards are randomly drawn (without replacement) from an ordinary deck of 52 playing cards, Y is the number of aces obtained in the first draw and X is the total number of aces obtained in both draws, find
(a) the joint probability distribution of X and Y;
(b) the marginal distribution of Y;
(c) the conditional distribution of X given $Y = 1$.

Problem 43.11 ‡
Let N_1 and N_2 represent the numbers of claims submitted to a life insurance company in April and May, respectively. The joint probability function of N_1 and N_2 is

$$P(n_1, n_2) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^{n_1-1} e^{-n_1}(1 - e^{-n_1})^{n_2-1}, & \text{for } n_1 = 1, 2, 3, \cdots \text{ and } n_2 = 1, 2, 3, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected number of claims that will be submitted to the company in May if exactly 2 claims were submitted in April.

Problem 43.12
Suppose that discrete random variables X and Y each take only the values 0 and 1. It is known that $P(X = 0|Y = 1) = 0.6$ and $P(X = 1|Y = 0) = 0.7$. Is it possible that X and Y are independent? Justify your conclusion.

Problem 43.13 ‡
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state (0 or 1) of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

X\ Y	0	1
0	0.800	0.025
1	0.050	0.125

Calculate $P_{Y|X}(y|1)$.

Problem 43.14 ‡
Let N denote the number of accidents occurring during one month on the northbound side of a highway and let S denote the number occurring on the southbound side.
Suppose that N and S are jointly distributed as indicated in the table.

$N\backslash S$	0	1	2	3 or more
0	0.04	0.06	0.10	0.04
1	0.10	0.18	0.08	0.03
2	0.12	0.06	0.05	0.02
3 or more	0.05	0.04	0.02	0.01

Calculate $P(N|N+S=2)$.

Problem 43.15 ‡
Let X be the annual number of hurricanes hitting Florida, and let Y be the annual number of hurricanes hitting Texas. X and Y are independent Poisson variables with respective means 1.70 and 2.30.
Calculate $P(X-Y|X+Y=3)$.

Problem 43.16
A Safety Officer for an auto insurance company in Connecticut was interested in learning how the extent of an individual's injury X in an automobile accident relates to the type of safety restraint Y the individual was wearing at the time of the accident. As a result, the Safety Officer used statewide ambulance and police records to compile the following two-way table of joint probabilities:

	Y			
X	None (0)	Belt Only (1)	Belt and Harness (2)	$p_X(x)$
None(0)	0.065	0.075	0.06	0.20
Minor(1)	0.175	0.16	0.115	0.45
Major(2)	0.135	0.10	0.065	0.30
Death(3)	0.025	0.015	0.01	0.05
$p_Y(y)$	0.40	0.35	0.25	1.00

If a randomly selected person wears no restraint, what is the probability of death?

Problem 43.17
The joint distribution function of X and Y is given in the following table:

$X\backslash Y$	-1	2.5	3	4.7
0	0.11	0.03	0.15	0.04
1	0.03	0.09	0.15	0.16
3	0.00	0.16	0.06	0.02

Calculate $P(0 \leq Y \leq 4|X = 1)$.

Problem 43.18
A box contain 4 reds, 3 whites and 2 blues. A random sample of 3 balls is chosen. Let R and W be the number of red and white balls chosen. What is the conditional probability of $W = 2$ given that $R = 1$?

Problem 43.19
The joint distribution function of X and Y is given in the following table:

$X\backslash Y$	-1	2.5	3	4.7
0	0.11	0.03	0.15	0.04
1	0.03	0.09	0.15	0.16
3	0.00	0.16	0.06	0.02

Calculate $F_{X|Y}(3|4.7)$.

Problem 43.20
Let X and Y be independent geometric random variables with common parameter p. Calculate the conditional distribution of X, given that $X+Y = n$.

Problem 43.21 ‡
The probability of x losses occurring in year 1 is $(0.5)^{x+1},\ x = 0, 1, 2, \cdots$. The probability of y losses in year 2 given x losses in year 1 is given by the table:

$X\backslash Y$	0	1	2	3	4^+
0	0.60	0.25	0.05	0.05	0.05
1	0.45	0.30	0.10	0.10	0.05
2	0.25	0.30	0.20	0.20	0.05
3	0.15	0.20	0.20	0.30	0.15
4^+	0.05	0.15	0.25	0.35	0.20

Calculate the probability of exactly 2 losses in 2 years.

Problem 43.22 ‡
A flood insurance company determines that N, the number of claims received in a month, is a random variable with $P(N = n) = \frac{2}{3^{n+1}}$, for $n = 0, 1, 2, \cdots$ The numbers of claims received in different months are mutually independent. Calculate the probability that more than three claims will be received during a consecutive two-month period, given that fewer than two claims were received in the first of the two months.

44 Conditional Distributions: Continuous Case

In this section, we develop the distribution of X given Y when both are continuous random variables. Unlike the discrete case, we cannot use simple conditional probability to define the conditional probability of an event given $Y = y$, because the conditioning event has probability 0 for any y. However, we motivate our approach by the following argument.
Suppose X and Y are two continuous random variables with joint density $f_{XY}(x, y)$. Let $f_{X|Y}(x|y)$ denote the probability density function of X given that $Y = y$. We define

$$\mathrm{P}(a < X < b|Y = y) = \int_a^b f_{X|Y}(x|y)dx.$$

Then for δ very small we have (See Remark 30.1)

$$\mathrm{P}(x \leq X \leq x + \delta|Y = y) \approx \delta f_{X|Y}(x|y).$$

On the other hand, for small ϵ we have

$$\begin{aligned}\mathrm{P}(x \leq X \leq x + \delta|Y = y) \approx & \mathrm{P}(x \leq X \leq x + \delta|y \leq Y \leq y + \epsilon) \\ = & \frac{\mathrm{P}(x \leq X \leq x + \delta, y \leq Y \leq y + \epsilon)}{\mathrm{P}(y \leq Y \leq y + \epsilon)} \\ \approx & \frac{\delta\epsilon f_{XY}(x, y)}{\epsilon f_Y(y)}.\end{aligned}$$

In the limit, as ϵ tends to 0, we are left with

$$\delta f_{X|Y}(x|y) \approx \frac{\delta f_{XY}(x, y)}{f_Y(y)}.$$

This suggests the following definition. The **conditional density function** of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$.
Note that

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{XY}(x, y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1.$$

Compare this definition with the discrete case where

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)}.$$

The **conditional cumulative distribution function** of X given $Y = y$ is defined by

$$F_{X|Y}(x|y) = \mathrm{P}(X \leq x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)dt.$$

From this definition, it follows

$$f_{X|Y}(x|y) = \frac{\partial}{\partial x} F_{X|Y}(x|y).$$

Example 44.1
Suppose X and Y have the following joint density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & |X| + |Y| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of X.
(b) Find the conditional distribution of Y given $X = \frac{1}{2}$.

Solution.
(a) Clearly, X only takes values in $(-1, 1)$. So $f_X(x) = 0$ if $|x| \geq 1$. Let $-1 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2} dy = \int_{-1+|x|}^{1-|x|} \frac{1}{2} dy = 1 - |x|.$$

(b) The conditional density of Y given $X = \frac{1}{2}$ is then given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \begin{cases} 1 & -\frac{1}{2} < y < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $f_{Y|X}$ follows a uniform distribution on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ ■

Example 44.2
Suppose that X is uniformly distributed on the interval $(0, 1)$ and that, given $X = x,\ Y$ is uniformly distributed on the interval $(1 - x, 1)$.
(a) Determine the joint density $f_{XY}(x, y)$.
(b) Find the probability $\mathrm{P}(Y \geq \frac{1}{2})$.

Solution.
Since X is uniformly distributed on $(0,1)$, we have $f_X(x) = 1, 0 < x < 1$ and 0 otherwise. Similarly, since, given $X = x$, Y is uniformly distributed on $(1-x, 1)$, the conditional density of Y given $X = x$ is $\frac{1}{1-(1-x)} = \frac{1}{x}$ on the interval $(1-x, 1)$ and 0 otherwise; i.e., $f_{Y|X}(y|x) = \frac{1}{x}, 1-x < y < 1$ for $0 < x < 1$ and 0 otherwise. Thus,

$$f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{x}, 0 < x < 1, 1-x < y < 1.$$

(b) Using Figure 44.1 we find

$$\begin{aligned} \mathrm{P}(Y \geq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_{1-x}^1 \frac{1}{x} dy dx + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \frac{1}{x} dy dx \\ &= \int_0^{\frac{1}{2}} \frac{1-(1-x)}{x} dx + \int_{\frac{1}{2}}^1 \frac{1/2}{x} dx \\ &= \frac{1+\ln 2}{2} \blacksquare \end{aligned}$$

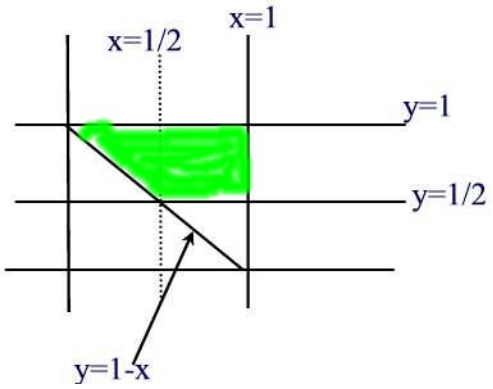


Figure 44.1

Example 44.3
The joint density of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{15}{2} x(2-x-y) & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the conditional density of X, given that $Y = y$ for $0 \leq y \leq 1$.

Solution.
The marginal density function of Y is

$$f_Y(y) = \int_0^1 \frac{15}{2} x(2 - x - y)dx = \frac{15}{2}\left(\frac{2}{3} - \frac{y}{2}\right)$$

for $0 \leq y \leq 1$ and 0 otherwise. Thus,

$$\begin{aligned} f_{X|Y}(x|y) =& \frac{f_{XY}(x,y)}{f_Y(y)} \\ =& \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}} \\ =& \frac{6x(2-x-y)}{4-3y} \end{aligned}$$

for $0 \leq x, y \leq 1$ and 0 otherwise ■

Example 44.4
The joint density function of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}}e^{-y}}{y} & x \geq 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathrm{P}(X > 1|Y = y)$.

Solution.
The marginal density function of Y is

$$f_Y(y) = e^{-y} \int_0^\infty \frac{1}{y} e^{-\frac{x}{y}} dx = e^{-y}\left[-e^{-\frac{x}{y}}\right]_0^\infty = e^{-y}.$$

for $y > 0$ and 0 otherwise. Thus,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{e^{-y}} = \frac{1}{y}e^{-\frac{x}{y}}$$

for $x \geq 0,\ y > 0$ and 0 otherwise. Hence,

$$\mathrm{P}(X > 1|Y = y) = \int_1^\infty \frac{1}{y} e^{-\frac{x}{y}} dx = -e^{-\frac{x}{y}}\,|_1^\infty = e^{-\frac{1}{y}}$$ ■

We end this section with the following theorem.

Theorem 44.1
Continuous random variables X and Y with $f_Y(y) > 0$ are independent if and only if

$$f_{X|Y}(x|y) = f_X(x).$$

Proof.
Suppose first that X and Y are independent. Then $f_{XY}(x,y) = f_X(x)f_Y(y)$. Thus,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x).$$

Conversely, suppose that $f_{X|Y}(x|y) = f_X(x)$. Then $f_{XY}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_X(x)f_Y(y)$. This shows that X and Y are independent ■

Example 44.5
Let X and Y be two continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & 0 \le y < x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $f_X(x)$, $f_Y(y)$ and $f_{X|Y}(x|1)$.
(b) Are X and Y independent?

Solution.
(a) We have

$$f_X(x) = \int_0^x \frac{1}{2}dy = \frac{x}{2}, \quad 0 \le x \le 2$$

and 0 otherwise. Likewise,

$$f_Y(y) = \int_y^2 \frac{1}{2}dx = \frac{1}{2}(2-y), \quad 0 \le y \le 2$$

and 0 otherwise.

$$f_{X|Y}(x|1) = \frac{f_{XY}(x,1)}{f_Y(1)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1, \quad 1 \le x \le 2$$

and 0 otherwise.
(b) Since $f_{X|Y}(x|1) \neq f_X(x)$, X and Y are dependent ■

Practice Problems

Problem 44.1
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 5x^2y & -1 \leq x \leq 1, 0 < y \leq |x| \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_{X|Y}(x|y)$, the conditional probability density function of X given $Y = y$. Sketch the graph of $f_{X|Y}(x|0.5)$.

Problem 44.2
Suppose that X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} 8xy & 0 \leq x < y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_{X|Y}(x|y)$, the conditional probability density function of X given $Y = y$.

Problem 44.3
Suppose that X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{3y^2}{x^3} & 0 \leq y < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_{Y|X}(y|x)$, the conditional probability density function of Y given $X = x$.

Problem 44.4
The joint density function of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the conditional density of X given $Y = y$ and that of Y given $X = x$.

Problem 44.5
Let X and Y be continuous random variables with conditional and marginal p.d.f.'s given by

$$f_X(x) = \frac{x^3e^{-x}}{6} I_{(0,\infty)}(x)$$

and

$$f_{Y|X}(y|x) = \frac{3y^2}{x^3} I_{(0,x)}(y)$$

where $I_A(x)$ is the indicator function of A.
(a) Find the joint p.d.f. of X and Y.
(b) Find the conditional p.d.f. of X given $Y = y$.

Problem 44.6
Suppose X, Y are two continuous random variables with joint probability density function

$$f_{XY}(x,y) = \begin{cases} 12xy(1-x) & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $f_{X|Y}(x|y)$. Are X and Y independent?
(b) Find $\mathrm{P}(Y < \frac{1}{2} | X > \frac{1}{2})$.

Problem 44.7
The joint probability density function of the random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3}x - y + 1 & 1 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional probability density function of X given $Y = y$.
(b) Find $\mathrm{P}(X < \frac{3}{2} | Y = \frac{1}{2})$.

Problem 44.8 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 24xy & 0 < x < 1, 0 < y < 1 - x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\mathrm{P}\left(Y < X | X = \frac{1}{3}\right)$.

Problem 44.9 ‡
Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)} & x > 1, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3 .

Problem 44.10 ‡
A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy.
Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f_{XY}(x, y) = 2(x + y)$ on the region where the density is positive.
Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

Problem 44.11 ‡
An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x < y < x + 1$.
If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.5?

Problem 44.12 ‡
You are given the following information about N, the annual number of claims for a randomly selected insured:

$$\begin{aligned} \mathrm{P}(N = 0) &= \frac{1}{2} \\ \mathrm{P}(N = 1) &= \frac{1}{3} \\ \mathrm{P}(N > 1) &= \frac{1}{6} \end{aligned}$$

Let S denote the total annual claim amount for an insured. When $N = 1, S$ is exponentially distributed with mean 5 . When $N > 1, S$ is exponentially distributed with mean 8 . Determine $\mathrm{P}(4 < S < 8)$.

Problem 44.13
Let Y have a uniform distribution on the interval $(0, 1)$, and let the conditional distribution of X given $Y = y$ be uniform on the interval $(0, \sqrt{y})$. What is the marginal density function of X for $0 < x < 1$?

Problem 44.14 ‡
The distribution of Y, given X, is uniform on the interval $[0, X]$. The marginal density of X is

$$f_X(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the conditional density of X, given $Y = y > 0$.

Problem 44.15
Suppose that X has a continuous distribution with p.d.f. $f_X(x) = 2x$ on $(0, 1)$ and 0 elsewhere. Suppose that Y is a continuous random variable such that the conditional distribution of Y given $X = x$ is uniform on the interval $(0, x)$. Find the mean and variance of Y.

Problem 44.16 ‡
An insurance policy is written to cover a loss X where X has density function

$$f_X(x) = \begin{cases} \frac{3}{8}x^2 & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The time T (in hours) to process a claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$.
Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

Problem 44.17 ‡
A machine has two components and fails when both components fail. The number of years from now until the first component fails, X, and the number of years from now until the machine fails, Y, are random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{18}e^{-\frac{(x+y)}{6}} & \text{for } 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_{Y|X}(y|2)$.

Problem 44.18 ‡
As a block of concrete is put under increasing pressure, engineers measure the pressure X at which the first fracture appears and the pressure Y at which the second fracture appears. X and Y are measured in tons per square inch and have joint density function

$$f_{XY}(x,y) = \begin{cases} 24x(1-y) & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the conditional density of the pressure at which the second fracture appears, given that the first fracture appears at 1/3 ton per square inch.

Problem 44.19

$$f_{XY}(x,y) = \begin{cases} k & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Problem 44.20
Let X and Y be two continuous random variables with joint density function $f_{XY}(x,y)$. Show that

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy}.$$

Problem 44.21 ‡
The elapsed time, T, between the occurrence and the reporting of an accident has probability density function

$$f_T(t) = \begin{cases} \frac{8t-t^2}{72} & \text{for } 0 < t < 6 \\ 0 & \text{otherwise.} \end{cases}$$

Given that $T = t$, the elapsed time between the reporting of the accident and payment by the insurer is uniformly distributed on $[2+t, 10]$.
Calculate the probability that the elapsed time between the occurrence of the accident and payment by the insurer is less than 4.

45 Joint Probability Distributions of Functions of Random Variables

Theorem 36.1 provided a result for finding the pdf of a function of one random variable: if $Y = g(X)$ is a function of the random variable X, where $g(x)$ is monotone and differentiable then the pdf of Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

An extension to functions of two random variables is given in the following theorem.

Theorem 45.1
Let X and Y be jointly continuous random variables with joint probability density function $f_{XY}(x, y)$ with $A = \{(x, y) : f_{XY}(x, y) > 0\}$ and 0 otherwise. Let $U = g_1(X, Y)$ and $V = g_2(X, Y)$. Suppose that $u = g_1(x, y)$ and $v = g_2(x, y)$ are 1-1 transformations from A onto B. That is, there is an inverse transformation $x = x(u, v)$ and $y = y(u, v)$. Furthermore, suppose that g_1 and g_2 have continuous partial derivatives at all points $(x, y) \in A$ and such that the Jacobian determinant

$$J(x, y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial y} \neq 0$$

for all $(x, y) \in A$. Then the random variables U and V are continuous random variables with joint density function given by

$$f_{UV}(u, v) = f_{XY}(g_1^{-1}(u, v), g_2^{-1}(u, v))|J(g_1^{-1}(u, v), g_2^{-1}(u, v))|$$

for all $(u, v) \in B$ and 0 otherwise.

Proof.
Since g_1 and g_2 are differentiable, the Inverse Function Theorem asserts that the functions $x = x(u, v) = g_1^{-1}(u, v)$ and $y = y(u, v) = g_2^{-1}(u, v)$ are two differentiable functions of u and v. We assume that the functions $x = x(u, v)$ and $y = y(u, v)$ take a point in the $uv-$plane to exactly one point in the $xy-$plane.
Let us see what happens to a small rectangle T in the $uv-$plane with sides

of lengths Δu and Δv as shown in Figure 45.1.

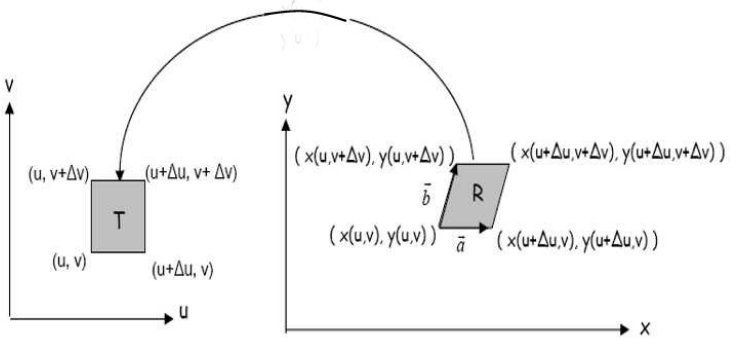


Figure 45.1

Since the side-lengths are small, by the Mean Value Theorem, we have

$$x(u+\Delta u, v) - x(u,v) \approx \frac{\partial x}{\partial u}(\alpha_1, v)\Delta u, \quad u \le \alpha_1 \le u + \Delta u$$

$$x(u+\Delta u, v+\Delta v) - x(u, v+\Delta v) \approx \frac{\partial x}{\partial u}(\alpha_2, v+\Delta v)\Delta u, \quad u \le \alpha_2 \le u + \Delta u$$

$$y(u, v+\Delta v) - x(u,v) \approx \frac{\partial y}{\partial v}(u, \beta_1)\Delta v, \quad v \le \beta_1 \le v + \Delta v$$

$$x(u+\Delta u, v+\Delta v) - x(u+\Delta u, v) \approx \frac{\partial y}{\partial v}(u+\Delta u, \beta_2)\Delta v, \quad v \le \beta_2 \le v + \Delta v.$$

Since Δu and Δv are small, each side of the rectangle in the $uv-$plane is transformed into a line segment in the $xy-$plane. The result is that the rectangle in the $uv-$plane is transformed into a parallelogram R in the $xy-$plane with sides in vector form

$$\vec{a} = \frac{\partial x}{\partial u}\Delta u\vec{i} + \frac{\partial y}{\partial u}\Delta u\vec{j}$$

and

$$\vec{b} = \frac{\partial x}{\partial v}\Delta v\vec{i} + \frac{\partial y}{\partial v}\Delta v\vec{j}$$

Now, the area of R is

$$\text{Area R} \approx ||\vec{a} \times \vec{b}|| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| \Delta u \Delta v.$$

Using determinant notation, we define the **Jacobian**, $\frac{\partial(x,y)}{\partial(u,v)}$, as follows

$$J(x,y) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Thus, we can write

$$\text{Area R} \approx |J(x,y)|\Delta u \Delta v.$$

Now, suppose we are integrating $f(x,y)$ over a region R. Partition R into mn small parallelograms. Then using Riemann sums we can write

$$\begin{aligned} \int\int_R f(x,y)dxdy &\approx \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij},y_{ij}) \cdot \text{Area of } \mathrm{R_{ij}} \\ &\approx \sum_{j=1}^{m}\sum_{i=1}^{n} f(g_1^{-1}(u_{ij},v_{ij}), g_2^{-1}(u_{ij},v_{ij}))|J(x,y)|\Delta u \Delta v \end{aligned}$$

where (u_{ij}, v_{ij}) in T_{ij} corresponds to a point (x_{ij}, y_{ij}) in R_{ij}. Now, letting $m, n \to \infty$ to obtain

$$\int\int_R f(x,y)dxdy = \int\int_T f(g_1^{-1}(u,v), g_2^{-1}(u,v))|J(x,y)|dudv.$$

Finally, we have

$$\begin{aligned} \mathrm{P}((X,Y) \in R) &= \int\int_R f_{XY}(x,y)dxdy \\ &= \int\int_T f_{XY}(g_1^{-1}(u,v), g_2^{-1}(u,v))|J(x,y)|dudv \\ &= \int\int_T f_{UV}(u,v)dudv = \mathrm{P}((U,V) \in T) \end{aligned}$$

where

$$f_{UV}(u,v) = f_{XY}(g_1^{-1}(u,v), g_2^{-1}(u,v))|J(x,y)| \ \blacksquare$$

Example 45.1
Let X and Y be jointly continuous random variables with density function $f_{XY}(x,y)$. Let $U = X + Y$ and $V = X - Y$. Find the joint density function of U and V.

Solution.
Let $u = g_1(x, y) = x+y$ and $v = g_2(x, y) = x-y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Moreover

$$J(x, y) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Thus,

$$f_{UV}(u, v) = \frac{1}{2} f_{XY}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \blacksquare$$

Example 45.2
Let X and Y be jointly continuous random variables with density function $f_{XY}(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$. Let $U = X + Y$ and $V = X - Y$. Find the joint density function of U and V.

Solution.
Since $J(x, y) = -\frac{1}{2}$, we have

$$f_{UV}(u, v) = \frac{1}{4\pi} e^{-\frac{(\frac{u+v}{2})^2+(\frac{u-v}{2})^2}{2}} = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}} \blacksquare$$

Example 45.3
Suppose that X and Y have joint density function given by

$$f_{XY}(x, y) = \begin{cases} 4xy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = \frac{X}{Y}$ and $V = XY$.
(a) Find the joint density function of U and V.
(b) Find the marginal density of U and V.
(c) Are U and V independent?

Solution.
(a) Now, if $u = g_1(x, y) = \frac{x}{y}$ and $v = g_2(x, y) = xy$ then solving for x and y we find $x = \sqrt{uv}$ and $y = \sqrt{\frac{v}{u}}$. Moreover,

$$J(x, y) = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \\ -\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2\sqrt{uv}} \end{vmatrix} = \frac{1}{2u}$$

By Theorem 45.1, we find

$$f_{UV}(u, v) = \frac{1}{2u} f_{XY}(\sqrt{uv}, \sqrt{\frac{v}{u}}) = \frac{2v}{u},\ 0 < uv < 1,\ 0 < \frac{v}{u} < 1$$

and 0 otherwise. The region where f_{UV} is defined is shown in Figure 45.2.
(b) The marginal density of U is

$$f_U(u) = \int_0^u \frac{2v}{u} dv = u, \quad 0 < u \leq 1$$
$$f_U(u) = \int_0^{\frac{1}{u}} \frac{2v}{u} dv = \frac{1}{u^3}, \quad u > 1$$

and 0 otherwise. The marginal density of V is

$$f_V(v) = \int_0^{\infty} f_{UV}(u,v) du = \int_v^{\frac{1}{v}} \frac{2v}{u} du = -4v \ln v, \quad 0 < v < 1$$

and 0 otherwise.
(c) Since $f_{UV}(u,v) \neq f_U(u) f_V(v)$, U and V are dependent ■

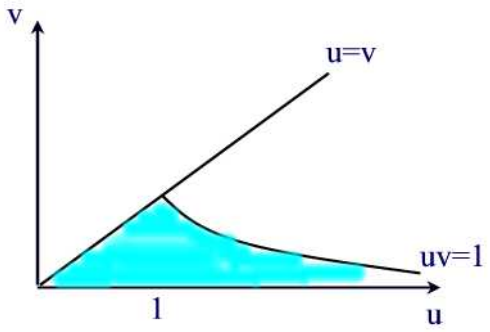


Figure 45.2

Practice Problems

Problem 45.1
Let X and Y be two random variables with joint pdf f_{XY}. Let $Z = aX + bY$ and $W = cX + dY$ where $ad - bc \neq 0$. Find the joint probability density function of Z and W.

Problem 45.2
Let X_1 and X_2 be two independent exponential random variables each having parameter λ. Find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_2}$.

Problem 45.3
Let X and Y be random variables with joint pdf $f_{XY}(x, y)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Phi = \tan^{-1}\left(\frac{Y}{X}\right)$ with $-\pi < \Phi \leq \pi$. Find $f_{R\Phi}(r, \phi)$.

Problem 45.4
Let X and Y be two random variables with joint pdf $f_{XY}(x, y)$. Let $Z = g(X, Y) = \sqrt{X^2 + Y^2}$ and $W = \frac{Y}{X}$. Find $f_{ZW}(z, w)$.

Problem 45.5
If X and Y are independent gamma random variables with parameters (α, λ) and (β, λ) respectively, compute the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$.

Problem 45.6
Let X_1 and X_2 be two continuous random variables with joint density function

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & x_1 \geq 0, x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. Find the joint density function of Y_1 and Y_2.

Problem 45.7
Let X_1 and X_2 be two independent normal random variables with parameters (0,1) and (0,4) respectively. Let $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - 3X_2$. Find $f_{Y_1Y_2}(y_1, y_2)$.

Problem 45.8
Let X be a uniform random variable on $(0, 2\pi)$ and Y an exponential random variable with $\lambda = 1$ and independent of X. Show that

$$U = \sqrt{2Y}\cos X \text{ and } V = \sqrt{2Y}\sin X$$

are independent standard normal random variables

Problem 45.9
Let X and Y be two random variables with joint density function f_{XY}. Compute the pdf of $U = X + Y$. What is the pdf in the case X and Y are independent? Hint: let $V = Y$.

Problem 45.10
Let X and Y be two random variables with joint density function f_{XY}. Compute the pdf of $U = Y - X$.

Problem 45.11
Let X and Y be two random variables with joint density function f_{XY}. Compute the pdf of $U = XY$. Hint: let $V = X$.

Problem 45.12
Let X and Y be two independent exponential distributions with mean 1. Find the distribution of $\frac{X}{Y}$.

Problem 45.13 ‡
A device containing two key components fails when, and only when, both components fail. The lifetimes, T_1 and T_2 of these components are independent with common density function

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The cost, X, of operating the device until failure is $2T_1 + T_2$. Let g be the density function for X.
Determine $g(x)$, for $x > 0$.

Problem 45.14 ‡
A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums, and let f be the density function of X.
Determine $f(x)$, where it is positive.

Properties of Expectation

We have seen that the expected value of a random variable is a weighted average of the possible values of X and also is the center of the distribution of the variable. Recall that the expected value of a discrete random variable X with probability mass function $p(x)$ is defined by

$$E(X) = \sum_x xp(x)$$

provided that the sum is finite.
For a continuous random variable X with probability density function $f(x)$, the expected value is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

provided that the improper integral is convergent.
In this chapter we develop and exploit properties of expected values.

46 Expected Value of a Function of Two Random Variables

In this section, we learn some equalities and inequalities about the expectation of random variables. Our goals are to become comfortable with the expectation operator and learn about some useful properties.
First, we introduce the definition of expectation of a function of two random variables: Suppose that X and Y are two random variables taking values in S_X and S_Y respectively. For a function $g : S_X \times S_Y \to \mathbb{R}$, the expected value of $g(X, Y)$ is

$$E(g(X, Y) = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y)p_{XY}(x, y).$$

if X and Y are discrete with joint probability mass function $p_{XY}(x,y)$ and

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)dxdy$$

if X and Y are continuous with joint probability density function $f_{XY}(x,y)$.

Example 46.1
Let X and Y be two discrete random variables with joint probability mass function:

$$p_{XY}(1,1) = \tfrac{1}{3}, p_{XY}(1,2) = \tfrac{1}{8}, p_{XY}(2,1) = \tfrac{1}{2}, p_{XY}(2,2) = \tfrac{1}{24}.$$

Find the expected value of $g(X,Y) = XY$.

Solution.
The expected value of the function $g(X,Y) = XY$ is calculated as follows:

$$\begin{aligned} E(g(X,Y)) =& E(XY) = \sum_{x=1}^{2}\sum_{y=1}^{2} xyp_{XY}(x,y) \\ =& (1)(1)(\frac{1}{3}) + (1)(2)(\frac{1}{8}) + (2)(1)(\frac{1}{2}) + (2)(2)(\frac{1}{24}) \\ =& \frac{7}{4} \blacksquare \end{aligned}$$

An important application of the above definition is the following result.

Proposition 46.1
The expected value of the sum/difference of two random variables is equal to the sum/difference of their expectations. That is,

$$E(X+Y) = E(X) + E(Y)$$

and

$$E(X-Y) = E(X) - E(Y).$$

Proof.
We prove the result for discrete random variables X and Y with joint probability mass function $p_{XY}(x,y)$. Letting $g(X,Y) = X \pm Y$ we have

$$\begin{aligned} E(X \pm Y) &= \sum_x \sum_y (x \pm y) p_{XY}(x,y) \\ &= \sum_x \sum_y x p_{XY}(x,y) \pm \sum_x \sum_y y p_{XY}(x,y) \\ &= \sum_x x \sum_y p_{XY}(x,y) \pm \sum_y y \sum_x p_{XY}(x,y) \\ &= \sum_x x p_X(x) \pm \sum_y y p_Y(y) \\ &= E(X) \pm E(Y). \end{aligned}$$

A similar proof holds for the continuous case where you just need to replace the sums by improper integrals and the joint probability mass function by the joint probability density function ■

Using mathematical induction one can easily extend the previous result to

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n), \quad E(X_i) < \infty.$$

Example 46.2
A group of N business executives throw their business cards into a jar. The cards are mixed, and each person randomly selects one. Find the expected number of people that select their own card.

Solution.
Let X = the number of people who select their own card. For $1 \leq i \leq N$ let

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ person chooses his own card} \\ 0 & \text{otherwise} \end{cases}$$

Then $E(X_i) = \mathrm{P}(X_i = 1) = \frac{1}{N}$ and

$$X = X_1 + X_2 + \cdots + X_N.$$

Hence,

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_N) = \left(\frac{1}{N}\right) N = 1 \blacksquare$$

Example 46.3 *(***Sample Mean)**
Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed random variables, each having a mean μ and variance σ^2. Define a new random variable by

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

We call $\overline{X}$ the **sample mean**. Find $E(\overline{X})$.

Solution.
The expected value of $\overline{X}$ is

$$E(\overline{X}) = E\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu.$$

Because of this result, when the distribution mean μ is unknown, the sample mean is often used in statistics to estimate it ■

The following property is known as the monotonicity property of the expected value.

Proposition 46.2
If X is a non-negative random variable then $E(X) \geq 0$. Thus, if X and Y are two random variables such that $X \geq Y$ then $E(X) \geq E(Y)$.

Proof.
We prove the result for the continuous case. We have

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{0}^{\infty} xf(x)dx \geq 0 \end{aligned}$$

since $f(x) \geq 0$ so the integrand is non-negative. Now, if $X \geq Y$ then $X - Y \geq 0$ so that by the previous proposition we can write $E(X) - E(Y) = E(X - Y) \geq 0$ ■

As a direct application of the monotonicity property we have

Proposition 46.3 *(Boole's Inequality)*
For any events $A_1, A_2, \cdots, A_n$ we have

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathrm{P}(A_i).$$

Proof.
For $i = 1, \cdots, n$ define

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$X = \sum_{i=1}^{n} X_i$$

so X denotes the number of the events A_i that occur. Also, let

$$Y = \begin{cases} 1 & \text{if } X \geq 1 \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

so Y is equal to 1 if at least one of the A_i occurs and 0 otherwise. Clearly, $X \geq Y$ so that $E(X) \geq E(Y)$. But

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P(A_i)$$

and

$$E(Y) = \mathrm{P}\{ \text{ at least one of the } A_i \text{ occur } \} = \mathrm{P}\left(\textstyle\bigcup_{i=1}^{n} A_i\right).$$

Thus, the result follows ■

Proposition 46.4
If X is a random variable with range $[a, b]$ then $a \leq E(X) \leq b$.

Proof.
Let $Y = X - a \geq 0$. Then $E(Y) \geq 0$. But $E(Y) = E(X) - E(a) = E(X) - a \geq 0$. Thus, $E(X) \geq a$. Similarly, let $Z = b - X \geq 0$. Then $E(Z) = b - E(X) \geq 0$ or $E(X) \leq b$ ■

We have determined that the expectation of a sum is the sum of the expectations. The same is not always true for products: in general, the expectation of a product need not equal the product of the expectations. But it is true in an important special case, namely, when the random variables are independent.

Proposition 46.5
If X and Y are independent random variables then for any functions h and g we have

$$E(g(X)h(Y)) = E(g(X))E(h(Y)).$$

In particular, $E(XY) = E(X)E(Y)$.

Proof.
We prove the result for the continuous case. The proof of the discrete case is similar. Let X and Y be two independent random variables with joint density function $f_{XY}(x, y)$. Then

$$\begin{aligned} E(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \left(\int_{-\infty}^{\infty} h(y)f_Y(y)dy\right)\left(\int_{-\infty}^{\infty} g(x)f_X(x)dx\right) \\ &= E(h(Y))E(g(X)) \ \blacksquare \end{aligned}$$

We next give a simple example to show that the expected values need not multiply if the random variables are not independent.

Example 46.4
Consider a single toss of a coin. We define the random variable X to be 1 if heads turns up and 0 if tails turns up, and we set $Y = 1 - X$. Thus X and Y are dependent. Show that $E(XY) \neq E(X)E(Y)$.

Solution.
Clearly, $E(X) = E(Y) = \frac{1}{2}$. But $XY = 0$ so that $E(XY) = 0 \neq E(X)E(Y)$ $\blacksquare$

Example 46.5
Suppose a box contains 10 green, 10 red and 10 black balls. We draw 10

balls from the box by sampling with replacement. Let X be the number of green balls, and Y be the number of black balls in the sample.
(a) Find $E(XY)$.
(b) Are X and Y independent? Explain.

Solution.
First we note that X and Y are binomial with $n = 10$ and $p = \frac{1}{3}$.
(a) Let X_i be 1 if we get a green ball on the i^{th} draw and 0 otherwise, and Y_j be the event that in j^{th} draw we got a black ball. Trivially, X_i and Y_j are independent if $1 \leq i \neq j \leq 10$. Moreover, $X_iY_i = 0$ for all $1 \leq i \leq 10$. Since $X = X_1 + X_2 + \cdots X_{10}$ and $Y = Y_1 + Y_2 + \cdots Y_{10}$ we have

$$XY = \sum \sum_{1 \leq i \neq j \leq 10} X_iY_j.$$

Hence,

$$E(XY) = \sum \sum_{1 \leq i \neq j \leq 10} E(X_iY_j) = \sum \sum_{1 \leq i \neq j \leq 10} E(X_i)E(Y_j) = 90 \times \frac{1}{3} \times \frac{1}{3} = 10.$$

(b) Since $E(X) = E(Y) = np = \frac{10}{3}$, we have $E(XY) \neq E(X)E(Y)$ so X and Y are dependent ■

The following inequality will be of importance in the next section.

Proposition 46.6 *(Markov's Inequality)*
If $X \geq 0$ and $c > 0$ then $\mathrm{P}(X \geq c) \leq \frac{E(X)}{c}$.

Proof.
Let $c > 0$. Define

$$I = \begin{cases} 1 & \text{if } X \geq c \\ 0 & \text{otherwise.} \end{cases}$$

Since $X \geq 0$, we have $I \leq \frac{X}{c}$. Taking expectations of both side we find $E(I) \leq \frac{E(X)}{c}$. Now the result follows since $E(I) = \mathrm{P}(X \geq c)$ ■

Example 46.6
Let X be a non-negative random variable. Let a be a positive constant. Prove that $\mathrm{P}(X \geq a) \leq \frac{E(e^{tX})}{e^{ta}}$ for all $t \geq 0$.

Solution.
Applying Markov's inequality we find

$$\mathrm{P}(X \geq a) = \mathrm{P}(tX \geq ta) = \mathrm{P}(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}} \blacksquare$$

As an important application of the previous result we have

Proposition 46.7
If $X \geq 0$ and $E(X) = 0$ then $\mathrm{P}(X = 0) = 1$.

Proof.
Since $E(X) = 0$, by the previous result we find $\mathrm{P}(X \geq c) = 0$ for all $c > 0$. But

$$\mathrm{P}(X > 0) = \mathrm{P}\left(\bigcup_{n=1}^{\infty}(X > \frac{1}{n})\right) \leq \sum_{n=1}^{\infty}\mathrm{P}(X > \frac{1}{n}) = 0.$$

Hence, $P(X > 0) = 0$. Since $X \geq 0$, we have $1 = \mathrm{P}(X \geq 0) = \mathrm{P}(X = 0) + \mathrm{P}(X > 0) = \mathrm{P}(X = 0)$ $\blacksquare$

Corollary 46.1
Let X be a random variable. If $Var(X) = 0$, then $\mathrm{P}(X = E(X)) = 1$. That is, $X = E(X)$.

Proof.
Suppose that $Var(X) = 0$. Since $(X - E(X))^2 \geq 0$ and $Var(X) = E((X - E(X))^2)$, by the previous result we have $P(X - E(X) = 0) = 1$. That is, $\mathrm{P}(X = E(X)) = 1$ $\blacksquare$

Example 46.7 *(Expected value of a Binomial Random Variable)*
Let X be a binomial random variable with parameters (n, p). Find $E(X)$.

Solution.
We have that X is the number of successes in n trials. For $1 \leq i \leq n$, let X_i denote the number of successes in the i^{th} trial. Then $E(X_i) = 0(1-p)+1p = p$. Since $X = X_1 + X_2 + \cdots + X_n$, we find $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np$ $\blacksquare$

Practice Problems

Problem 46.1
Let X and Y be independent random variables, both being equally likely to be any of the numbers $1, 2, \cdots, m$. Find $E(|X - Y|)$.

Problem 46.2
Let X and Y be random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 1 & 0 < x < 1, x < y < x + 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(XY)$.

Problem 46.3
Let X and Y be two independent uniformly distributed random variables in [0,1]. Find $E(|X - Y|)$.

Problem 46.4
Let X and Y be continuous random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 2(x + y) & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X^2Y)$ and $E(X^2 + Y^2)$.

Problem 46.5
Suppose that $E(X) = 5$ and $E(Y) = -2$. Find $E(3X + 4Y - 7)$.

Problem 46.6
Suppose that X and Y are independent, and that $E(X) = 5$, $E(Y) = -2$. Find $E[(3X - 4)(2Y + 7)]$.

Problem 46.7
Let X and Y be two independent random variables that are uniformly distributed on the interval $(0, L)$. Find $E(|X - Y|)$.

Problem 46.8
Ten married couples are to be seated at five different tables, with four people at each table. Assume random seating, what is the expected number of married couples that are seated at the same table?

Problem 46.9
John and Katie randomly, and independently, choose 3 out of 10 objects. Find the expected number of objects
(a) chosen by both individuals.
(b) not chosen by either individual.
(c) chosen exactly by one of the two.

Problem 46.10
If $E(X) = 1$ and Var(X) = 5 find
(a) $E[(2 + X)^2]$
(b) $\text{Var}(4 + 3X)$

Problem 46.11 ‡
Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6, 0 < t_2 < 6, t_1 + t_2 < 10$, and zero otherwise.
Determine $E[T_1+T_2]$, the expected time between a car accident and payment of the claim.

Problem 46.12 ‡
Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \leq t_1 \leq t_2 \leq L$, where L is a positive constant.
Determine the expected value of the sum of the squares of T_1 and T_2.

Problem 46.13
Let X and Y be two independent random variables with $\mu_X = 1, \mu_Y = -1, \sigma_X^2 = \frac{1}{2}$, and $\sigma_Y^2 = 2$. Compute $E[(X + 1)^2(Y - 1)^2]$.

Problem 46.14 ‡
A machine consists of two components, whose lifetimes have the joint density function

$$f(x, y) = \begin{cases} \frac{1}{50} & \text{for } x > 0, y > 0, x + y < 10 \\ 0 & \text{otherwise.} \end{cases}$$

The machine operates until both components fail. Calculate the expected operational time of the machine.

Problem 46.15 ‡
A city with borders forming a square with sides of length 1 has its city hall located at the origin when a rectangular coordinate system is imposed on the city so that two sides of the square are on the positive axes. The density function of the population is

$$f(x,y) = \begin{cases} 1.5(x^2+y^2) & \text{for } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

A resident of the city can travel to the city hall only along a route whose segments are parallel to the city borders.
Calculate the expected value of the travel distance to the city hall of a randomly chosen resident of the city.

Problem 46.16
Let X and Y be independent random variables and α, β, and γ be arbitrary constants. Show that

$$\text{Var}(\alpha X + \beta Y + \gamma) = \alpha^2 \text{Var}(X) + \beta^2 \text{Var}(Y).$$

Problem 46.17 ‡
The profit for a new product is given by $Z = 3X - Y - 5$ where X and Y are independent random variables with $\text{Var}(X) = 1$ and $\text{Var}(Y) = 2$.
Calculate $\text{Var}(Z)$.

Problem 46.18 ‡
Two random variables X and Y have the joint density function

$$f(x,y) = \begin{cases} kx & \text{for } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that X and Y are independent.

Problem 46.19 ‡
Let X and Y be two random variables with joint density function

$$f(x,y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \le x \le 1, x \le y \le 2x \\ 0 & \text{otherwise.} \end{cases}$$

Show that X and Y are dependent.

Problem 46.20 ‡
Two claimants place calls simultaneously to an insurer's claims call center. The times X and Y, in minutes, that elapse before the respective claimants get to speak with call center representatives are independently and identically distributed. The moment generating function of each random variable is

$$M(t) = \left(\frac{1}{1-1.5t}\right)^2, \ t < \frac{2}{3}.$$

Find the standard deviation of $X + Y$.

Problem 46.21 ‡
The return on two investments, X and Y, follows the joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}, & 0 < |x| + |y| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(X)$.

Problem 46.22 ‡
Random variables $X \geq 0$ and $Y \geq 0$ are uniformly distributed on the region bounded by the x and y axes, and the curve $y = 1 - x^2$. Calculate $E(XY)$.

Problem 46.23 ‡
A dental insurance company pays 100% of the cost of fillings and 70% of the cost of root canals. Fillings and root canals cost 50 and 500 each, respectively. The tables below show the probability distributions of the annual number of fillings and annual number of root canals for each of the company's policyholders.

# of fillings	0	1	2	3
Probability	0.60	0.20	0.15	0.05

# of root canals	0	1
Probability	0.80	0.20

Calculate the expected annual payment per policyholder for fillings and root canals.

Problem 46.24 ‡
Let X denote the loss amount sustained by an insurance company's policyholder in an auto collision. Let Z denote the portion of X that the insurance

company will have to pay. An actuary determines that X and Z are independent with respective density and probability functions

$$f(x) = \begin{cases} \frac{1}{8}e^{-\frac{x}{8}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$P(Z = z) = \begin{cases} 0.45, & z = 1 \\ 0.55, & \text{otherwise.} \end{cases}$$

Calculate the variance of the insurance company's claim payment ZX.

Problem 46.25 ‡

An insurance company will cover losses incurred from tornadoes in a single calendar year. However, the insurer will only cover losses for a maximum of three separate tornadoes during this time frame. Let X be the number of tornadoes that result in at least 50 million in losses, and let Y be the total number of tornadoes. The joint probability function for X and Y is

$$p_{XY}(x, y) = \begin{cases} c(x + 2y), & x = 0, 1, 2, 3,\ y = 0, 1, 2, 3,\ x \leq y \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant. Calculate the expected number of tornadoes that result in fewer than 50 million in losses.

47 Covariance and Variance of Sums

So far, We have discussed the absence or presence of a relationship between two random variables, i.e., independence or dependence. But if there is in fact a relationship, the relationship may be either weak or strong. In this section, we introduce a measure that quantifies this difference in the strength of a relationship between two random variables.
The **Covariance** between X and Y is defined by

$$\mathrm{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))].$$

Thus, if $\mathrm{Cov}(X,Y) > 0$ and X is above (respectively below) its mean then by the monotonicity of the expectation Y is also above (respectively below) its mean. Likewise, if $\mathrm{Cov}(X,Y) < 0$ and X is above (respectively below) its mean then Y is below (respectively above) its mean.
An alternative expression of covariance that is sometimes more convenient is

$$\begin{aligned}\mathrm{Cov}(X,Y) =& E(XY - E(X)Y - XE(Y) + E(X)E(Y)) \\ =& E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\ =& E(XY) - E(X)E(Y).\end{aligned}$$

Recall that for independent X, Y we have $E(XY) = E(X)E(Y)$ and so $\mathrm{Cov}(X,Y) = 0$. However, the converse statement is false as there exist random variables that have covariance 0 but are dependent. For example, let X be a random variable such that

$$\mathrm{P}(X = 0) = \mathrm{P}(X = 1) = \mathrm{P}(X = -1) = \frac{1}{3}$$

and define

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Thus, Y depends on X.
Clearly, $XY = 0$ so that $E(XY) = 0$. Also,

$$E(X) = (0 + 1 - 1)\frac{1}{3} = 0$$

and thus

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$$

Useful facts of Covariance are collected in the next result.

Theorem 47.1
(a) $\text{Cov}(X,Y) = \text{Cov}(Y,X)$ (Symmetry)
(b) $\text{Cov}(X,X) = \text{Var}(X)$
(c) $\text{Cov}(aX,Y) = a\text{Cov}(X,Y)$
(d) $\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \text{Cov}(X_i,Y_j)$

Proof.
(a) $\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = E(YX) - E(Y)E(X) = \text{Cov}(Y,X)$.
(b) $\text{Cov}(X,X) = E(X^2) - (E(X))^2 = Var(X)$.
(c) $\text{Cov}(aX,Y) = E(aXY) - E(aX)E(Y) = aE(XY) - aE(X)E(Y) = a(E(XY) - E(X)E(Y)) = a\text{Cov}(X,Y)$.
(d) First note that $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i)$ and $E\left[\sum_{j=1}^{m} Y_j\right] = \sum_{j=1}^{m} E(Y_j)$. Then

$$\begin{aligned}
\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) &= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} E(X_i)\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} E(Y_j)\right)\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - E(X_i))\sum_{j=1}^{m}(Y_j - E(Y_j))\right] \\
&= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i - E(X_i))(Y_j - E(Y_j))\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} E[(X_i - E(X_i))(Y_j - E(Y_j))] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \text{Cov}(X_i,Y_j) \blacksquare
\end{aligned}$$

Example 47.1
Given that $E(X) = 5, E(X^2) = 27.4, E(Y) = 7, E(Y^2) = 51.4$ and $\text{Var}(X+Y) = 8$, find $\text{Cov}(X+Y, X+1.2Y)$.

Solution.
By definition,

$$\text{Cov}(X+Y, X+1.2Y) = E((X+Y)(X+1.2Y)) - E(X+Y)E(X+1.2Y).$$

Using the properties of expectation and the given data, we get

$$\begin{aligned} E(X+Y)E(X+1.2Y) =& (E(X)+E(Y))(E(X)+1.2E(Y)) \\ =& (5+7)(5+(1.2)\cdot 7) = 160.8 \\ E((X+Y)(X+1.2Y)) =& E(X^2)+2.2E(XY)+1.2E(Y^2) \\ =& 27.4+2.2E(XY)+(1.2)(51.4) \\ =& 2.2E(XY)+89.08. \end{aligned}$$

Thus,

$$\mathrm{Cov}(X+Y, X+1.2Y) = 2.2E(XY)+89.08-160.8 = 2.2E(XY)-71.72.$$

To complete the calculation, it remains to find $E(XY)$. To this end, we make use of the still unused relation $\mathrm{Var}(X+Y)=8$. We have,

$$\begin{aligned} 8 =& \mathrm{Var}(X+Y) = E((X+Y)^2)-(E(X+Y))^2 \\ =& E(X^2)+2E(XY)+E(Y^2)-(E(X)+E(Y))^2 \\ =& 27.4+2E(XY)+51.4-(5+7)^2 = 2E(XY)-65.2. \end{aligned}$$

so $E(XY)=36.6$. Substituting this above gives $\mathrm{Cov}(X+Y, X+1.2Y) = (2.2)(36.6)-71.72 = 8.8$ ■

Example 47.2
Show that if X and Y are independent then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y).$$

Solution.
We have

$$\begin{aligned} \mathrm{Var}(X+Y) =& E[(X+Y)^2]-(E(X+Y))^2 = E(X^2+2XY+Y^2)-(E(X)+E(Y))^2 \\ =& E(X^2+2XY+Y^2)-E(X)^2-2E(X)E(Y)-E(Y)^2 \\ =& [E(X^2)-E(X)^2]+[E(Y^2)-E(Y)^2]+2[E(XY)-E(X)E(Y)] \\ =& [E(X^2)-E(X)^2]+[E(Y^2)-E(Y)^2]+2[E(X)E(Y)-E(X)E(Y)] \\ =& \mathrm{Var}(X)+\mathrm{Var}(Y) \ \blacksquare \end{aligned}$$

Practice Problems

Problem 47.1
If X and Y are independent and identically distributed with mean μ and variance σ^2, find $E[(X-Y)^2]$.

Problem 47.2
Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\text{Cov}(X,Y)$.

Problem 47.3
Suppose that X and Y are random variables with $\text{Cov}(X,Y) = 3$. Find $\text{Cov}(2X-5, 4Y+2)$.

Problem 47.4
Let X and Y be two random variables. Show that

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y).$$

Problem 47.5
A salesperson salary consists of two parts a commission, X, and a fixed income Y. so that the total salary is $X+Y$. Suppose that $\text{Var}(X) = 5,000$, $\text{Var}(Y) = 10,000$, and $\text{Var}(X+Y) = 17,000$.
If X is increased by a flat amount of 100, and Y is increased by 10%, what is the variance of the total salary after these increases?

Problem 47.6
Suppose the joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} 1 & 0 < x < 1,\ x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the covariance of X and Y.

Problem 47.7
Let X and Z be independent random variables with X uniformly distributed on $(-1,1)$ and Z uniformly distributed on $(0, 0.1)$. Let $Y = X^2 + Z$. Then X and Y are dependent.
(a) Find the joint pdf of X and Y.
(b) Find the covariance of X and Y.

Problem 47.8
Let the random variable Θ be uniformly distributed on $[0, 2\pi]$. Consider the random variables $X = \cos\Theta$ and $Y = \sin\Theta$. Show that $\text{Cov}(X, Y) = 0$ even though X and Y are dependent. Thus, sometimes there might be strong relationship between X and Y even though $\text{Cov}(X, Y) = 0$.

Problem 47.9
Let X and Y be continuous random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 3x & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X, Y)$.

Problem 47.10 ‡
An insurance policy pays a total medical benefit consisting of two parts for each claim. Let X represent the part of the benefit that is paid to the surgeon, and let Y represent the part that is paid to the hospital. The variance of X is 5000, the variance of Y is 10,000, and the variance of the total benefit, $X + Y$, is 17,000.
Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim.
Calculate the variance of the total benefit after these revisions have been made.

Problem 47.11 ‡
A joint density function is given by

$$f_{XY}(x, y) = \begin{cases} kx & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X, Y)$

Problem 47.12 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} \frac{8}{3}xy & 0 \leq x \leq 1, x \leq y \leq 2x \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X, Y)$

Problem 47.13 ‡
Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval (0, 12) . Given $X = x$, Y is uniformly distributed on the interval $(0, x)$.
Determine $\text{Cov}(X, Y)$ according to this model.

Problem 47.14 ‡
Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X) = 5, E(X^2) = 27.4, E(Y) = 7, E(Y^2) = 51.4$, and $\text{Var}(X + Y) = 8$.
Let $C_1 = X+Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge.
Calculate $\text{Cov}(C_1, C_2)$.

Problem 47.15
The following table gives the joint probability distribution of two random variables X and Y.

X \ Y	0	1	2
0	0.25	0.08	0.05
1	0.12	0.20	0.10
2	0.03	0.07	0.10

(a) Give the marginal distributions of X and Y.
(b) Find $E(X)$ and $E(Y)$.
(c) Find $\text{Cov}(X, Y)$.
(d) Find $E(100X + 75Y)$.

Problem 47.16
Let X, Y and Z be random variables with means 1,2 and 3, respectively, and variances 4,5, and 9, respectively. Also, $\text{Cov}(X, Y) = 2, \text{Cov}(X, Z) = 3$, and $\text{Cov}(Y, Z) = 1$. What are the mean and variance, respectively, of the random variable $W = 3X + 2Y - Z$?

Problem 47.17
Let X_1, X_2, X_3 be uniform random variables on the interval $(0, 1)$ with $\text{Cov}(X_i, X_j) = \frac{1}{24}$ for $i, j \in \{1, 2, 3\},\ i \neq j$. Calculate the variance of $X_1 + 2X_2 - X_3$.

Problem 47.18
Let X and Y be discrete random variables with joint distribution defined by the following table

Y\ X	2	3	4	5	$p_Y(y)$
0	0.05	0.05	0.15	0.05	0.30
1	0.40	0	0	0	0.40
2	0.05	0.15	0.10	0	0.30
$p_X(x)$	0.50	0.20	0.25	0.05	1

For this joint distribution, $E(X) = 2.85,\ E(Y) = 1$. Calculate $\text{Cov}(X, Y)$.

Problem 47.19
Two cards are drawn without replacement from a pack of cards. The random variable X measures the number of heart cards drawn, and the random variable Y measures the number of club cards drawn. Find the covariance of X and Y.

Problem 47.20 ‡
The joint probability density function of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} \frac{x+y}{8} & 0 < x < 2, 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(0.5(X + Y))$.

Problem 47.21 ‡
Points scored by a game participant can be modeled by $Z = 3X + 2Y - 5$. X and Y are independent random variables with $\text{Var}(X) = 3$ and $\text{Var}(Y) = 4$. Calculate $\text{Var}(Z)$.

Problem 47.22 ‡
For a random variable X, we define the **coefficient of variation** to be the number $CV(X) = \frac{\sigma_X}{E(X)}$. Two independent random variables X and Y have the same mean. The coefficients of variation of X and Y are 3 and 4 respectively. Calculate the coefficient of variation of $0.5(X + Y)$.

Problem 47.23 ‡
Let X be a random variable that takes on the values $-1, 0$, and 1 with equal probabilities. Let $Y = X^2$. Which of the following is true?

(A) $\text{Cov}(X,Y) > 0$; the random variables X and Y are dependent.
(B) $\text{Cov}(X,Y) > 0$; the random variables X and Y are independent.
(C) $\text{Cov}(X,Y) = 0$; the random variables X and Y are dependent.
(D) $\text{Cov}(X,Y) = 0$; the random variables X and Y are independent.
(E) $\text{Cov}(X,Y) < 0$; the random variables X and Y are dependent.

Problem 47.24 ‡
Annual windstorm losses, X and Y, in two different regions are independent, and each is uniformly distributed on the interval [0,10]. Calculate the covariance of X and Y, given that $X + Y < 10$.

48 The Coefficient of Correlation

In this section, we introduce a number, called the coefficient of correlation, that measures the linear dependence of two random variables. For this purpose, let X and Y be two random variables with $\text{Var}(X) > 0$ and $\text{Var}(Y) > 0$. We define the **coefficient of correlation** to be the number

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

This number satisfies the following property.

Theorem 48.1
The coefficient of correlation is a number between -1 and 1 inclusive. That is, $|\rho(X,Y)| \leq 1$.

Proof.
We first show that $\rho(X,Y) \geq -1$. Let $\sigma_X^2 = \text{Var}(X)$ and $\sigma_Y^2 = \text{Var}(Y)$. We have

$$\begin{aligned} 0 \leq & Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \\ = & \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} + \frac{2Cov(X,Y)}{\sigma_X \sigma_Y} \\ = & 2[1 + \rho(X,Y)] \end{aligned}$$

implying that $-1 \leq \rho(X,Y)$. Likewise, to show that $\rho(X,Y) \leq 1$, we proceed as follows

$$\begin{aligned} 0 \leq & Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) \\ = & \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} - \frac{2Cov(X,Y)}{\sigma_X \sigma_Y} \\ = & 2[1 - \rho(X,Y)] \end{aligned}$$

implying that $\rho(X,Y) \leq 1$ ■

Example 48.1
Show that $\rho(X,Y) = \pm 1$ if and only if $Y = aX + b$ for some constants a and b.

Solution.
First notice that $\rho(X,Y) = \pm 1$ is equivalent to $\text{Cov}(X,Y)^2 = \text{Var}(X)\text{Var}(Y)$. If $\rho(X,Y) = 1$ then $\text{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 0$. This implies that $\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = E\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right)$ (See Corollary 46.1.)This is equivalent to $Y = a+bX$ where $a = -\sigma_Y E\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right)$ and $b = \frac{\sigma_Y}{\sigma_X} > 0$. If $\rho(X,Y) = -1$then $\text{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = 0$. This implies that $\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = E\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)$ which is equivalent to $Y = a+bX$ where $a = \sigma_Y E\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right)$ and $b = -\frac{\sigma_Y}{\sigma_X} < 0$.
Conversely, suppose that $Y = a + bX$. Then $\text{Var}(Y) = b^2\text{Var}(X)$ and

$$\begin{aligned}\text{Cov}(X,Y) =& E[X(a+bX)] - E(X)E(a+bX) = aE(X) + bE(X^2) - aE(X) - bE(X)^2 \\ =& b\text{Var}(X).\end{aligned}$$

Hence,

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{b\text{Var}(X)}{|b|\text{Var}(X)} = \frac{b}{|b|}.$$

If $b > 0$ then $\rho(X,Y) = 1$ and if $b < 0$ then $\rho(X,Y) = -1$ ■

As stated in the introduction to this section, the correlation coefficient is a measure of the degree of linearity between X and Y. A value of $\rho(X,Y)$ near $+1$ or -1 indicates a high degree of linearity between X and Y, whereas a value near 0 indicates a lack of such linearity.
Correlation is a scaled version of covariance; note that the two concepts always have the same sign (positive, negative, or 0). When the sign is positive, the variables X and Y are said to be **positively correlated** and this indicates that Y tends to increase when X does; when the sign is negative, the variables are said to be **negatively correlated** and this indicates that Y tends to decrease when X increases; and when the sign is 0, the variables are said to be **uncorrelated**.
Figure 48.1 shows some examples of data pairs and their correlation.

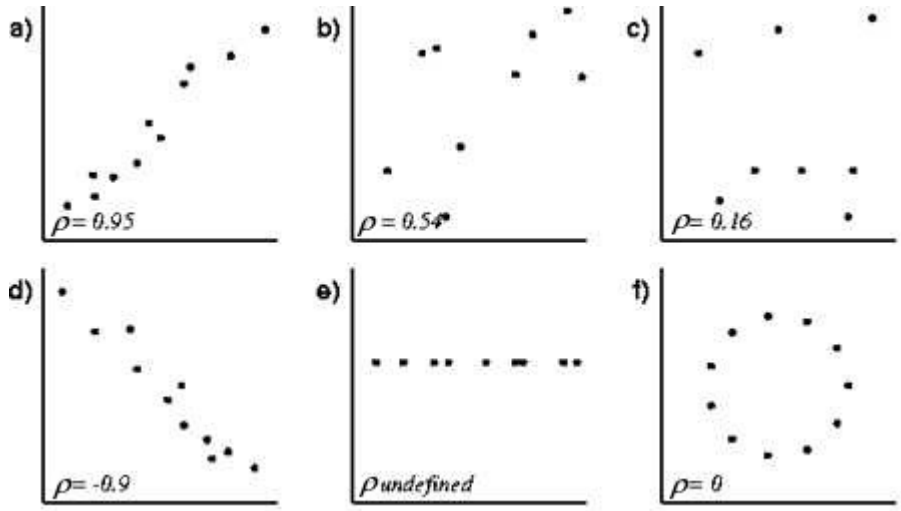


Figure 48.1

Example 48.2
Suppose the joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} 1 & 0 < x < 1,\ x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the covariance and correlation of X and Y..

Solution.
We first find the marginal pdf's for X and Y. Since

$$f_X(x) = \int_x^{x+1} dy = 1,\ 0 < x < 1$$

then X is uniform on (0,1) and therefore

$$E(X) = \tfrac{1}{2} \quad \text{and} \quad \text{Var(X)}=\tfrac{1}{12}$$

Now, if $0 < y < 1$

$$f_Y(y) = \int_0^y dx = y,$$

and if $1 < y < 2$

$$f_Y(y) = \int_{y-1}^1 dx = 2 - y.$$

and so

$$E(Y) = \int_0^1 y^2 dy + \int_1^2 y(2-y)dy = 1$$

and

$$E(Y^2) = \int_0^1 y^3 dy + \int_1^2 y^2(2-y)dy = \frac{7}{6}.$$

Thus,

$$Var(Y) = \frac{7}{6} - 1 = \frac{1}{6}.$$

Also,

$$E(XY) = \int_0^1 \int_x^{x+1} xydydx = \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \frac{7}{12}.$$

Hence,

$$Cov(X,Y) = \frac{7}{12} - \frac{1}{2} = \frac{1}{12}$$

and

$$\rho(X,Y) = \frac{1/12}{\sqrt{1/12}\sqrt{1/6}} = \frac{\sqrt{2}}{2} \blacksquare$$

Practice Problems

Problem 48.1
Two cards are drawn without replacement from a pack of cards. The random variable X measures the number of heart cards drawn, and the random variable Y measures the number of club cards drawn. Find the coefficient of correlation of X and Y.

Problem 48.2
Let X and Z be independent random variables with X uniformly distributed on $(-1, 1)$ and Z uniformly distributed on $(0, 0.1)$. Let $Y = X^2 + Z$. Then X and $Y$are dependent.
(a) Find the joint pdf of X and Y.
(b) Find the the coefficient of correlation of X and Y.

Problem 48.3
If X_1, X_2, X_3, X_4 are (pairwise) uncorrelated random variables each having mean 0 and variance 1, compute the correlations of
(a) $X_1 + X_2$ and $X_2 + X_3$.
(b) $X_1 + X_2$ and $X_3 + X_4$.

Problem 48.4
Let X be uniformly distributed on $[-1, 1]$ and $Y = X^2$. Show that X and Y are uncorrelated even though Y depends functionally on X (the strongest form of dependence).

Problem 48.5
Let X and Y be continuous random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 3x & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y)$ and $\rho(X, Y)$.

Problem 48.6 ‡
A joint density function is given by

$$f_{XY}(x, y) = \begin{cases} kx & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\rho(X, Y)$.

Problem 48.7
Let X_1, X_2, and X_3 be independent random variables each with mean 0 and variance 1. Let $X = 2X_1 - X_3$ and $Y = 2X_2 + X_3$. Find $\rho(X, Y)$.

Problem 48.8
The coefficient of correlation between random variables X and Y is $\frac{1}{3}$, and $\sigma_X^2 = a, \sigma_Y^2 = 4a$. The random variable Z is defined to be $Z = 3X - 4Y$, and it is found that $\sigma_Z^2 = 114$. Find a.

Problem 48.9
Given n independent random variables $X_1, X_2, \cdots, X_n$ each having the same variance σ^2. Define $U = 2X_1 + X_2 + \cdots + X_{n-1}$ and $V = X_2 + X_3 + \cdots + X_{n-1} + 2X_n$. Find $\rho(U, V)$.

Problem 48.10
The following table gives the joint probability distribution of two random variables X and Y.

X \ Y	1	2	3	$p_X(x)$
1	0.25	0.25	0.00	0.5
2	0.00	0.25	0.25	0.5
$p_Y(y)$	0.25	0.5	0.25	1

Find $\rho(X, Y)$.

Problem 48.11
Let X and Y be two random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} 2 & 0 < x \leq y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\rho(X, Y)$.

Problem 48.12
The following table gives the joint probability distribution of two random variables X and Y.

Y \ X	1	2	3	$p_Y(y)$
1	0.01	0.02	0.25	0.28
2	0.02	0.03	0.20	0.25
3	0.02	0.10	0.05	0.17
4	0.15	0.10	0.05	0.30
$p_X(x)$	0.20	0.25	0.55	1

Find $\rho(X, Y)$.

Problem 48.13 ‡
An actuary analyzes a company's annual personal auto claims, M, and annual commercial auto claims, N. The analysis reveals that $\text{Var}(M) = 1600$, $\text{Var}(N) = 900$, and the correlation between M and N is 0.64. Calculate $\text{Var}(M + N)$.

49 Conditional Expectation

Since conditional probability measures are probability measures (that is, they possess all of the properties of unconditional probability measures), conditional expectations inherit all of the properties of regular expectations.
Let X and Y be random variables. In the discrete case, we define **conditional expectation** of X given that $Y = y$ by

$$E(X|Y = y\} = \sum_x x\mathrm{P}(X = x|Y = y) = \sum_x x p_{X|Y}(x|y)$$

where $p_{X|Y}$ is the conditional probability mass function of X, given that $Y = y$ which is given by

$$p_{X|Y}(x|y) = \mathrm{P}(X = x|Y = y) = \frac{p_{XY}(x, y)}{p_Y(y)}.$$

In the continuous, case we have

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

where

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

Example 49.1
Suppose X and Y are discrete random variables with values 1, 2, 3, 4 and joint pmf given by

$$p_{XY}(x, y) = \begin{cases} \frac{1}{16} & \text{if } x = y \\ \frac{2}{16} & \text{if } x < y \\ 0 & \text{if } x > y \end{cases}$$

for $x, y = 1, 2, 3, 4$.
(a) Find the joint probability distribution of X and Y in tabular form.
(b) Find the conditional expectation of Y given that $X = 3$.

Solution.
(a) The joint probability distribution is given in tabular form

X \ Y	1	2	3	4	$p_X(x)$
1	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{7}{16}$
2	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{5}{16}$
3	0	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$
4	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$
$p_Y(y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	1

(b) We have

$$E(Y|X=3) = \sum_{y=1}^{4} y p_{Y|X}(y|3) = \frac{p_{XY}(3,1)}{p_X(3)} + \frac{2p_{XY}(3,2)}{p_X(3)} + \frac{3p_{XY}(3,3)}{p_X(3)} + \frac{4p_{XY}(3,4)}{p_X(3)}$$
$$= 3 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} = \frac{11}{3} \blacksquare$$

Example 49.2
Suppose that the joint density of X and Y is given by

$$f_{XY}(x,y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y}, \quad x, y > 0.$$

Compute $E(X|Y=y)$.

Solution.
The conditional density is found as follows

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx} = \frac{(1/y)e^{-\frac{x}{y}}e^{-y}}{\int_0^{\infty}(1/y)e^{-\frac{x}{y}}e^{-y}dx}$$
$$= \frac{(1/y)e^{-\frac{x}{y}}}{\int_0^{\infty}(1/y)e^{-\frac{x}{y}}dx} = \frac{1}{y}e^{-\frac{x}{y}}.$$

Hence,

$$E(X|Y=y) = \int_0^{\infty} \frac{x}{y} e^{-\frac{x}{y}} dx = -\left[xe^{-\frac{x}{y}} \Big|_0^{\infty} - \int_0^{\infty} e^{-\frac{x}{y}} dx \right]$$
$$= -\left[xe^{-\frac{x}{y}} + ye^{-\frac{x}{y}} \right]_0^{\infty} = y \blacksquare$$

Example 49.3
Let Y be a random variable with a density f_Y given by

$$f_Y(y) = \begin{cases} \frac{\alpha-1}{y^\alpha} & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 1$. Given $Y = y$, let X be a random variable which is Uniformly distributed on $(0, y)$.
(a) Find the marginal distribution of X.
(b) Calculate $E(Y|X = x)$ for every $x > 0$.

Solution.
The joint density function is given by

$$f_{XY}(x, y) = \begin{cases} \frac{\alpha-1}{y^{\alpha+1}} & 0 < x < y,\ y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Observe that X only takes positive values, thus $f_X(x) = 0,\ x \leq 0$. For $0 < x < 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_1^{\infty} f_{XY}(x, y)dy = \frac{\alpha - 1}{\alpha}.$$

For $x \geq 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_x^{\infty} f_{XY}(x, y)dy = \frac{\alpha - 1}{\alpha x^\alpha}.$$

(b) For $0 < x < 1$, we have

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{\alpha}{y^{\alpha+1}}, \quad y > 1.$$

Hence,

$$E(Y|X = x) = \int_1^{\infty} \frac{y\alpha}{y^{\alpha+1}}dy = \alpha \int_1^{\infty} \frac{dy}{y^\alpha} = \frac{\alpha}{\alpha - 1}.$$

If $x \geq 1$ then

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{\alpha x^\alpha}{y^{\alpha+1}}, \quad y > x.$$

Hence,

$$E(Y|X = x) = \int_x^{\infty} y\frac{\alpha x^\alpha}{y^{\alpha+1}}dy = \frac{\alpha x}{\alpha - 1} \ \blacksquare$$

Notice that if X and Y are independent then $p_{X|Y}(x|y) = p_X(x)$ so that $E(X|Y = y) = E(X)$.

Practice Problems

Problem 49.1
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X|Y=y)$ and $E(Y|X=x)$.

Problem 49.2
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} \frac{3y^2}{x^3} & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y|X=x)$.

Problem 49.3
Let X and Y be discrete random variables with conditional mass function

$$p_{Y|X}(y|2) = \begin{cases} 0.2 & y = 1 \\ 0.3 & y = 2 \\ 0.5 & y = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E(Y|X=2)$.

Problem 49.4
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} \frac{21}{4}x^2y & x^2 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y|X=x)$.

Problem 49.5
Let X and Y be discrete random variables with joint probability mass function defined by the following table

$X\backslash Y$	1	2	3	$p_X(x)$
1	1/9	1/9	0	2/9
2	1/3	0	1/6	1/2
3	1/9	1/18	1/9	5/18
$p_Y(y)$	5/9	1/6	5/18	1

Compute $E(X|Y = i)$ for $i = 1, 2, 3$. Are X and Y independent?

Problem 49.6 ‡
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$\begin{aligned}
\mathrm{P}(X = 0, Y = 0) &= 0.800 \\
\mathrm{P}(X = 1, Y = 0) &= 0.050 \\
\mathrm{P}(X = 0, Y = 1) &= 0.025 \\
\mathrm{P}(X = 1, Y = 1) &= 0.125.
\end{aligned}$$

Calculate $E(Y|X = 1)$.

Problem 49.7
The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f_{XY}(x, y) = \begin{cases} 2x & 0 < x < 1, x < y < x + 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional expectation of Y given that $X = x$?

Problem 49.8 ‡
An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

$X\backslash Y$	0	1	2	$p_X(x)$
0	0.12	0.13	0.05	0.30
1	0.06	0.15	0.15	0.36
2	0.05	0.12	0.10	0.27
3	0.02	0.03	0.02	0.07
$p_Y(y)$	0.25	0.43	0.32	1

where X is the number of tornadoes in county Q and Y that of county P. Calculate the conditional expectation of the annual number of tornadoes in county Q, given that there are no tornadoes in county P.

Problem 49.9
Let X and Y be two continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < x < 1$, find $\mathrm{E}(Y|X = x)$.

Problem 49.10 ‡
New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y) = \begin{cases} 0.50 & \text{for } 0 < x, y < 0.5 \\ 1.25 & \text{for } 0 < x < 0.5, 0.5 < y < 1 \\ 1.50 & \text{for } 0.5 < x < 1, 0 < y < 0.5 \\ 0.75 & \text{for } 0.5 < x < 1, 0.5 < y < 1. \end{cases}$$

Calculate $\mathrm{E}(Y|X = 0.75)$.

Problem 49.11 ‡
The number of severe storms that strike city J in a year follows a binomial distribution with $n = 5$ and $p = 0.6$. Given that m severe storms strike city J in a year, the number of severe storms that strike city K in the same year is: m with probability 1/2, $m + 1$ with probability 1/3, and $m + 2$ with probability 1/6.
Calculate the expected number of severe storms that strike city J in a year during which 5 severe storms strike city K.

Problem 49.12 ‡
Ten cards from a deck of playing cards are in a box: two diamonds, three spades, and five hearts. Two cards are randomly selected without replacement.
Calculate the expected number of diamonds selected, given that no spade is selected.

Problem 49.13
Let X and Y be two independent Poisson random variables with common parameter λ. Let $Z = X + Y$. Find $E(X|Z = z)$. Hint: Example 43.3 and Example 5.4.

Problem 49.14
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} cxy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
(b) Are X and Y independent?
(c) Find $E(Y|X = x)$.

Problem 49.15
A standard deck has 52 cards, with values $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$ (4 cards of each value). Choose 5 cards randomly from the deck, without replacement. Given that exactly 2 Jack cards (i.e., value "J") appear, find the expected number of Ace cards (i.e., value "A") that appear.

Problem 49.16
A machine has a random supply Y at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). It is not resupplied during the day; hence, $X \le Y$. It has been observed that X, Y have joint density

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & 0 \le x \le y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E(X|Y = y)$ for a fixed $0 \le y \le 3$.

Problem 49.17
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X|Y = 2)$.

Problem 49.18
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} ce^{-y} & 0 \leq x \leq y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
(b) Are X and Y independent?
(c) Find the marginal densities of X and Y.
(d) Find $E(Y|X = x)$.
(e) Find $E(X|Y = y)$.

Problem 49.19
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0 < x < 2, x < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
(b) Find the marginal density of X and and the conditional density of Y given X.
(c) Find $E(Y|X = x)$.

Problem 49.20 ‡
A fair die is rolled repeatedly. Let X be the number of rolls needed to obtain a 5 and Y the number of rolls needed to obtain a 6. Calculate $E(X|Y = 2)$.

Problem 49.21 ‡
A driver and a passenger are in a car accident. Each of them independently has probability 0.3 of being hospitalized. When a hospitalization occurs, the loss is uniformly distributed on $[0, 1]$. When two hospitalizations occur, the losses are independent.
Calculate the expected number of people in the car who are hospitalized, given that the total loss due to hospitalizations from the accident is less than 1.

Problem 49.22 ‡
The number of burglaries occurring on Burlington Street during a one-year period is Poisson distributed with mean 1. Calculate the expected number of burglaries on Burlington Street in a one-year period, given that there are at least two burglaries.

Problem 49.23 ‡
The lifetime of a machine part is exponentially distributed with a mean of five years. Calculate the mean lifetime of the part, given that it survives less than ten years.

Problem 49.24 ‡
An individual experiences a loss due to property damage and a loss due to bodily injury. Losses are independent and uniformly distributed on the interval [0,3]. Calculate the expected loss due to bodily injury, given that at least one of the losses is less than 1.

Problem 49.25 ‡
Annual windstorm losses, X and Y, in two different regions are independent, and each is uniformly distributed on the interval [0,10]. Calculate the covariance of X and Y, given that $X + Y < 10$.

50 Double Expectation

Similar to the unconditional case, we want to define the conditional expectation of a function of a random variable. For this, let $g(x)$ be any function, the conditional expected value of g given $Y = y$ is, in the continuous case,

$$E(g(X)|Y = y) = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

if the integral exists. For the discrete case, we have a sum instead of an integral. That is, the conditional expectation of g given $Y = y$ is

$$E(g(X)|Y = y) = \sum_{x} g(x) p_{X|Y}(x|y).$$

The proof of this result is identical to the unconditional case. See Theorem 16.1 and Theorem 29.2.

Example 50.1
Let X be a random variable that is uniformly distributed in $(0, 1)$ and $Y|X = x$ be uniformly distributed in $(0, x)$. Find $E(Y^2|X = x)$.

Solution.
We have,

$$\begin{aligned} E(Y^2|X = x) &= \int_{-\infty}^{x} y^2 f_{Y|X}(y|x) dy = \int_{0}^{x} y^2 f_{Y|X}(y|x) dy \\ &= \int_{0}^{x} \frac{y^2}{x} dy = \frac{x^2}{3} \end{aligned}$$

for $0 < x < 1$ and 0 otherwise ■

Next, let $\phi_X(y) = E(X|Y = y)$ denote the function of the random variable Y whose value at $Y = y$ is $E(X|Y = y)$. Clearly, $\phi_X(y)$ is a random variable. We denote this random variable by $E(X|Y)$. The expectation of this random variable is just the expectation of X as shown in the following theorem.

Theorem 50.1 *(Double Expectation Property)*

$$E(X) = E(E(X|Y)).$$

Proof.
We give a proof in the case X and Y are continuous random variables.

$$\begin{aligned} E(E(X|Y)) &= \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \right) f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dx dy = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY}(x,y) dy dx \\ &= \int_{-\infty}^{\infty} x f_X(x) dx = E(X) \ \blacksquare \end{aligned}$$

In the case X and Y are discrete random variables, we have the version

$$E(X) = E(X|Y=y_1) p_Y(y_1) + E(X|Y=y_2) p_Y(y_2) + \cdots .$$

Example 50.2
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} \frac{3y^2}{x^3} & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y)$ in two different ways.

Solution.
The marginal density of X is

$$f_X(x) = \int_0^x \frac{3y^2}{x^3} dy = 1, \quad 0 < x < 1$$

and 0 otherwise. Likewise, the marginal density of Y is

$$f_Y(y) = \int_y^1 \frac{3y^2}{x^3} dx = \frac{3}{2}(1-y^2), \quad 0 < y < 1$$

and 0 otherwise. Hence,

$$E(Y) = \int_0^1 y f_Y(y) dy = \frac{3}{2} \int_0^1 y(1-y^2) dy = \frac{3}{8}.$$

Next, we find conditional density of Y given $X = x$

$$f_{Y|X}(x|y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{3y^2}{x^3}, \quad 0 < x < y < 1$$

and 0 otherwise. Hence,

$$E(Y|X=x) = \int_0^x \frac{3y^3}{x^3} dx = \frac{3}{4}x.$$

Thus,

$$E(Y) = E(E(Y|X)) = \int_0^1 E(Y|X=x) f_X(x) dx = \int_0^1 \frac{3}{4} x dx = \frac{3}{8} \blacksquare$$

Computing Probabilities by Conditioning

Suppose we want to know the probability of some event, A. Suppose also that knowing Y gives us some useful information about whether or not A occurred.

Define an indicator random variable

$$X = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Then

$$\mathrm{P}(A) = E(X)$$

and for any random variable Y

$$E(X|Y=y) = \mathrm{P}(A|Y=y).$$

Thus, by the double expectation property we have

$$\begin{aligned} \mathrm{P}(A) = & E(X) = \sum_y E(X|Y=y)\mathrm{P}(Y=y) \\ = & \sum_y \mathrm{P}(A|Y=y) p_Y(y) \end{aligned}$$

in the discrete case and

$$\mathrm{P}(A) = \int_{-\infty}^{\infty} \mathrm{P}(A|Y=y) f_Y(y) dy$$

in the continuous case.

Example 50.3
Let X be a random variable with density function

$$f_X(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y|X$ be a random variable that is uniformly distributed in $(0, x),\ x \geq 0$. Compute the probability of the even $E = \{Y \leq 2\}$ by conditioning on the value of X.

Solution.
We have

$$P(E) = \int_0^\infty P(E|X = x) f_X(x) dx.$$

Since

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

we can write

$$P(E|X = x) = \int_0^2 f_{Y|X}(y|x) dy = \begin{cases} 1 & 0 < x < 2 \\ \frac{2}{x} & x \geq 2. \end{cases}$$

Thus,

$$\begin{aligned} P(E) = \int_0^\infty P(E|X = x) f_X(x) dx &= \int_0^2 xe^{-x} dx + \int_2^\infty \frac{2}{x} xe^{-x} dx \\ &= 1 - e^{-2} \ \blacksquare \end{aligned}$$

Example 50.4
Let X and Y be two independent continuous random variables. Show that

$$P(X + Y < a) = \int_{-\infty}^\infty F_Y(a - x) f_X(x) dx.$$

Solution.
We have

$$\begin{aligned}
\mathrm{P}(X+Y<a) &= \int_{-\infty}^{\infty} \mathrm{P}(X+Y<a|X=x) f_X(x)dx \\
&= \int_{-\infty}^{\infty} \mathrm{P}(Y<a-X|X=x) f_X(x)dx \\
&= \int_{-\infty}^{\infty} \mathrm{P}(Y<a-x) f_X(x)dx \quad (\text{ by independence}) \\
&= \int_{-\infty}^{\infty} F_Y(a-x) f_X(x)dx \ \blacksquare
\end{aligned}$$

Practice Problems

Problem 50.1
Let $a(X)$ and $b(X)$ be random variables that are functions of X. Let Y be any random variable. Show that

$$E(a(X) + b(X)Y|X = x) = a(x) + b(x)E(Y|X = x).$$

Problem 50.2
Let X and Y be two random variables and $g : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function. Show that

$$E[E(g(X, Y)|X)] = E[g(X, Y)].$$

Problem 50.3
A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume that the miner is at all times equally likely to choose any one of the doors, find the expected time for him to get to safety.

Problem 50.4
Let Y be a uniform random variable on $(0, 1)$ and suppose that $(X|Y = y)$ is binomial with parameters (n, y). Find $P(X = 0)$.

Problem 50.5
Suppose that X and Y have joint distribution

$$f_{XY}(x, y) = \begin{cases} \frac{21}{4}x^2y & x^2 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y)$ in two ways.

Problem 50.6
Suppose that $E(X|Y) = 18 - \frac{3}{5}Y$ and $E(Y|X) = 10 - \frac{1}{3}X$. Find $E(X)$ and $E(Y)$.

Problem 50.7
Let X be an exponential random variable with $\lambda = 5$ and $Y|X$ a uniformly distributed random variable on $(-3, X)$. Find $E(Y)$.

Problem 50.8
In a mall, a survey found that the number of people who pass by JC Penney between 4:00 and 5:00 pm is a Poisson random variable with parameter $\lambda = 100$. Assume that each person may enter the store, independently of the other person, with a given probability $p = 0.15$. What is the expected number of people who enter the store during the given period?

Problem 50.9
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$\begin{aligned}
\mathrm{P}(X = 0, Y = 0) &= 0.800 \\
\mathrm{P}(X = 1, Y = 0) &= 0.050 \\
\mathrm{P}(X = 0, Y = 1) &= 0.025 \\
\mathrm{P}(X = 1, Y = 1) &= 0.125.
\end{aligned}$$

Calculate $E(Y|X = 1)$ and $E(Y^2|X = 1)$.

Problem 50.10
The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f_{XY}(x, y) = \begin{cases} 2x & 0 < x < 1, x < y < x + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(Y|X = x)$ and $E(Y^2|X = x)$.

Problem 50.11
An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

$X \backslash Y$	0	1	2	$P_X(x)$
0	0.12	0.13	0.05	0.30
1	0.06	0.15	0.15	0.36
2	0.05	0.12	0.10	0.27
3	0.02	0.03	0.02	0.07
$p_Y(y)$	0.25	0.43	0.32	1

where X is the number of tornadoes in county Q and Y that of county P. Calculate $E(X|Y=0)$ and $E(X^2|Y=0)$.

Problem 50.12
Let X and Y be two continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < x < 1$, find $E(Y|X=x)$ and $E(Y^2|X=x)$.

Problem 50.13
Suppose that the number of stops X in a day for a UPS delivery truck driver is Poisson with mean λ and that the expected distance driven by the driver Y, given that there are $X = x$ stops, has a normal distribution with a mean of αx miles, and a standard deviation of βx miles. Find the mean of the number of miles driven per day.

Problem 50.14
New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y) = \begin{cases} 0.50 & \text{for } 0 < x, y < 0.5 \\ 1.25 & \text{for } 0 < x < 0.5, 0.5 < y < 1 \\ 1.50 & \text{for } 0.5 < x < 1, 0 < y < 0.5 \\ 0.75 & \text{for } 0.5 < x < 1, 0.5 < y < 1. \end{cases}$$

Calculate $E(Y^2|X=0.75)$.

Problem 50.15
There are some bulbs in a box. 30% of them are style A bulbs, which can last 10 hours with standard deviation 1 hour; the other are style B bulbs, which can last 15 hours with standard deviation 2 hour. If you choose one bulb randomly, what is the expectation of its lifetime?

Problem 50.16
Let $X_1, X_2, \cdots$, be independent random variables with the same mean. Let N be a non-negative integer valued random variable that is independent of the X_i's. Show that

$$E[\sum_{i=1}^{N} X_i] = E(N)E(X_1).$$

Problem 50.17
Suppose that the number of people entering a department store on a given day is a random variable with mean 50. Suppose further that the amounts of money spent by these customers are independent random variables having a common mean of 8 dollars. Finally, suppose also that the amount of money spent by a customer is also independent of the total number of customers who enter the store. What is the expected amount of money spent in the store on a given day?

Problem 50.18
Suppose that the number of times N we roll a die is a Poisson random variable with $\lambda = 10$. Let X_i denote the the value of the i^{th} roll. Find the expected number from the N rolls.

Problem 50.19
Let X and Y be independent exponentially distributed random variables with parameters μ and λ respectively. Using conditioning, find $\Pr(Y < X)$.

Problem 50.20
Let X and Y be independent exponential random variables with parameter λ. Show that $X + Y$ is a Gamma random variable with parameters $(2, \lambda)$.

Problem 50.21 ‡
The amount of a claim that a car insurance company pays out follows an exponential distribution. By imposing a deductible of d, the insurance company reduces the expected claim payment by 10%.
Calculate the percentage reduction on the variance of the claim payment.

Problem 50.22 ‡
In a large population of patients, 20% have early stage cancer, 10% have advanced stage cancer, and the other 70% do not have cancer. Six patients

from this population are randomly selected.
Calculate the expected number of selected patients with advanced stage cancer, given that at least one of the selected patients has early stage cancer.

Problem 50.23 ‡
A motorist just had an accident. The accident is minor with probability 0.75 and is otherwise major. Let b be a positive constant. If the accident is minor, then the loss amount follows a uniform distribution on the interval $[0, b]$. If the accident is major, then the loss amount follows a uniform distribution on the interval $[b, 3b]$. The median loss amount due to this accident is 672. Calculate the mean loss amount due to this accident.

Problem 50.24 ‡
For a certain insurance company, 10% of its policies are Type A, 50% are Type B, and 40% are Type C. The annual number of claims for an individual Type A, Type B, and Type C policy follow Poisson distributions with respective means 1, 2, and 10. Let X represent the annual number of claims of a randomly selected policy. Calculate the variance of X.

Problem 50.25 ‡
The number of tornadoes in a given year follows a Poisson distribution with mean 3. Calculate the variance of the number of tornadoes in a year given that at least one tornado occurs.

Problem 50.26 ‡
An actuary is studying hurricane models. A year is classified as a high, medium, or low hurricane year with probabilities 0.1, 0.3, and 0.6, respectively. The numbers of hurricanes in high, medium, and low years follow Poisson distributions with means 20, 15, and 10, respectively. Calculate the variance of the number of hurricanes in a randomly selected year.

51 Conditional Variance

In this section, we introduce the concept of conditional variance. Just as we have defined the conditional expectation of X given that $Y = y$, we can define the conditional variance of X given Y as follows

$$\text{Var}(X|Y = y) = E[(X - E(X|Y))^2|Y = y].$$

Note that the conditional variance is a random variable since it is a function of Y.

Example 51.1
Let X and Y be two random variables with joint density function given by

$X\backslash Y$	0	1	2	$p_X(x)$
0	1/8	2/8	1/8	4/8
1	2/8	1/8	1/8	4/8
$p_Y(y)$	3/8	3/8	2/8	1

What is the conditional variance of Y given $X = 0$?

Solution.
We first find $E(Y|X = 0)$:

$$\begin{aligned} E(Y|X=0) =& \sum_{y=0}^{2} y p_{Y|0}(y|0) = \sum_{y=0}^{2} y \frac{p_{XY}(0,y)}{p_X(0)} \\ =& (0)\frac{\frac{1}{8}}{\frac{4}{8}} + (1)\frac{\frac{2}{8}}{\frac{4}{8}} + (2)\frac{\frac{1}{8}}{\frac{4}{8}} = 1. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Var}(Y|Y=0) =& E[(Y - E(Y|X=0))^2|X=0] = E[(Y-1)^2|X=0] \\ =& \sum_{y=0}^{2}(y-1)^2 p_{Y|0}(y|0) = \sum_{y=0}^{2}(y-1)^2 \frac{p_{XY}(0,y)}{p_X(0)} \\ =& (0-1)^2\left(\frac{1}{4}\right) + (1-1)^2\left(\frac{2}{4}\right) + (2-1)^2\left(\frac{1}{4}\right) = 0.5 \ \blacksquare \end{aligned}$$

Next, we introduce some of the major properties of the conditional variance.

Theorem 51.1
Let X and Y be random variables. Then
(a) $\text{Var}(X|Y) = E(X^2|Y) - [E(X|Y)]^2$.
(b) $E(\text{Var}(X|Y)) = E[E(X^2|Y) - (E(X|Y))^2] = E(X^2) - E[(E(X|Y))^2]$.
(c) $\text{Var}(E(X|Y)) = E[(E(X|Y))^2] - (E(X))^2$.
(d) Law of Total Variance: $\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E(X|Y))$.

Proof.
(a) We have

$$\begin{aligned}\text{Var}(X|Y) =& E[(X - E(X|Y))^2|Y] \\ =& E[(X^2 - 2XE(X|Y) + (E(X|Y))^2|Y] \\ =& E(X^2|Y) - 2E(X|Y)E(X|Y) + (E(X|Y))^2 \\ =& E(X^2|Y) - [E(X|Y)]^2.\end{aligned}$$

(b) Taking E of both sides of the result in (a) we find

$$E(\text{Var}(X|Y)) = E[E(X^2|Y) - (E(X|Y))^2] = E(X^2) - E[(E(X|Y))^2].$$

(c) Since $E(E(X|Y)) = E(X)$ we have

$$\text{Var}(E(X|Y)) = E[(E(X|Y))^2] - (E(X))^2.$$

(d) The result follows by adding the two equations in (b) and (c) ■

Example 51.2
Suppose that X and Y have joint distribution

$$f_{XY}(x,y) = \begin{cases} \frac{3y^2}{x^3} & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X)$, $E(X^2)$, $Var(X)$, $E(Y|X)$, $Var(Y|X)$, $E[Var(Y|X)]$, $Var[E(Y|X)]$, and $Var(Y)$.

Solution.
First we find marginal density functions.

$$f_X(x) = \int_0^x \frac{3y^2}{x^3} dy = 1, \quad 0 < x < 1$$

$$f_Y(y) = \int_y^1 \frac{3y^2}{x^3} dx = \frac{3}{2}(1-y^2), \quad 0 < y < 1$$

Now,

$$E(X) = \int_0^1 x dx = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 dx = \frac{1}{3}$$

Thus,

$$Var(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Next, we find conditional density of Y given $X = x$

$$f_{Y|X}(x|y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{3y^2}{x^3}, \quad 0 < x < y < 1$$

Hence,

$$E(Y|X=x) = \int_0^x \frac{3y^3}{x^3} dy = \frac{3}{4}x$$

and

$$E(Y^2|X=x) = \int_0^x \frac{3y^4}{x^3} dy = \frac{3}{5}x^2$$

Thus,

$$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2 = \frac{3}{5}x^2 - \frac{9}{16}x^2 = \frac{3}{80}x^2$$

Also,

$$Var[E(Y|X)] = Var\left(\frac{3}{4}x\right) = \frac{9}{16}Var(X) = \frac{9}{16} \times \frac{1}{12} = \frac{3}{64}$$

and

$$E[Var(Y|X)] = E\left(\frac{3}{80}X^2\right) = \frac{3}{80}E(X^2) = \frac{3}{80} \times \frac{1}{3} = \frac{1}{80}.$$

Finally,

$$Var(Y) = Var[E(Y|X)] + E[Var(Y|X)] = \frac{19}{320} \blacksquare$$

Two random variables X and Y are said to have **bivariate normal distribution** if

(a) X and Y are normal random variables;
(b) $X|Y = y$ is a normal random variable with mean $\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$ and variance $\text{Var}(X|Y) = \sigma_X^2(1 - \rho^2)$;
(c) $Y|x = x$ is a normal random variable with mean $\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ and variance $\text{Var}(Y|X) = \sigma_Y^2(1 - \rho^2)$.

Example 51.3
Let X and Y be two bivariate normal random variables with $E(X) = 0, E(Y) = -1$, $E(XY) = 1$, $E(Y|X = 2) = 1$, and $E(X|Y = 0) = \frac{1}{16}$. Calculate $\text{Var}(Y|X = -2)$.

Solution.
Since X and Y have the bivariate normal distribution, we have

$$\text{Var}(Y|X = -2) = (1 - \rho^2)\text{Var}(Y).$$

Now,

$$\begin{aligned}
\rho =& \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \cdot \sigma_Y} = \frac{1}{\sigma_X \cdot \sigma_Y} \\
E(Y|X = 2) =& \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = -1 + \frac{1}{\sigma_X \cdot \sigma_Y}\frac{\sigma_Y}{\sigma_X}(2) \\
=& -1 + \frac{2}{\sigma_X^2} = 1 \\
\sigma_X =& 1 \\
E(X|Y = 0) =& \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y) = 0 + \frac{1}{\sigma_X \cdot \sigma_Y}(0 - (-1)) = \frac{1}{\sigma_Y^2} = \frac{1}{16} \\
\sigma_Y =& 4 \\
\rho =& \frac{1}{1(4)} = \frac{1}{4}.
\end{aligned}$$

Hence,

$$\text{Var}(Y|X = -2) = (1 - \rho^2)\text{Var}(Y) = \left(1 - \frac{1}{16}\right)(16) = 15 \blacksquare$$

Practice Problems

Problem 51.1 ‡
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$\begin{aligned}
\mathrm{P}(X=0,Y=0) &= 0.800 \\
\mathrm{P}(X=1,Y=0) &= 0.050 \\
\mathrm{P}(X=0,Y=1) &= 0.025 \\
\mathrm{P}(X=1,Y=1) &= 0.125.
\end{aligned}$$

Calculate $\mathrm{Var}(Y|X=1)$.

Problem 51.2 ‡
The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f_{XY}(x,y) = \begin{cases} 2x & 0<x<1, x<y<x+1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X=x$?

Problem 51.3 ‡
An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

$X \backslash Y$	0	1	2	$P_X(x)$
0	0.12	0.13	0.05	0.30
1	0.06	0.15	0.15	0.36
2	0.05	0.12	0.10	0.27
3	0.02	0.03	0.02	0.07
$p_Y(y)$	0.25	0.43	0.32	1

where X is the number of tornadoes in county Q and Y that of county P. Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P.

Problem 51.4
Let X be a random variable with mean 3 and variance 2, and let Y be a random variable such that for every x, the conditional distribution of Y given $X = x$ has a mean of x and a variance of x^2. What is the variance of the marginal distribution of Y?

Problem 51.5
Let X and Y be two continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < x < 1$, find $\text{Var}(Y|X = x)$.

Problem 51.6
Suppose that the number of stops X in a day for a UPS delivery truck driver is Poisson with mean λ and that the distance driven by the driver Y, given that there are $X = x$ stops, has a normal distribution with a mean of αx miles, and a standard deviation of βx miles. Find the mean and variance of the number of miles driven per day.

Problem 51.7 ‡
The number of workplace injuries, N, occurring in a factory on any given day is Poisson distributed with mean λ. The parameter λ is a random variable that is determined by the level of activity in the factory, and is uniformly distributed on the interval $[0, 3]$.
Calculate $\text{Var}(N)$.

Problem 51.8 ‡
New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y) = \begin{cases} 0.50 & \text{for } 0 < x, y < 0.5 \\ 1.25 & \text{for } 0 < x < 0.5, 0.5 < y < 1 \\ 1.50 & \text{for } 0.5 < x < 1, 0 < y < 0.5 \\ 0.75 & \text{for } 0.5 < x < 1, 0.5 < y < 1. \end{cases}$$

Calculate $\text{Var}(Y|X = 0.75)$.

Problem 51.9 ‡
A motorist makes three driving errors, each independently resulting in an accident with probability 0.25.
Each accident results in a loss that is exponentially distributed with mean 0.80. Losses are mutually independent and independent of the number of accidents. The motorist's insurer reimburses 70% of each loss due to an accident.
Calculate the variance of the total unreimbursed loss the motorist experiences due to accidents resulting from these driving errors.

Problem 51.10 ‡
The number of hurricanes that will hit a certain house in the next ten years is Poisson distributed with mean 4.
Each hurricane results in a loss that is exponentially distributed with mean 1000. Losses are mutually independent and independent of the number of hurricanes.
Calculate the variance of the total loss due to hurricanes hitting this house in the next ten years.

Problem 51.11 ‡
The intensity of a hurricane is a random variable that is uniformly distributed on the interval $[0, 3]$. The damage from a hurricane with a given intensity y is exponentially distributed with a mean equal to y.
Calculate the variance of the damage from a random hurricane.

Problem 51.12 ‡
On Main Street, a driver's speed just before an accident is uniformly distributed on $[5, 20]$. Given the speed, the resulting loss from the accident is exponentially distributed with mean equal to three times the speed.
Calculate the variance of a loss due to an accident on Main Street.

Problem 51.13
Let X and Y be random variables with joint probability function given below.

$Y \backslash X$	0	1	2	3	4	$P_Y(y)$
0	0.02	0.02	0.00	0.10	0.00	0.14
1	0.02	0.04	0.10	0.00	0.00	0.16
2	0.02	0.06	0.00	0.10	0.00	0.18
3	0.02	0.08	0.10	0.00	0.05	0.25
4	0.02	0.10	0.00	0.10	0.05	0.27
$p_X(x)$	0.10	0.30	0.20	0.30	0.10	1

Calculate $\text{Var}(Y|X=2)$.

Problem 51.14
Let X and Y be random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{4}{5}(xy+x+y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(Y|X=x)$.

Problem 51.15
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{2}{\pi} & x^2+y^2 \le 1,\ y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(Y|X=x)$.

Problem 51.16
Let X and Y be two random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} e^{-y} & 0 \le x \le y \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(Y|X=x)$.

Problem 51.17 ‡
The joint probability density for X and Y is

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+2y)}, & x>0, y>0 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the variance of Y given that $X>3$ and $Y>3$.

Problem 51.18 ‡
A fire in an apartment building results in a loss, X, to the owner and a loss, Y, to the tenants. The variables X and Y have a bivariate normal distribution with $E(X)=40,\ \text{Var}(X)=76,\ E(Y)=30,\ \text{Var}(Y)=32$, and $\text{Var}(X|Y=28.5)=57$. Calculate $\text{Var}(Y|X=25)$.

Problem 51.19 ‡
A machine has two components and fails when both components fail. The number of years from now until the first component fails, X, and the number of years from now until the machine fails, Y, are random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{18}e^{-\frac{(x+y)}{6}} & \text{for } 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(Y|X = 2)$.

Problem 51.20 ‡
The annual profits that company A and company B earn follow a bivariate normal distribution. Company A's annual profit has mean 2000 and standard deviation 1000. Company B's annual profit has mean 3000 and standard deviation 500. The correlation coefficient between these annual profits is 0.80.
Calculate the probability that company B's annual profit is less than 3900, given that company A's annual profit is 2300.

Problem 51.21 ‡
The returns on two investments, X and Y, follow the joint probability density function

$$f_{XY}(x,y) = \begin{cases} k, & 0 < |x| + |y| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the maximum value of $\text{Var}(Y|X = x), \ -1 < x < 1$.

Problem 51.22 ‡
Let X be the annual number of hurricanes hitting Florida, and let Y be the annual number of hurricanes hitting Texas. X and Y are independent Poisson variables with respective means 1.70 and 2.30.
Calculate $\text{Var}(X - Y|X + Y = 3)$.

Moment Generating Functions and the Central Limit Theorem

For a positive integer n and a random variable X, we call $E(X^n)$ **moments**. As you have already experienced in some cases, the mean $E(X)$ and the variance $\text{Var}(X) = E(X^2) - (E(X))^2$, which are functions of moments, are sometimes difficult to find. Special functions, called moment-generating functions, can sometimes make finding the mean and variance of a random variable simpler. In this Chapter, we'll learn what a moment-generating function is, and then we'll learn how to use these functions in finding moments and functions of moments such as the mean and the variance. Also, we will use moment generating functions to identify the distribution of a random variable. We conclude this section with an important probability result known as the Central Limit Theorem (CLT).

52 Moment Generating Functions

The **moment generating function** of a random variable X, denoted by $M_X(t)$, is defined as

$$M_X(t) = E[e^{tX}]$$

provided that the expectation exists for t in some neighborhood of 0.
For a discrete random variable with a pmf $\text{p}_\text{X}(x)$ we have

$$M_X(t) = \sum_x e^{tx} p_X(x)(x)$$

and for a continuous random variable with pdf f,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Example 52.1
Let X be a discrete random variable with pmf given by the following table

x	1	2	3	4	5
$p_X(x)$	0.15	0.20	0.40	0.15	0.10

Find $M_X(t)$.

Solution.
We have

$$M_X(t) = 0.15e^t + 0.20e^{2t} + 0.40e^{3t} + 0.15e^{4t} + 0.10e^{5t} \blacksquare$$

Example 52.2
Let X be the uniform random variable on the interval $[a, b]$. Find $M_X(t)$.

Solution.
We have

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a}dx = \frac{1}{t(b-a)}[e^{tb} - e^{ta}] \blacksquare$$

As the name suggests, the moment generating function can be used to generate moments $E(X^n)$ for $n = 1, 2, \cdots$. Our first result shows how to use the moment generating function to calculate moments.

Proposition 52.1

$$E(X^n) = M_X^n(0)$$

where

$$M_X^n(0) = \frac{d^n}{dt^n}M_X(t)\Big|_{t=0}.$$

Proof.
We prove the result for a continuous random variable X with pdf f. The discrete case is shown similarly. In what follows we always assume that we can differentiate under the integral sign. This interchangeability of differentiation and expectation is not very limiting, since all of the distributions we will consider enjoy this property. We have

$$\begin{aligned}\frac{d}{dt}M_X(t) =& \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}f(x)dx = \int_{-\infty}^{\infty}\left(\frac{d}{dt}e^{tx}\right)f(x)dx \\ =& \int_{-\infty}^{\infty} xe^{tx}f(x)dx = E[Xe^{tX}]\end{aligned}$$

Hence,

$$\frac{d}{dt}M_X(t)\mid_{t=0} = E[Xe^{tX}]\mid_{t=0} = E(X).$$

By induction on n we find

$$\frac{d^n}{dt^n}M_X(t)\mid_{t=0} = E[X^n e^{tX}]\mid_{t=0} = E(X^n) \blacksquare$$

We next compute $M_X(t)$ for some common distributions.

Example 52.3
Let X be a binomial random variable with parameters n and p. Find the expected value and the variance of X using moment generating functions.

Solution.
We can write

$$\begin{aligned} M_X(t) =& E(e^{tX}) = \sum_{k=0}^{n} e^{tk}\,{}_nC_k p^k(1-p)^{n-k} \\ =& \sum_{k=0}^{n} {}_nC_k(pe^t)^k(1-p)^{n-k} = (pe^t+1-p)^n. \end{aligned}$$

Differentiating yields

$$\frac{d}{dt}M_X(t) = npe^t(pe^t+1-p)^{n-1}.$$

Thus,

$$E(X) = \frac{d}{dt}M_X(t)\mid_{t=0} = np.$$

To find $E(X^2)$, we differentiate a second time to obtain

$$\frac{d^2}{dt^2}M_X(t) = n(n-1)p^2e^{2t}(pe^t+1-p)^{n-2} + npe^t(pe^t+1-p)^{n-1}.$$

Evaluating at $t=0$, we find

$$E(X^2) = M_X''(0) = n(n-1)p^2 + np.$$

The variance of X is

$$Var(X) = E(X^2) - (E(X))^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p) \blacksquare$$

Example 52.4
Let X be a Poisson random variable with parameter λ. Find the expected value and the variance of X using moment generating functions.

Solution.
We can write

$$\begin{aligned} M_X(t) =& E(e^{tX}) = \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{e^{tn} \lambda^n}{n!} \\ =& e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}. \end{aligned}$$

Differentiating for the first time, we find

$$M'_X(t) = \lambda e^t e^{\lambda(e^t-1)}.$$

Thus,

$$E(X) = M'_X(0) = \lambda.$$

Differentiating a second time, we find

$$M''_X(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}.$$

Hence,

$$E(X^2) = M''_X(0) = \lambda^2 + \lambda.$$

The variance is then

$$Var(X) = E(X^2) - (E(X))^2 = \lambda \blacksquare$$

Example 52.5
Let X be an exponential random variable with parameter λ. Find the expected value and the variance of X using moment generating functions.

Solution.
We can write

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx = \frac{\lambda}{\lambda - t}$$

where $t < \lambda$. Differentiation twice yields

$$M'_X(t) = \frac{\lambda}{(\lambda - t)^2} \quad \text{and} \quad M''_X(t) = \frac{2\lambda}{(\lambda - t)^3}.$$

Hence,

$$E(X) = M'_X(0) = \frac{1}{\lambda} \quad \text{and} \quad E(X^2) = M''_X(0) = \frac{2}{\lambda^2}.$$

The variance of X is given by

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{\lambda^2} \blacksquare$$

Practice Problems

Problem 52.1
Let X be a discrete random variable with range $\{1, 2, \cdots, n\}$ so that its pmf is given by $p_X(j) = \frac{1}{n}$ for $1 \leq j \leq n$. Find $E(X)$ and $Var(X)$ using moment generating functions.

Problem 52.2
Let X be a geometric distribution function with $p_X(n) = p(1-p)^{n-1}$. Find the expected value and the variance of X using moment generating functions.

Problem 52.3
The following problem exhibits a random variable with no moment generating function. Let X be a random variable with pmf given by

$$p_X(n) = \frac{6}{\pi^2 n^2}, \quad n = 1, 2, \cdots$$

Show that $M_X(t)$ does not exist in any neighborhood of 0.

Problem 52.4
Let X be a gamma random variable with parameters α and λ. Find the expected value and the variance of X using moment generating functions.

Problem 52.5
Let X be a random variable with pdf given by

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find $M_X(t)$.

Problem 52.6 ‡
Let X_1, X_2, X_3 be three independent discrete random variables with common probability mass function

$$\mathrm{P}(x) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{2}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the moment generating function $M(t)$, of $Y = X_1 X_2 X_3$.

Problem 52.7
Suppose a random variable X has moment generating function

$$M_X(t) = \left(\frac{2+e^t}{3}\right)^9.$$

Find the variance of X.

Problem 52.8
Let X be a random variable with density function

$$f(x) = \begin{cases} (k+1)x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the moment generating function of X

Problem 52.9
If the moment generating function for the random variable X is $M_X(t) = \frac{1}{t+1}$, find $E[(X-2)^3]$.

Problem 52.10
Suppose that X is a random variable with moment generating function $M_X(t) = \sum_{j=0}^{\infty} \frac{e^{(tj-1)}}{j!}$. Find $\mathrm{P}(X=2)$.

Problem 52.11
Find the moment generating function of the standard normal distribution.

Problem 52.12
If X has a standard normal distribution and $Y = e^X$, what is the k^{th} moment of Y?

Problem 52.13
The random variable X has an exponential distribution with parameter b. It is found that $M_X(-b^2) = 0.2$. Find b.

Problem 52.14 ‡
The value of a piece of factory equipment after three years of use is $100(0.5)^X$ where X is a random variable having moment generating function

$$M_X(t) = \frac{1}{1-2t} \text{ for } t < \frac{1}{2}.$$

Calculate the expected value of this piece of equipment after three years of use.

Problem 52.15
Let X be a random variable with density function

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

for a fixed $\lambda > 0$. Find $M_X(t)$. What is the domain of $M_X(t)$?

Problem 52.16
Let X be a random variable with cumulative distribution

$$F(x) = 1 + 3e^{-4x} - 4e^{-3x}$$

for $x \geq 0$ and 0 otherwise. Find $M_X(t)$.

Problem 52.17
Let X be a random variable, and let a and b be finite constants. Show that

$$M_{aX+b}(t) = e^{bt} M_X(at).$$

Problem 52.18
Let X be a normal random variable with parameters μ and σ^2. Find the expected value and the variance of X using moment generating functions.

Problem 52.19 ‡
An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1 - 2500t)^4}$$

Determine the standard deviation of the claim size for this class of accidents.

Problem 52.20 ‡
Let X represent the number of policies sold by an agent in a day. The moment generating function of X is

$$M_X(t) = 0.45e^t + 0.35e^{2t} + 0.15e^{3t} + 0.05e^{4t}, \quad -\infty < t < \infty.$$

Calculate the standard deviation of X.

Problem 52.21 ‡
A homeowners insurance policy covers losses due to theft, with a deductible of 3. Theft losses are uniformly distributed on $[0, 10]$. Determine the moment generating function, $M(t)$, for $t \neq 0$, of the claim payment on a theft.

53 Moment Generating Functions of Sums of Independent RVs

Another use of moment generating functions is to identify which probability mass(density) function a random variable X follows. The moment generating function uniquely determines the distribution as shown in the following result.

Theorem 53.1
If random variables X and Y both have moment generating functions $M_X(t)$ and $M_Y(t)$ that exist in some neighborhood of zero and if $M_X(t) = M_Y(t)$ for all t in this neighborhood, then X and Y have the same distributions.

The general proof of this is an inversion problem involving Laplace transform theory and is omitted.

Example 53.1
Identify the random variable whose moment generating function is given by

$$M_X(t) = \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{15}.$$

Solution.
Using Example 52.3, X is the binomial random variable with $p = \frac{3}{4}$ and $n = 15$ ■

Moment generating functions are also useful in establishing the distribution of sums of independent random variables. Before we look at some examples, we first establish a relationship between the moment generating function of a sum of independent random variables and the moment generating functions of its component variables. Let $X_1, X_2, \cdots, X_N$ be independent random variables. Then, the moment generating function of $Y = X_1 + \cdots + X_N$ is

$$M_Y(t) = E(e^{t(X_1+X_2+\cdots+X_n)}) = E(e^{X_1 t} \cdots e^{X_N t})$$

$$= \prod_{k=1}^{N} E(e^{X_k t}) = \prod_{k=1}^{N} M_{X_k}(t)$$

where we used Proposition 46.5.

Example 53.2
If X and Y are independent binomial random variables with parameters (n, p) and (m, p), respectively, what is the pmf of $X + Y$?

Solution.
We have

$$\begin{aligned} M_{X+Y}(t) =& M_X(t)M_Y(t) \\ =& (pe^t + 1 - p)^n (pe^t + 1 - p)^m \\ =& (pe^t + 1 - p)^{n+m} \end{aligned}$$

where we used Example 52.3. Since $(pe^t+1-p)^{n+m}$ is the moment generating function of a binomial random variable having parameters $m+n$ and p, $X+Y$ is a binomial random variable with this same pmf ■

Example 53.3
If X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively, what is the pmf of $X + Y$?

Solution.
We have

$$\begin{aligned} M_{X+Y}(t) =& M_X(t)M_Y(t) \\ =& e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \\ =& e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

where we used Example 52.4. Since $e^{(\lambda_1+\lambda_2)(e^t-1)}$ is the moment generating function of a Poisson random variable having parameter $\lambda_1 + \lambda_2$, $X + Y$ is a Poisson random variable with this same pmf ■

Example 53.4
If X and Y are independent normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively, what is the distribution of $X + Y$?

Solution.
Using independence and Problem 52.17, we have

$$\begin{aligned} M_{X\pm Y}(t) =& M_X(t)M_Y(t) \\ =& e^{\frac{\sigma_1^2 t^2}{2} + \mu_1 t} \cdot e^{\frac{\sigma_2^2 t^2}{2} + \mu_2 t} \\ =& e^{\frac{(\sigma_1^2+\sigma_2^2)t^2}{2} + (\mu_1+\mu_2)t} \end{aligned}$$

which is the moment generating function of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Because the moment generating function uniquely determines the distribution then $X+Y$ is a normal random variable with the same distribution ■

Joint Moment Generating Functions
For any random variables $X_1, X_2, \cdots, X_n$, the joint moment generating function is defined by

$$M(t_1, t_2, \cdots, t_n) = E(e^{t_1X_1+t_2X_2+\cdots+t_nX_n}).$$

Example 53.5
Let X and Y be two independent normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively. Find the joint moment generating function of $X + Y$ and $X - Y$.

Solution.
The joint moment generating function is

$$\begin{aligned} M(t_1, t_2) =& E(e^{t_1(X+Y)+t_2(X-Y)}) = E(e^{(t_1+t_2)X+(t_1-t_2)Y}) \\ =& E(e^{(t_1+t_2)X})E(e^{(t_1-t_2)Y}) = M_X(t_1 + t_2)M_Y(t_1 - t_2) \\ =& e^{(t_1+t_2)\mu_1+\frac{1}{2}(t_1+t_2)^2\sigma_1^2}e^{(t_1-t_2)\mu_2+\frac{1}{2}(t_1-t_2)^2\sigma_2^2} \\ =& e^{(t_1+t_2)\mu_1+(t_1-t_2)\mu_2+\frac{1}{2}(t_1^2+t_2^2)\sigma_1^2+\frac{1}{2}(t_1^2+t_2^2)\sigma_2^2+t_1t_2(\sigma_1^2-\sigma_2^2)} \end{aligned}$$ ■

Example 53.6
Let X and Y be two random variables with joint distribution function

$$f_{XY}(x, y) = \begin{cases} e^{-x-y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the joint moment generating function, find $E(XY), E(X), E(Y)$.

Solution.
We note first that $f_{XY}(x, y) = f_X(x)f_Y(y)$ so that X and Y are independent. Thus, the moment generating function is given by

$$M(t_1, t_2) = E(e^{t_1X+t_2Y}) = E(e^{t_1X})E(e^{t_2Y}) = \frac{1}{1-t_1}\frac{1}{1-t_2}.$$

Thus,

$$E(XY) = \frac{\partial^2}{\partial t_2 \partial t_1} M(t_1, t_2)\Big|_{(0,0)} = \frac{1}{(1-t_1)^2(1-t_2)^2}\Big|_{(0,0)} = 1$$
$$E(X) = \frac{\partial}{\partial t_1} M(t_1, t_2)\Big|_{(0,0)} = \frac{1}{(1-t_1)^2(1-t_2)}\Big|_{(0,0)} = 1$$
$$E(Y) = \frac{\partial}{\partial t_2} M(t_1, t_2)\Big|_{(0,0)} = \frac{1}{(1-t_1)(1-t_2)^2}\Big|_{(0,0)} = 1 \blacksquare$$

Example 53.7
In the morning hours, suppose that the number of people who enter the post office is a Poisson random variable with mean λ and each person is, independently of everything, female with probability p and male with probability $1-p$. Let X and Y represent the numbers of females and males, respectively, entering the post office. Determine the joint moment generating function $M_{XY}(s,t)$.

Solution.
First, note that $X+Y$ is a Poisson random variable with mean λ. Also, $X|X+Y$ is a binomial random variable with parameters $n = X+Y$ and p. We have

$$\begin{aligned}
M_{XY}(s,t) =& E(e^{sX+tY}) = E[E(e^{sX+tY}|X+Y)] \\
=& E[E(e^{t(X+Y)+(s-t)X}|X+Y)] = E[e^{t(X+Y)}E(e^{(s-t)X}|X+Y)] \\
=& E[e^{t(X+Y)}(pe^{(s-t)}+1-p)^{X+Y}] \\
=& E[e^{t(X+Y)}e^{(X+Y)\ln(pe^{(s-t)}+1-p)}] \\
=& E[e^{(X+Y)[t+\ln(pe^{(s-t)}+1-p)]}] = M_{X+Y}(t+\ln(pe^{(s-t)}+1-p)) \\
=& e^{\lambda(e^{t+\ln(pe^{(s-t)}+1-p)}-1)} \\
=& e^{\lambda(e^t[pe^{(s-t)}+1-p]-1)} \\
=& e^{\lambda(pe^s+(1-p)e^t-1)} \blacksquare
\end{aligned}$$

Practice Problems

Problem 53.1
Let X be an exponential random variable with parameter λ. Find the moment generating function of $Y = 3X - 2$.

Problem 53.2
Identify the random variable whose moment generating function is given by

$$M_Y(t) = e^{-2t}\left(\frac{3}{4}e^{3t} + \frac{1}{4}\right)^{15}.$$

Problem 53.3
The moment generating function of X is $M_X(t) = e^{2e^t-2}$and that of Y is $M_Y(t) = \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{10}$. Find $E(XY)$ if X and Y are independent.

Problem 53.4
Let X and Y be independent random variables with density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

Using moment generating functions, find the probability density function of $aX + bY$ where a and b are constants.

Problem 53.5 ‡
Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$. The errors from the two instruments are independent of each other.
Calculate the probability that the average value of the two measurements is within $0.005h$ of the height of the tower.

Problem 53.6
Using moment generating functions, show that the sum of n independently exponential random variable each with parameter λ is a gamma random variable with parameters n and λ.

Problem 53.7 ‡
X and Y are independent random variables with common moment generating function $M(t) = e^{\frac{t^2}{2}}$. Let $W = X + Y$ and $Z = X - Y$. Determine the joint moment generating function, $M(t_1, t_2)$ of W and Z.

Problem 53.8 ‡
A company insures homes in three cities, J, K, and L . Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent.
The moment generating functions for the loss distributions of the cities are:

$$\begin{aligned} M_J(t) =& (1-2t)^{-3} \\ M_K(t) =& (1-2t)^{-2.5} \\ M_L(t) =& (1-2t)^{-4.5} \end{aligned}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

Problem 53.9
Let $X_1, X_2, \cdots, X_n$ be independent geometric random variables each with parameter p. Define $Y = X_1 + X_2 + \cdots X_n$.
(a) Find the moment generating function of $X_i,\ 1 \leq i \leq n$.
(b) Find the moment generating function of a negative binomial random variable with parameters (n, p).
(c) Show that Y defined above is a negative binomial random variable with parameters (n, p).

Problem 53.10
Let X_1 and X_2 be two random variables with joint density function

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the moment generating function $M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$.

Problem 53.11
The moment generating function for the joint distribution of random variables X and Y is $M(t_1, t_2) = \frac{1}{3(1-t_2)} + \frac{2}{3}e^{t_1} \cdot \frac{2}{(2-t_2)},\ \ t_2 < 1$. Find $\text{Var}(X)$.

Problem 53.12
Let X and Y be two independent random variables with moment generating functions

$$M_X(t) = e^{t^2+2t} \text{ and } M_Y(t) = e^{3t^2+t}$$

Determine the moment generating function of $X + 2Y$.

Problem 53.13
Let X_1 and X_2 be random variables with joint moment generating function

$$M(t_1, t_2) = 0.3 + 0.1e^{t_1} + 0.2e^{t_2} + 0.4e^{t_1+t_2}$$

What is $E(2X_1 - X_2)$?

Problem 53.14
Suppose X and Y are random variables whose joint distribution has moment generating function

$$M_{XY}(t_1, t_2) = \left(\frac{1}{4}e^{t_1} + \frac{3}{8}e^{t_2} + \frac{3}{8}\right)^{10}$$

for all t_1, t_2. Find the covariance between X and Y.

Problem 53.15
Independent random variables X, Y and Z are identically distributed. Let $W = X+Y$. The moment generating function of W is $M_W(t) = (0.7+0.3e^t)^6$. Find the moment generating function of $V = X + Y + Z$.

Problem 53.16 ‡
Let X and Y be identically distributed independent random variables such that the moment generating function of $X + Y$ is

$$M(t) = 0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t}, \ -\infty < t < \infty.$$

Calculate $\mathrm{P}(X \leq 0)$.

Problem 53.17 ‡
An insurance company insures two types of cars, economy cars and luxury cars. The damage claim resulting from an accident involving an economy car has normal $N(7, 1)$ distribution, the claim from a luxury car accident has

normal $N(20, 6)$ distribution.
Suppose the company receives three claims from economy car accidents and one claim from a luxury car accident. Assuming that these four claims are mutually independent, what is the probability that the total claim amount from the three economy car accidents exceeds the claim amount from the luxury car accident?

Problem 53.18
Let $X_1, X_2, \cdots, X_n$ be n independent identically distributed random variables and let $\overline{X} = \frac{X_1+X_2+\cdots+X_n}{n}$. Find $M_{\overline{X}}(t)$.

Problem 53.19
Let X_1, X_2, and X_3 be three independent gamma random variables with parameters $\alpha = 7$ and $\lambda = 5$. Find the distribution of the random variable $Y = X_1 + X_2 + X_3$.

Problem 53.20
Let X_1, X_2, and X_3 be three independent gamma random variables with parameters $\alpha = 7$ and $\lambda = 5$. Find the distribution of the random variable $\overline{X} = \frac{X_1+X_2+X_3}{3}$.

Problem 53.21 ‡
For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000.
For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000.
The total claim amounts of the two companies are independent. Calculate the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount.

Problem 53.22 ‡
A certain brand of refrigerator has a useful life that is normally distributed with mean 10 years and standard deviation 3 years. The useful lives of these refrigerators are independent. Calculate the probability that the total useful life of two randomly selected refrigerators will exceed 1.9 times the useful life of a third randomly selected refrigerator.

Problem 53.23 ‡
Two independent estimates are to be made on a building damaged by fire. Each estimate is normally distributed with mean $10b$ and variance b^2 .
Calculate the probability that the first estimate is at least 20 percent higher than the second.

Problem 53.24 ‡
The random variable X has moment generating function $M(t)$. Determine which of the following is the moment generating function of some random variable.
i) $M(t)M(5t)$
ii) $2M(t)$
iii) $e^t M(t)$.

Problem 53.25 ‡
A delivery service owns two cars that consume 15 and 30 miles per gallon. Fuel costs 3 per gallon. On any given business day, each car travels a number of miles that is independent of the other and is normally distributed with mean 25 miles and standard deviation 3 miles. Calculate the probability that on any given business day, the total fuel cost to the delivery service will be less than 7.

Problem 53.26 ‡
Every day, the 30 employees at an auto plant each have probability 0.03 of having one accident and zero probability of having more than one accident. Given there was an accident, the probability of it being major is 0.01. All other accidents are minor. The numbers and severities of employee accidents are mutually independent. Let X and Y represent the numbers of major accidents and minor accidents, respectively, occurring in the plant today. Determine the joint moment generating function $M_{XY}(s,t)$.

Problem 53.27 ‡
In a group of 15 health insurance policyholders diagnosed with cancer, each policyholder has probability 0.90 of receiving radiation and probability 0.40 of receiving chemotherapy. Radiation and chemotherapy treatments are independent events for each policyholder, and the treatments of different policyholders are mutually independent. The policyholders in this group all have the same health insurance that pays 2 for radiation treatment and 3 for

chemotherapy treatment.
Calculate the variance of the total amount the insurance company pays for the radiation and chemotherapy treatments for these 15 policyholders.

54 The Central Limit Theorem

The central limit theorem is one of the most remarkable theorems among the limit theorems. This theorem says that the sum of a large number of independent identically distributed random variables is well-approximated by a normal random variable.

Theorem 54.1 *(CLT)*
Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, each with mean μ and variance σ^2. Then,

$$P\left(\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right)\leq a\right)\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{-\frac{x^2}{2}}dx$$

as $n\to\infty$.

The Central Limit Theorem says that regardless of the underlying distribution of the variables X_i, so long as they are independent, the distribution of $\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right)$ converges to the same, normal, distribution.
The central limit theorem suggests approximating the random variable

$$\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right)$$

with a standard normal random variable. This implies that the sample mean has approximately a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.
Also, a sum of n independent and identically distributed random variables with common mean μ and variance σ^2 can be approximated by a normal distribution with mean $n\mu$ and variance $n\sigma^2$.

Example 54.1
Let $X_i,\ i=1,2,\cdots,48$ be independent random variables that are uniformly distributed on the interval $[-0.5, 0.5]$. Find the approximate probability $\mathrm{P}(|\overline{X}|\leq 0.05)$, where $\overline{X}$ is the arithmetic average of the $X_i's$.

Solution.
Since each X_i is uniformly distributed on $[-0.5, 0.5]$, its mean is $\mu=0$ and its variance is $\sigma^2=\int_{-0.5}^{0.5}x^2dx=\frac{x^3}{3}\Big]_{-0.5}^{0.5}=\frac{1}{12}$. By the Central Limit

Theorem, $\overline{X}$ has approximate distribution $N(\mu, \frac{\sigma^2}{n}) = N(0, \frac{1}{24^2})$. Thus $24\overline{X}$ is approximately standard normal, so

$$\begin{aligned}\mathrm{P}(|\overline{X}| \leq 0.05) \approx & \mathrm{P}(24 \cdot (-0.05) \leq 24\overline{X} \leq 24 \cdot (0.05)) \\ = & \Phi(1.2) - \Phi(-1.2) = 2\Phi(1.2) - 1 = 0.7698 \ \blacksquare\end{aligned}$$

Example 54.2
Let X_1, X_2, X_3, X_4 be a random sample of size 4 from a normal distribution with mean 2 and variance 10, and let $\overline{X}$ be the sample mean. Determine a such that $\mathrm{P}(\overline{X} \leq a) = 0.90$.

Solution.
The sample mean $\overline{X}$ is normal with mean $\mu = 2$ and variance $\frac{\sigma^2}{n} = \frac{10}{4} = 2.5$, and standard deviation $\sqrt{2.5} \approx 1.58$, so

$$0.90 = \mathrm{P}(\overline{X} \leq a) = P\left(\frac{\overline{X} - 2}{1.58} < \frac{a-2}{1.58}\right) = \Phi\left(\frac{a-2}{1.58}\right).$$

From the normal table, we get $\frac{a-2}{1.58} = 1.28$, so $a = 4.02$ $\blacksquare$

Example 54.3
Assume that the weights of individuals are independent and normally distributed with a mean of 160 pounds and a standard deviation of 30 pounds. Suppose that 25 people squeeze into an elevator that is designed to hold 4300 pounds.
(a) What is the probability that the load (total weight) exceeds the design limit?
(b) What design limit is exceeded by 25 occupants with probability 0.001?

Solution.
(a) Let X be an individual's weight. Then, X has a normal distribution with $\mu = 160$ pounds and $\sigma = 30$ pounds. Let $Y = X_1 + X_2 + \cdots + X_{25}$, where X_i denotes the i^{th} person's weight. Then, Y has a normal distribution with $E(Y) = 25E(X) = 25 \cdot (160) = 4000$ pounds and $\mathrm{Var}(Y) = 25\mathrm{Var}(X) = 25 \cdot (900) = 22500$. Now, the desired probability is

$$\begin{aligned}\mathrm{P}(Y > 4300) = & P\left(\frac{Y - 4000}{\sqrt{22500}} > \frac{4300 - 4000}{\sqrt{22500}}\right) \\ = & \mathrm{P}(Z > 2) = 1 - \mathrm{P}(Z \leq 2) \\ = & 1 - 0.9772 = 0.0228\end{aligned}$$

(b) We want to find x such that $\mathrm{P}(Y > x) = 0.001$. Note that

$$\begin{aligned}\mathrm{P}(Y > x) =& P\left(\frac{Y - 4000}{\sqrt{22500}} > \frac{x - 4000}{\sqrt{22500}}\right) \\ =& P\left(Z > \frac{x - 4000}{\sqrt{22500}}\right) = 0.01\end{aligned}$$

It is equivalent to $P\left(Z \leq \frac{x-4000}{\sqrt{22500}}\right) = 0.999$. From the normal Table we find $\mathrm{P}(Z \leq 3.09) = 0.999$. So $(x - 4000)/150 = 3.09$. Solving for x we find $x \approx 4463.5$ pounds ■

Practice Problems

Problem 54.1
Letter envelopes are packaged in boxes of 100. It is known that, on average, the envelopes weigh 1 ounce, with a standard deviation of 0.05 ounces. What is the probability that 1 box of envelopes weighs more than 100.4 ounces?

Problem 54.2
In the SunBelt Conference men basketball league, the standard deviation in the distribution of players' height is 2 inches. A random group of 25 players are selected and their heights are measured. Estimate the probability that the average height of the players in this sample is within 1 inch of the conference average height.

Problem 54.3
A radio battery manufacturer claims that the lifespan of its batteries has a mean of 54 days and a standard deviation of 6 days. A random sample of 50 batteries were picked for testing. Assuming the manufacturer's claims are true, what is the probability that the sample has a mean lifetime of less than 52 days?

Problem 54.4
If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40, inclusive.

Problem 54.5
Let $X_i, i = 1, 2, \cdots, 10$ be independent random variables each uniformly distributed over (0,1). Calculate an approximation to $\mathrm{P}(\sum_{i=1}^{10} X_i > 6)$.

Problem 54.6
Suppose that $X_i, i = 1, \cdots, 100$ are exponentially distributed random variables with parameter $\lambda = \frac{1}{1000}$. Let $\overline{X} = \frac{\sum_{i=1}^{100} X_i}{100}$. Approximate $\mathrm{P}(950 \leq \overline{X} \leq 1050)$.

Problem 54.7
A baseball team plays 100 independent games. It is found that the probability of winning a game is 0.8. Estimate the probability that team wins at least 90 games.

Problem 54.8
A small auto insurance company has 10,000 automobile policyholders. It has found that the expected yearly claim per policyholder is \$240 with a standard deviation of \$800. Estimate the probability that the total yearly claim exceeds \$2.7 million.

Problem 54.9
Let $X_1, X_2, \cdots, X_n$ be n independent random variables each with mean 100 and standard deviation 30. Let X be the sum of these random variables. Find n such that $\mathrm{P}(X > 2000) \geq 0.95$.

Problem 54.10 ‡
A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250.
Calculate the approximate 90th percentile for the distribution of the total contributions received.

Problem 54.11 ‡
An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2. Assume the numbers of claims filed by distinct policyholders are independent of one another.
What is the approximate probability that there is a total of between 2450 and 2600 claims during a one-year period?

Problem 54.12 ‡
A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1 . A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes.
What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772?

Problem 54.13 ‡
Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y :

E(X) =	50
E(Y) =	20
Var(X) =	50
Var(Y) =	30
Cov (X,Y) =	10

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period.
Approximate the value of P($T < 7100$).

Problem 54.14 ‡
The total claim amount for a health insurance policy follows a distribution with density function

$$f(x) = \begin{cases} \frac{1}{1000} e^{-\frac{x}{1000}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The premium for the policy is set at 100 over the expected total claim amount. If 100 policies are sold, what is the approximate probability that the insurance company will have claims exceeding the premiums collected?

Problem 54.15 ‡
A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

(i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.
(ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.
(iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.
Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

Problem 54.16
(a) Give the approximate sampling distribution for the following quantity based on random samples of independent observations:

$$\overline{X} = \frac{\sum_{i=1}^{100} X_i}{100}, \quad E(X_i) = 100, \quad \text{Var}(X_i) = 400.$$

(b) What is the approximate probability the sample mean will be between 96 and 104?

Problem 54.17 ‡
In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from -2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people.
What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

Problem 54.18 ‡
Claims filed under auto insurance policies follow a normal distribution with mean 19,400 and standard deviation 5,000.
Calculate the probability that the average of 25 randomly selected claims exceeds 20,000.

Problem 54.19 ‡
The amounts of automobile losses reported to an insurance company are mutually independent, and each loss is uniformly distributed between 0 and 20,000. The company covers each such loss subject to a deductible of 5,000.
Calculate the probability that the total payout on 200 reported losses is between 1,000,000 and 1,200,000.

Problem 54.20 ‡
Claim amounts at an insurance company are independent of one another. In year one, claim amounts are modeled by a normal random variable X with mean 100 and standard deviation 25. In year two, claim amounts are modeled by the random variable $Y = 1.04X + 5$.
Calculate the probability that a random sample of 25 claim amounts in year two average between 100 and 110.

Problem 54.21 ‡
At the start of a week, a coal mine has a high-capacity storage bin that is half full. During the week, 20 loads of coal are added to the storage bin. Each load of coal has a volume that is normally distributed with mean 1.50 cubic yards and standard deviation 0.25 cubic yards.
During the same week, coal is removed from the storage bin and loaded into 4 railroad cars. The amount of coal loaded into each railroad car is normally distributed with mean 7.25 cubic yards and standard deviation 0.50 cubic yards.
The amounts added to the storage bin or removed from the storage bin are mutually independent. Calculate the probability that the storage bin contains more coal at the end of the week than it had at the beginning of the week.

Problem 54.22 ‡
A company provides a death benefit of 50,000 for each of its 1000 employees. There is a 1.4% chance that any one employee will die next year, independent of all other employees. The company establishes a fund such that the probability is at least 0.99 that the fund will cover next year's death benefits. Calculate, using the Central Limit Theorem, the smallest amount of money, rounded to the nearest 50 thousand, that the company must put into the fund.

Problem 54.23 ‡
An investor invests 100 dollars in a stock. Each month, the investment has probability 0.5 of increasing by 1.10 dollars and probability 0.5 of decreasing by 0.90 dollars. The changes in price in different months are mutually independent. Calculate the probability that the investment has a value greater than 91 dollars at the end of month 100.

Sample Exam 1

Duration: 3 hours

Problem 1 ‡
A survey of a group's viewing habits over the last year revealed the following information

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

(A) 24% (B) 36% (C) 41% (D) 52% (E) 60%

Problem 2 ‡
A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.
(ii) 22% have low blood pressure.
(iii) 15% have an irregular heartbeat.
(iv) Of those with an irregular heartbeat, one-third have high blood pressure.
(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

(A) 2%
(B) 5%
(C) 8%
(D) 9%
(E) 20

Problem 3 ‡

A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers.

Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers.

A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

(A) 0.20
(B) 0.25
(C) 0.35
(D) 0.42
(E) 0.57

Problem 4 ‡

A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool.

This month, the insurance company writes four new policies for adults earning their first driver's license.

Calculate the probability that these four will contain at least two more high-risk drivers than low-risk drivers.

(A) 0.006
(B) 0.012
(C) 0.018
(D) 0.049
(E) 0.073

Problem 5 ‡
A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $\frac{3}{5}$.
The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months.
Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.
(A) 0.01
(B) 0.12
(C) 0.23
(D) 0.29
(E) 0.41

Problem 6 ‡
An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003e^{-\frac{x}{2}}, & 0 < x < 15 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected claim payment.

(A) 320
(B) 328
(C) 352
(D) 380
(E) 540

Problem 7 ‡
The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250 . In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $[0, 1500]$.
Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

(A) 361
(B) 403
(C) 433
(D) 464
(E) 521

Problem 8 ‡
Claim amounts for wind damage to insured homes are mutually independent random variables with common density function

$$f(x) = \begin{cases} \frac{3}{x^4} & x > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where x is the amount of a claim in thousands. Suppose 3 such claims will be made. Calculate the expected value of the largest of the three claims.

(A) 2025
(B) 2700
(C) 3232
(D) 3375
(E) 4500

Problem 9 ‡
In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from -2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people.
What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

(A) 0.14

(B) 0.38
(C) 0.57
(D) 0.77
(E) 0.88

Problem 10 ‡
Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \leq t_1 \leq t_2 \leq L$, where L is a positive constant.
Determine the expected value of the sum of the squares of T_1 and T_2.

(A) $\frac{L^2}{3}$
(B) $\frac{L^2}{2}$
(C) $\frac{2L^2}{3}$
(D) $\frac{3L^2}{4}$
(E) L^2

Problem 11 ‡
Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X) = 5, E(X^2) = 27.4, E(Y) = 7, E(Y^2) = 51.4$, and $\text{Var}(X + Y) = 8$.
Let $C_1 = X+Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge.
Calculate $\text{Cov}(C_1, C_2)$.

(A) 8.80
(B) 9.60
(C) 9.76
(D) 11.52
(E) 12.32

Problem 12 ‡
A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f_{XY}(x, y) = \begin{cases} 6[1 - (x + y)] & x > 0, y > 0, x + y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

(A) 0.360
(B) 0.480
(C) 0.488
(D) 0.512
(E) 0.520

Problem 13 ‡
The amounts of automobile losses reported to an insurance company are mutually independent, and each loss is uniformly distributed between 0 and 20,000. The company covers each such loss subject to a deductible of 5,000. Calculate the probability that the total payout on 200 reported losses is between 1,000,000 and 1,200,000.

(A) 0.0803
(B) 0.1051
(C) 0.1799
(D) 0.8201
(E) 0.8575

Problem 14 ‡
Let X and Y be identically distributed independent random variables such that the moment generating function of $X + Y$ is

$$M(t) = 0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t}, \ -\infty < t < \infty.$$

Calculate $\mathrm{P}(X \leq 0)$.

(A) 0.33
(B) 0.34
(C) 0.50
(D) 0.67
(E) 0.70

Problem 15 ‡
The amount of a claim that a car insurance company pays out follows an

exponential distribution. By imposing a deductible of d, the insurance company reduces the expected claim payment by 10%.
Calculate the percentage reduction on the variance of the claim payment.

(A) 1%
(B) 5%
(C) 10%
(D) 20%
(E) 25%

Problem 16 ‡
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{p-1}{x^p}, & x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the value of p such that $E(X) = 2$.

(A) 1
(B) 2.5
(C) 3
(D) 5
(E) There is no such p.

Problem 17 ‡
Damages to a car in a crash are modeled by a random variable with density function

$$f(x) = \begin{cases} c(x^2 - 60x + 800), & 0 < x < 20 \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant. A particular car is insured with a deductible of 2. This car was involved in a crash with resulting damages in excess of the deductible. Calculate the probability that the damages exceeded 10.

(A) 0.12
(B) 0.16
(C) 0.20
(D) 0.26
(E) 0.78

Problem 18 ‡
In a group of 25 factory workers, 20 are low-risk and five are high-risk. Two of the 25 factory workers are randomly selected without replacement.
Calculate the probability that exactly one of the two selected factory workers is low-risk.

(A) 0.160
(B) 0.167
(C) 0.320
(D) 0.333
(E) 0.633

Problem 19 ‡
The number of policies that an agent sells has a Poisson distribution with modes at 2 and 3. K is the smallest number such that the probability of selling more than K policies is less than 25%. Calculate K.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Problem 20 ‡
On any given day, a certain machine has either no malfunctions or exactly one malfunction. The probability of malfunction on any given day is 0.40. Machine malfunctions on different days are mutually independent. Calculate the probability that the machine has its third malfunction on the fifth day, given that the machine has not had three malfunctions in the first three days.

(A) 0.064
(B) 0.138
(C) 0.148
(D 0.230
(E) 0.246

Problem 21 ‡
A policyholder purchases automobile insurance for two years. Define the following events:

F = the policyholder has exactly one accident in year one.
G = the policyholder has one or more accidents in year two.

Define the following events:
i) The policyholder has exactly one accident in year one and has more than one accident in year two.
ii) The policyholder has at least two accidents during the two-year period.
iii) The policyholder has exactly one accident in year one and has at least one accident in year two.
iv) The policyholder has exactly one accident in year one and has a total of two or more accidents in the two-year period.
v) The policyholder has exactly one accident in year one and has more accidents in year two than in year one.
Determine the number of events from the above list of five that are the same as $F \cap G$.

(A) None
(B) Exactly one
(C) Exactly two
(D) Exactly three
(E) All

Problem 22 ‡
For a certain health insurance policy, losses are uniformly distributed on the interval $[0, b]$. The policy has a deductible of 180 and the expected value of the un-reimbursed portion of a loss is 144. Calculate b.

(A) 236
(B) 288
(C) 388
(D) 450
(E) 468

Problem 23 ‡
The random variables X and Y have joint probability function $p(x, y)$ for $x = 0, 1$ and $y = 0, 1, 2$. Suppose $3p(1, 1) = p(1, 2)$, and $p(1, 1)$ maximizes the variance of XY. Calculate the probability that X or Y is 0.

(A) 11/25

(B) 23/50
(C) 23/49
(D) 26/49
(E) 14/25

Problem 24 ‡
In a group of 15 health insurance policyholders diagnosed with cancer, each policyholder has probability 0.90 of receiving radiation and probability 0.40 of receiving chemotherapy. Radiation and chemotherapy treatments are independent events for each policyholder, and the treatments of different policyholders are mutually independent. The policyholders in this group all have the same health insurance that pays 2 for radiation treatment and 3 for chemotherapy treatment.
Calculate the variance of the total amount the insurance company pays for the radiation and chemotherapy treatments for these 15 policyholders.

(A) 13.5
(B) 37.8
(C) 108.0
(D) 202.5
(E) 567.0

Problem 25 ‡
Each week, a subcommittee of four individuals is formed from among the members of a committee comprising seven individuals. Two subcommittee members are then assigned to lead the subcommittee, one as chair and the other as secretary.
Calculate the maximum number of consecutive weeks that can elapse without having the subcommittee contain four individuals who have previously served together with the same subcommittee chair.

(A) 70
(B) 140
(C) 210
(D) 420
(E) 840

Problem 26 ‡
A machine has two parts labeled A and B. The probability that part A works

for one year is 0.8 and the probability that part B works for one year is 0.6. The probability that at least one part works for one year is 0.9.
Calculate the probability that part B works for one year, given that part A works for one year.

(A) 1/2
(B) 3/5
(C) 5/8
(D) 3/4
(E) 5/6

Problem 27 ‡
The time until the next car accident for a particular driver is exponentially distributed with a mean of 200 days.
Calculate the probability that the driver has no accidents in the next 365 days, but then has at least one accident in the 365-day period that follows this initial 365-day period.

(A) 0.026
(B) 0.135
(C) 0.161
(D) 0.704
(E) 0.839

Problem 28 ‡
Losses under an insurance policy have the density function

$$f(x) = \begin{cases} 0.25e^{-0.25x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The deductible is 1 for each loss.
Calculate the median amount that the insurer pays a policyholder for a loss under the policy.

(A) 1.77
(B) 2.08
(C) 2.12
(D) 2.77
(E) 3.12

Problem 29 ‡
The loss L due to a boat accident is exponentially distributed. Boat insurance policy A covers up to 1 unit for each loss. Boat insurance policy B covers up to 2 units for each loss.
The probability that a loss is fully covered under policy B is 1.9 times the probability that it is fully covered under policy A.
Calculate the variance of L.

(A) 0.1
(B) 0.4
(C) 2.4
(D) 9.5
(E) 90.1

Problem 30 ‡
The random variable X has moment generating function $M(t)$. Determine which of the following is the moment generating function of some random variable.
i) $M(t)M(5t)$
ii) $2M(t)$
iii) $e^t M(t)$.

(A) at most one of i, ii, and iii
(B) i and ii only
(C) i and iii only
(D) ii and iii only
(E) i, ii, and iii

Problem 31 ‡
The number of workplace injuries, N, occurring in a factory on any given day is Poisson distributed with mean λ. The parameter λ is a random variable that is determined by the level of activity in the factory, and is uniformly distributed on the interval $[0, 3]$.
Calculate $\text{Var}(N)$.

(A) λ
(B) 2λ
(C) 0.75

(D) 0.150
(E) 2.25

Problem 32 ‡
The intensity of a hurricane is a random variable that is uniformly distributed on the interval $[0, 3]$. The damage from a hurricane with a given intensity y is exponentially distributed with a mean equal to y.
Calculate the variance of the damage from a random hurricane.

(A) 1.73
(B) 1.94
(C) 3.00
(D) 3.75
(E) 6.00

Answers

1. D
2. E
3. D
4. D
5. D
6. B
7. B
8. A
9. D
10. C
11. A
12. C
13. D
14. E
15. A
16. C
17. D
18. D
19. D
20. C
21. C
22. D
23. C
24. B
25. B
26. C
27. B
28. A
29. E
30. C
31. B
32. D

Sample Exam 2

Duration: 3 hours

Problem 1 ‡
The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35% . Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

(A) 0.05
(B) 0.12
(C) 0.18
(D) 0.25
(E) 0.35

Problem 2 ‡
An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is $\frac{1}{3}$. What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

(A) 0.280
(B) 0.311
(C) 0.467
(D) 0.484
(E) 0.700

Problem 3 ‡
An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

(A) 0.06
(B) 0.16
(C) 0.19
(D) 0.22
(E) 0.25

Problem 4 ‡
The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & 0 < x < 20 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16 ?

(A) 1/25
(B) 1/9

(C) 1/8
(D) 1/3
(E) 3/7

Problem 5 ‡
An insurance policy pays 100 per day for up to 3 days of hospitalization and 50 per day for each day of hospitalization thereafter.
The number of days of hospitalization, X, is a discrete random variable with probability function

$$p(k) = \begin{cases} \frac{6-k}{15} & k = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the expected payment for hospitalization under this policy.

(A) 123
(B) 210
(C) 220
(D) 270
(E) 367

Problem 6 ‡
An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1 + x)^{-4}$, where $0 < x < \infty$ and 0 otherwise.
Determine the company's expected monthly claims.

(A) 1/6
(B) 1/3
(C) 1/2
(D) 1
(E) 3

Problem 7 ‡
A baseball team has scheduled its opening game for April 1. If it rains on April 1, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed.
The insurance company determines that the number of consecutive days of

rain beginning on April 1 is a Poisson random variable with mean 0.6 .
What is the standard deviation of the amount the insurance company will have to pay?

(A) 668
(B) 699
(C) 775
(D) 817
(E) 904

Problem 8 ‡
A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f_{XY}(x,y) = \begin{cases} \frac{x+y}{8} & 0 < x, y < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the device fails during its first hour of operation?

(A) 0.125
(B) 0.141
(C) 0.391
(D) 0.625
(E) 0.875

Problem 9 ‡
The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively.
What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

(A) $\frac{1}{18}(1 - e^{-\frac{2}{3}} - e^{-\frac{1}{2}} + e^{-\frac{7}{6}})$
(B) $\frac{1}{18}e^{-\frac{7}{6}}$
(C) $1 - e^{-\frac{2}{3}} - e^{-\frac{1}{2}} + e^{-\frac{7}{6}}$
(D) $1 - e^{-\frac{2}{3}} - e^{-\frac{1}{2}} + e^{-\frac{1}{3}})$
(E) $1 - \frac{1}{3}e^{-\frac{2}{3}} - \frac{1}{6}e^{-\frac{1}{2}} + \frac{1}{8}e^{-\frac{7}{6}}$

Problem 10 ‡
Let X_1, X_2, X_3 be three independent discrete random variables with common probability mass function

$$\mathrm{P}(x) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{2}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the moment generating function $M(t)$, of $Y = X_1X_2X_3$.

(A) $\frac{19}{27} + \frac{8}{27}e^t$
(B) $1 + 2e^t$
(C) $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^3$
(D) $\frac{1}{27} + \frac{8}{27}e^{3t}$
(E) $\frac{1}{3} + \frac{2}{3}e^{3t}$

Problem 11 ‡
A device containing two key components fails when, and only when, both components fail. The lifetimes, X and Y, of these components are independent with common density function $f(t) = e^{-t}, t > 0$. The cost, Z, of operating the device until failure is $2X + Y$.
Find the probability density function of Z.

(A) $e^{-\frac{z}{2}} - e^{-z}$
(B) $2(e^{-\frac{z}{2}} - e^{-z})$
(C) $\frac{z^2e^{-z}}{2}$
(D) $\frac{e^{-\frac{z}{2}}}{2}$
(E) $\frac{e^{-\frac{z}{3}}}{3}$

Problem 12 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} 15y & x^2 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal density function of Y.

(A)

$$f_Y(y) = \begin{cases} 15y & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(B)

$$f_Y(y) = \begin{cases} \frac{15y^2}{2} & x^2 < y < x \\ 0, & \text{otherwise.} \end{cases}$$

(C)

$$f_Y(y) = \begin{cases} \frac{15y^2}{2} & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(D)

$$f_Y(y) = \begin{cases} 15y^{\frac{3}{2}}(1 - y^{\frac{1}{2}}) & x^2 < y < x \\ 0, & \text{otherwise.} \end{cases}$$

(E)

$$f_Y(y) = \begin{cases} 15y^{\frac{3}{2}}(1 - y^{\frac{1}{2}}) & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Problem 13 ‡

An insurance agent offers his clients auto insurance, homeowners insurance and renters insurance. The purchase of homeowners insurance and the purchase of renters insurance are mutually exclusive. The profile of the agent's clients is as follows:

i) 17% of the clients have none of these three products.

ii) 64% of the clients have auto insurance.

iii) Twice as many of the clients have homeowners insurance as have renters insurance.

iv) 35% of the clients have two of these three products.

v) 11% of the clients have homeowners insurance, but not auto insurance.

Calculate the percentage of the agent's clients that have both auto and renters insurance.

(A) 7%
(B) 10%
(C) 16%
(D) 25%
(E) 28%

Problem 14 ‡
A machine consists of two components, whose lifetimes have the joint density function

$$f(x,y) = \begin{cases} \frac{1}{50} & \text{for } x > 0, y > 0, x + y < 10 \\ 0 & \text{otherwise.} \end{cases}$$

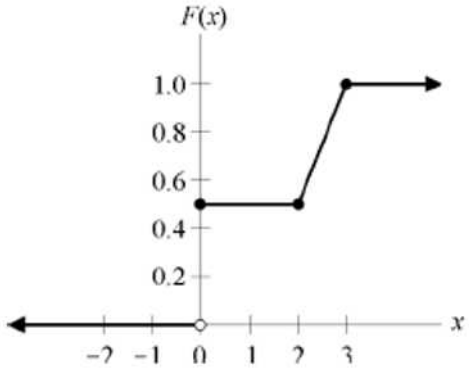
The machine operates until both components fail. Calculate the expected operational time of the machine.

(A) 1.7
(B) 2.5
(C) 3.3
(D) 5.0
(E) 6.7

Problem 15 ‡
The number of hurricanes that will hit a certain house in the next ten years is Poisson distributed with mean 4.
Each hurricane results in a loss that is exponentially distributed with mean 1000. Losses are mutually independent and independent of the number of hurricanes.
Calculate the variance of the total loss due to hurricanes hitting this house in the next ten years.

(A) 4,000,000
(B) 4,004,000
(C) 8,000,000
(D) 16,000,000
(E) 20,000,000

Problem 16 ‡
The figure below shows the cumulative distribution function of a random variable, X.



Calculate $E(X)$.

(A) 0.00
(B) 0.50
(C) 1.00
(D) 1.25
(E) 2.50

Problem 17 ‡
Two fair dice, one red and one blue, are rolled.
Let I be the event that the number rolled on the red die is odd.
Let J be the event that the number rolled on the blue die is odd.
Let H be the event that the sum of the numbers rolled on the two dice is odd.
Determine which of the following is true.

(A) I, J, and H are not mutually independent, but each pair is independent.
(B) I, J, and H are mutually independent.
(C) Exactly one pair of the three events is independent.
(D) Exactly two of the three pairs are independent.
(E) No pair of the three events is independent.

Problem 18 ‡
The proportion X of yearly dental claims that exceed 200 is a random variable

with probability density function

$$f(x) = \begin{cases} 60x^3(1-x)^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate Var $\left[\frac{X}{1-X}\right]$.

(A) 149/900
(B) 10/7
(C) 6
(D) 8
(E) 10

Problem 19 ‡
Two fair dice are tossed. One die is red and one die is green. Calculate the probability that the sum of the numbers on the two dice is an odd number given that the number that shows on the red die is larger than the number that shows on the green die.

(A) 1/4
(B) 5/12
(C) 3/7
(D) 1/2
(E) 3/5

Problem 20 ‡
In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4.
i) 75% of the patients in the group have stage 2 or lower.
ii) 80% of the patients in the group have stage 1 or higher.
iii) 80% of the patients in the group have stage 0, 1, 3, or 4.
One patient from the group is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

(A) 0.20
(B) 0.25
(C) 0.35
(D) 0.48
(E) 0.65

Problem 21 ‡
An insurance company categorizes its policyholders into three mutually exclusive groups: high-risk, medium-risk, and low-risk. An internal study of the company showed that 45% of the policyholders are low-risk and 35% are medium-risk. The probability of death over the next year, given that a policyholder is high-risk is two times the probability of death of a medium-risk policyholder. The probability of death over the next year, given that a policyholder is medium-risk is three times the probability of death of a low-risk policyholder. The probability of death of a randomly selected policyholder, over the next year, is 0.009.
Calculate the probability of death of a policyholder over the next year, given that the policyholder is high-risk.

(A) 0.0025
(B) 0.0200
(C) 0.1215
(D) 0.2000
(E) 0.3750

Problem 22 ‡
The working lifetime, in years, of a particular model of bread maker is normally distributed with mean 10 and variance 4.
Calculate the 12th percentile of the working lifetime, in years.

(A) 5.30
(B) 7.65
(C) 8.41
(D) 12.35
(E) 14.70

Problem 23 ‡
As a block of concrete is put under increasing pressure, engineers measure the pressure X at which the first fracture appears and the pressure Y at which the second fracture appears. X and Y are measured in tons per square inch and have joint density function

$$f_{XY}(x,y) = \begin{cases} 24x(1-y) & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the average pressure (in tons per square inch) at which the second fracture appears, given that the first fracture appears at 1/3 ton per square inch.

(A) 4/9
(B) 5/9
(C) 2/3
(D) 3/4
(E) 80/81

Problem 24 ‡
In a large population of patients, 20% have early stage cancer, 10% have advanced stage cancer, and the other 70% do not have cancer. Six patients from this population are randomly selected.
Calculate the expected number of selected patients with advanced stage cancer, given that at least one of the selected patients has early stage cancer.

(A) 0.403
(B) 0.500
(C) 0.547
(D) 0.600
(E) 0.625

Problem 25 ‡
Bowl I contains eight red balls and six blue balls. Bowl II is empty. Four balls are selected at random, without replacement, and transferred from bowl I to bowl II. One ball is then selected at random from bowl II.
Calculate the conditional probability that two red balls and two blue balls were transferred from bowl I to bowl II, given that the ball selected from bowl II is blue.

(A) 0.21
(B) 0.24
(C) 0.43
(D) 0.49
(E) 0.57

Problem 26 ‡
Six claims are to be randomly selected from a group of thirteen different

claims, which includes two workers compensation claims, four homeowners claims and seven auto claims.
Calculate the probability that the six claims selected will include one workers compensation claim, two homeowners claims and three auto claims.

(A) 0.025
(B) 0.107
(C) 0.153
(D) 0.245
(E) 0.643

Problem 27 ‡
The annual profit of a life insurance company is normally distributed. The probability that the annual profit does not exceed 2000 is 0.7642. The probability that the annual profit does not exceed 3000 is 0.9066.
Calculate the probability that the annual profit does not exceed 1000.

(A) 0.1424
(B) 0.3022
(C) 0.5478
(D) 0.6218
(E) 0.7257

Problem 28 ‡
A company has purchased a policy that will compensate for the loss of revenue due to severe weather events. The policy pays 1000 for each severe weather event in a year after the first two such events in that year. The number of severe weather events per year has a Poisson distribution with mean 1. Calculate the expected amount paid to this company in one year.

(A) 80
(B) 104
(C) 368
(D) 512
(E) 632

Problem 29 ‡
Losses, X, under an insurance policy are exponentially distributed with mean

10. For each loss, the claim payment Y is equal to the amount of the loss in excess of a deductible $d > 0$.
Calculate $\text{Var}(Y)$.

(A) $100 - d$
(B) $(10 - d)^2$
(C) $100e^{-\frac{d}{10}}$
(D) $100(2e^{-\frac{d}{10}} - e^{-\frac{d}{5}})$
(E) $(10 - d)^2(2e^{-\frac{d}{10}} - e^{-\frac{d}{5}})$

Problem 30 ‡
The number of boating accidents X a policyholder experiences this year is modeled by a Poisson random variable with variance 0.10. An insurer reimburses only the first accident. Let Y be the number of unreimbursed accidents the policyholder experiences this year and let p be the probability function of Y. Determine $p(y)$.

(A)
$$p(y) = \begin{cases} 1.1e^{-0.1}, & y = 0 \\ \frac{(0.1)^{y+1}}{(y+1)!}e^{-0.1}, & y = 1, 2, 3, \cdots \end{cases}$$

(B)
$$p(y) = \begin{cases} 1.1e^{-0.1}, & y = 0 \\ \frac{(0.1)^{y}}{y!}e^{-0.1}, & y = 1, 2, 3, \cdots \end{cases}$$

(C)
$$p(y) = \begin{cases} 1.1e^{-0.1}, & y = 0 \\ \frac{(0.1)^{y-1}}{(y-1)!}e^{-0.1}, & y = 1, 2, 3, \cdots \end{cases}$$

(D) $p(y) = \frac{(0.1)^{y+1}}{(y+1)!}e^{-0.1}, \ y = 0, 1, 2, 3, \cdots$
(E) $p(y) = \frac{(0.1)^{y-1}}{(y-1)!}e^{-0.1}, \ y = 0, 1, 2, 3, \cdots$

Problem 31 ‡
Let X denote the loss amount sustained by an insurance company's policyholder in an auto collision. Let Z denote the portion of X that the insurance company will have to pay. An actuary determines that X and Z are independent with respective density and probability functions
$$f(x) = \begin{cases} \frac{1}{8}e^{-\frac{x}{8}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$P(Z=z)=\begin{cases}0.45, & z=1\\ 0.55, & \text{otherwise.}\end{cases}$$

Calculate the variance of the insurance company's claim payment ZX.

(A) 13.0
(B) 15.8
(C) 28.8
(D) 35.2
(E) 44.6

Problem 32 ‡
Random variables X and Y have joint distribution

	$X=0$	$X=1$	$X=2$
$Y=0$	1/15	a	2/15
$Y=1$	a	b	a
$Y=2$	2/15	a	1/15

Let a be the value that minimizes the variance of X. Calculate the variance of Y.

(A) 2/5
(B) 8/15
(C) 16/25
(D) 2/3
(E) 7/10

Answers

1. A
2. C
3. D
4. B
5. C
6. C
7. B
8. D
9. C
10. A
11. A
12. E
13. B
14. D
15. C
16. D
17. A
18. C
19. E
20. C
21. B
22. B
23. B
24. C
25. D
26. D
27. C
28. B
29. D
30. A
31. E
32. A

Sample Exam 3

Duration: 3 hours

Problem 1 ‡
You are given $P(A \cup B) = 0.7$ and $P(A \cup B^c) = 0.9$. Determine $P(A)$.

(A) 0.2
(B) 0.3
(C) 0.4
(D) 0.6
(E) 0.8

Problem 2 ‡
In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0, p_{n+1} = \frac{1}{5}p_n$, where p_n represents the probability that the policyholder files n claims during the period.
Under this assumption, what is the probability that a policyholder files more than one claim during the period?

(A) 0.04
(B) 0.16
(C) 0.20
(D) 0.80
(E) 0.96

Problem 3 ‡
The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}, \quad n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

(A) 1/3
(B) 2/5
(C) 1/2
(D) 3/5
(E) 5/6

Problem 4 ‡
The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ with probability density function f, where $f(x)$ is proportional to $(10 + x)^{-2}$.
Calculate the probability that the lifetime of the machine part is less than 6.

(A) 0.04
(B) 0.15
(C) 0.47
(D) 0.53
(E) 0.94

Problem 5 ‡
Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & -2 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

(A) 1/5
(B) 3/5
(C) 1
(D) 28/15
(E) 12/5

Problem 6 ‡
An insurance policy is written to cover a loss, X, where X has the density function

$$f(x) = \begin{cases} \frac{1}{1000} & 0 \leq x \leq 1000 \\ 0 & \text{otherwise.} \end{cases}$$

The policy has a deductible, d, and the expected payment under the policy is 25% of what it would be with no deductible.
Calculate d.

(A) 250
(B) 375
(C) 500
(D) 625
(E) 750

Problem 7 ‡
An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is:

$$f(x) = \begin{cases} ce^{-0.004x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

(A) 161
(B) 165
(C) 173
(D) 182
(E) 250

Problem 8 ‡
A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0 < s < 1$ and $0 < t < 1$.
Determine which of the following represents the probability that the device fails during the first half hour of operation.

(A) $\int_0^{0.5} \int_0^{0.5} f(s,t)dsdt$
(B) $\int_0^{1} \int_0^{0.5} f(s,t)dsdt$

(C) $\int_{0.5}^{1}\int_{0.5}^{1} f(s,t)dsdt$
(D) $\int_{0}^{0.5}\int_{0}^{1} f(s,t)dsdt + \int_{0}^{1}\int_{0}^{0.5} f(s,t)dsdt$
(E) $\int_{0}^{0.5}\int_{0.5}^{1} f(s,t)dsdt + \int_{0}^{1}\int_{0}^{0.5} f(s,t)dsdt$

Problem 9 ‡
The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f_{XY}(x,y) = \begin{cases} \frac{6}{125000}(50-x-y) & 0 < x < 50-y < 50 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both components are still functioning 20 months from now?

(A) $\frac{6}{125,000}\int_{0}^{20}\int_{0}^{20}(50-x-y)dydx$
(B) $\frac{6}{125,000}\int_{20}^{30}\int_{20}^{50-x}(50-x-y)dydx$
(C) $\frac{6}{125,000}\int_{20}^{30}\int_{20}^{50-x-y}(50-x-y)dydx$
(D) $\frac{6}{125,000}\int_{20}^{50}\int_{20}^{50-x}(50-x-y)dydx$
(E) $\frac{6}{125,000}\int_{20}^{50}\int_{20}^{50-x-y}(50-x-y)dydx$

Problem 10 ‡
An insurance policy pays a total medical benefit consisting of two parts for each claim. Let X represent the part of the benefit that is paid to the surgeon, and let Y represent the part that is paid to the hospital. The variance of X is 5000, the variance of Y is 10,000, and the variance of the total benefit, $X+Y$, is 17,000.
Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim.
Calculate the variance of the total benefit after these revisions have been made.

(A) 18,200
(B) 18,800
(C) 19,300
(D) 19,520
(E) 20,670

Problem 11 ‡
A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent. Let X denote the ratio of claims to premiums.
What is the density function of X?

(A) $\frac{1}{2x+1}$
(B) $\frac{2}{(2x+1)^2}$
(C) e^{-x}
(D) $2e^{-2x}$
(E) xe^{-x}

Problem 12 ‡
An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x < y < x + 1$.
If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.5?

(A) 3/8
(B) 1/2
(C) 3/4
(D) 7/8
(E) 15/16

Problem 13 ‡
The cumulative distribution function for health care costs experienced by a policyholder is modeled by the function

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{100}}, & \text{for } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The policy has a deductible of 20. An insurer reimburses the policyholder for 100% of health care costs between 20 and 120 less the deductible. Health

care costs above 120 are reimbursed at 50%. Let G be the cumulative distribution function of reimbursements given that the reimbursement is positive. Calculate $G(115)$.

(A) 0.683
(B) 0.727
(C) 0.741
(D) 0.757
(E) 0.777

Problem 14 ‡
A driver and a passenger are in a car accident. Each of them independently has probability 0.3 of being hospitalized. When a hospitalization occurs, the loss is uniformly distributed on $[0, 1]$. When two hospitalizations occur, the losses are independent.
Calculate the expected number of people in the car who are hospitalized, given that the total loss due to hospitalizations from the accident is less than 1.

(A) 0.510
(B) 0.534
(C) 0.600
(D) 0.628
(E) 0.800

Problem 15 ‡
A motorist makes three driving errors, each independently resulting in an accident with probability 0.25.
Each accident results in a loss that is exponentially distributed with mean 0.80. Losses are mutually independent and independent of the number of accidents. The motorist's insurer reimburses 70% of each loss due to an accident.
Calculate the variance of the total unreimbursed loss the motorist experiences due to accidents resulting from these driving errors.

(A) 0.0432
(B) 0.0756
(C) 0.1782

(D) 0.2520
(E) 0.4116

Problem 16 ‡
Two fair dice are rolled. Let X be the absolute value of the difference between the two numbers on the dice. Calculate the probability that $X < 3$.

(A) 2/9
(B) 1/3
(C) 4/9
(D) 5/9
(E) 2/3

Problem 17 ‡
An urn contains four fair dice. Two have faces numbered 1, 2, 3, 4, 5, and 6; one has faces numbered 2, 2, 4, 4, 6, and 6; and one has all six faces numbered 6. One of the dice is randomly selected from the urn and rolled. The same die is rolled a second time. Calculate the probability that a 6 is rolled both times.

(A) 0.174
(B) 0.250
(C) 0.292
(D) 0.380
(E) 0.417

Problem 18 ‡
This year, a medical insurance policyholder has probability 0.70 of having no emergency room visits, 0.85 of having no hospital stays, and 0.61 of having neither emergency room visits nor hospital stays.
Calculate the probability that the policyholder has at least one emergency room visit and at least one hospital stay this year.

(A) 0.045
(B) 0.060
(C) 0.390
(D) 0.667
(E) 0.840

Problem 19 ‡
In 1982 Abby's mother scored at the 93rd percentile in the math SAT exam. In 1982 the mean score was 503 and the variance of the scores was 9604.
In 2008 Abby took the math SAT and got the same numerical score as her mother had received 26 years before. In 2008 the mean score was 521 and the variance of the scores was 10,201.
Math SAT scores are normally distributed and stated in multiples of ten. Calculate the percentile for Abby's score.

(A) 89th
(B) 90th
(C) 91st
(D) 92nd
(E) 93rd

Problem 20 ‡
A car is new at the beginning of a calendar year. The time, in years, before the car experiences its first failure is exponentially distributed with mean 2. Calculate the probability that the car experiences its first failure in the last quarter of some calendar year.

(A) 0.081
(B) 0.088
(C) 0.102
(D) 0.205
(E) 0.250

Problem 21 ‡
A policy covers a gas furnace for one year. During that year, only one of three problems can occur:
i) The igniter switch may need to be replaced at a cost of 60. There is a 0.10 probability of this.
ii) The pilot light may need to be replaced at a cost of 200. There is a 0.05 probability of this.
iii) The furnace may need to be replaced at a cost of 3000. There is a 0.01 probability of this.
Calculate the deductible that would produce an expected claim payment of 30.

(A) 100
(B) At least 100 but less than 150
(C) At least 150 but less than 200
(D) At least 200 but less than 250
(E) At least 250

Problem 22 ‡
The profits of life insurance companies A and B are normally distributed with the same mean. The variance of company B's profit is 2.25 times the variance of company A's profit. The 14th percentile of company A's profit is the same as the pth percentile of company B's profit. Calculate p.

(A) 5.3
(B) 9.3
(C) 21.0
(D) 23.6
(E) 31.6

Problem 23 ‡
The number of severe storms that strike city J in a year follows a binomial distribution with $n = 5$ and $p = 0.6$. Given that m severe storms strike city J in a year, the number of severe storms that strike city K in the same year is: m with probability 1/2, $m + 1$ with probability 1/3, and $m + 2$ with probability 1/6.
Calculate the expected number of severe storms that strike city J in a year during which 5 severe storms strike city K.

(A) 3.5
(B) 3.7
(C) 3.9
(D) 4.0
(E) 5.7

Problem 24 ‡
Four distinct integers are chosen randomly and without replacement from the first twelve positive integers. Let X be the random variable representing the second largest of the four selected integers, and let $p(x)$ be the probability

mass function of X. Determine $p(x)$, for integer values of x, where $p(x) > 0$.

(A) $\frac{(x-1)(x-2)(12-x)}{990}$
(B) $\frac{(x-1)(x-2)(12-x)}{495}$
(C) $\frac{(x-1)(11-x)(12-x)}{495}$
(D) $\frac{(x-1)(11-x)(12-x)}{990}$
(E) $\frac{(10-x)(11-x)(12-x)}{990}$

Problem 25 ‡
An actuary has done an analysis of all policies that cover two cars. 70% of the policies are of type A for both cars, and 30% of the policies are of type B for both cars. The number of claims on different cars across all policies are mutually independent. The distributions of the number of claims on a car are given in the following table.

Number of Claims	Type A	Type B
0	40%	25%
1	30%	25%
2	20%	25%
3	10%	25%

Four policies are selected at random. Calculate the probability that exactly one of the four policies has the same number of claims on both covered cars.

(A) 0.104
(B) 0.250
(C) 0.285
(D) 0.417
(E) 0.739

Problem 26 ‡
A drawer contains four pairs of socks, with each pair a different color. One sock at a time is randomly drawn from the drawer until a matching pair is obtained. Calculate the probability that the maximum number of draws is required.

(A) 0.0006
(B) 0.0095

(C) 0.0417
(D) 0.1429
(E) 0.2286

Problem 27 ‡
Individuals purchase both collision and liability insurance on their automobiles. The value of the insured's automobile is V. Assume the loss L on an automobile claim is a random variable with cumulative distribution function

$$F(\ell) = \begin{cases} \frac{3}{4}\left(\frac{\ell}{V}\right)^3, & 0 \le \ell < V \\ 1 - \frac{1}{10}e^{-\frac{\ell - V}{V}}, & \ell \ge V. \end{cases}$$

Calculate the probability that the loss on a randomly selected claim is greater than the value of the automobile.

(A) 0.00
(B) 0.10
(C) 0.25
(D) 0.75
(E) 0.90

Problem 28 ‡
A company provides each of its employees with a death benefit of 100. The company purchases insurance that pays the cost of total death benefits in excess of 400 per year. The number of employees who will die during the year is a Poisson random variable with mean 2. Calculate the expected annual cost to the company of providing the death benefits, excluding the cost of the insurance.

(A) 171
(B) 189
(C) 192
(D) 200
(E) 208

Problem 29 ‡
For a certain insurance company, 10% of its policies are Type A, 50% are Type B, and 40% are Type C. The annual number of claims for an individual Type A, Type B, and Type C policy follow Poisson distributions with

respective means 1, 2, and 10. Let X represent the annual number of claims of a randomly selected policy. Calculate the variance of X.

(A) 5.10
(B) 16.09
(C) 21.19
(D) 42.10
(E) 47.20

Problem 30 ‡
Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$. Let Y be the smallest integer greater than or equal to X. Determine the probability function of Y.

(A) $1 - e^{-\lambda y},\ y = 1, 2, 3, \cdots$
(B) $e^{-\lambda y}(e^{\lambda} - 1),\ y = 1, 2, 3, \cdots$
(C) $e^{-\lambda y}(1 - e^{-\lambda}),\ y = 1, 2, 3, \cdots$
(D) $\lambda e^{-\lambda y}\ y = 1, 2, 3, \cdots$
(E) $\frac{\lambda^y e^{-\lambda}}{y!},\ y = 1, 2, 3, \cdots$

Problem 31 ‡
A city with borders forming a square with sides of length 1 has its city hall located at the origin when a rectangular coordinate system is imposed on the city so that two sides of the square are on the positive axes. The density function of the population is

$$f(x, y) = \begin{cases} 1.5(x^2 + y^2) & \text{for } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

A resident of the city can travel to the city hall only along a route whose segments are parallel to the city borders.
Calculate the expected value of the travel distance to the city hall of a randomly chosen resident of the city.

(A) 1/2

(B) 3/4
(C) 1
(D) 5/4
(E) 3/2

Problem 32 ‡
Annual windstorm losses, X and Y, in two different regions are independent, and each is uniformly distributed on the interval [0,10]. Calculate the covariance of X and Y, given that $X + Y < 10$.

(A) $-50/9$
(B) $-25/9$
(C) 0
(D) 25/9
(E) 50/9

Answers

1. D
2. A
3. B
4. C
5. D
6. C
7. C
8. E
9. B
10. C
11. B
12. D
13. B
14. B
15. B
16. E
17. C
18. B
19. B
20. D
21. C
22. D
23. C
24. A
25. D
26. E
27. B
28. C
29. C
30. B
31. D
32. B

Sample Exam 4

Duration: 3 hours

Problem 1 ‡
An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn.

(A) 4
(B) 20
(C) 24
(D) 44
(E) 64

Problem 2 ‡
An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $\frac{1}{4}, \frac{1}{3}$, and , $\frac{5}{12}$ respectively.
Determine the probability that a randomly chosen employee will choose no supplementary coverage.

(A) 0
(B) 47/144

(C) 1/2
(D) 97/144
(E) 7/9

Problem 3 ‡
A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease.
Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

(A) 0.324
(B) 0.657
(C) 0.945
(D) 0.950
(E) 0.995

Problem 4 ‡
A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by

$$V = 100000Y$$

where Y is a random variable with density function

$$f(y) = \begin{cases} k(1-y)^4 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.
What is the conditional probability that V exceeds 40,000, given that V exceeds 10,000?

(A) 0.08
(B) 0.13
(C) 0.17
(D) 0.20
(E) 0.51

Problem 5 ‡
A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device has the density function

$$f(t) = \begin{cases} \frac{1}{3}e^{-\frac{t}{3}} & 0 \leq x \leq \infty \\ 0 & \text{otherwise.} \end{cases}$$

Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$.
Calculate $E(X)$.

(A) $2 + \frac{1}{3}e^{-6}$
(B) $2 - 2e^{-\frac{2}{3}} + 5e^{-\frac{4}{3}}$
(C) 3
(D) $2 + 3e^{-\frac{2}{3}}$
(E) 5

Problem 6 ‡
An actuary determines that the claim size for a certain class of accidents is a continuous random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1 - 2500t)^4}.$$

Calculate the standard deviation of the claim size for this class of accidents.

(A) 1,340
(B) 5,000
(C) 8,660
(D) 10,000
(E) 11,180

Problem 7 ‡
The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.
Calculate the probability that the component will work without failing for at least five hours.

(A) 0.07

(B) 0.29
(C) 0.38
(D) 0.42
(E) 0.57

Problem 8 ‡
A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250.
Calculate the approximate 90th percentile for the distribution of the total contributions received.

(A) 6,328,000
(B) 6,338,000
(C) 6,343,000
(D) 6,784,000
(E) 6,977,000

Problem 9 ‡
An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days.
What is the probability that the next claim will be a Deluxe Policy claim?

(A) 0.172
(B) 0.223
(C) 0.400
(D) 0.487
(E) 0.500

Problem 10 ‡
A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold. Given the following

information

$$P(X=0,Y=0) = \frac{1}{6}$$
$$P(X=1,Y=0) = \frac{1}{12}$$
$$P(X=1,Y=1) = \frac{1}{6}$$
$$P(X=2,Y=0) = \frac{1}{12}$$
$$P(X=2,Y=1) = \frac{1}{3}$$
$$P(X=2,Y=2) = \frac{1}{6}$$

What is the variance of X?

(A) 0.47
(B) 0.58
(C) 0.83
(D) 1.42
(E) 2.58

Problem 11 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} 24xy & 0 < x < 1, 0 < y < 1-x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\mathrm{P}\left(Y < X | X = \frac{1}{3}\right)$.

(A) 1/27
(B) 2/27
(C) 1/4
(D) 1/3
(E) 4/9

Problem 12 ‡
An insurance policy is written to cover a loss X where X has density function

$$f_X(x) = \begin{cases} \frac{3}{8}x^2 & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The time T (in hours) to process a claim of size x, where $0 \leq x \leq 2$, is uniformly distributed on the interval from x to $2x$.
Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

(A) 0.17
(B) 0.25
(C) 0.32
(D) 0.58
(E) 0.83

Problem 13 ‡
The value of a piece of factory equipment after three years of use is $100(0.5)^X$ where X is a random variable having moment generating function

$$M_X(t) = \frac{1}{1-2t} \text{ for } t < \frac{1}{2}.$$

Calculate the expected value of this piece of equipment after three years of use.

(A) 12.5
(B) 25.0
(C) 41.9
(D) 70.7
(E) 83.8

Problem 14 ‡
Each time a hurricane arrives, a new home has a 0.4 probability of experiencing damage. The occurrences of damage in different hurricanes are independent. Calculate the mode of the number of hurricanes it takes for the home to experience damage from two hurricanes.

(A) 2
(B) 3
(C) 4
(D) 5
(E) 6

Problem 15 ‡
An automobile insurance company issues a one-year policy with a deductible

of 500. The probability is 0.8 that the insured automobile has no accident and 0.0 that the automobile has more than one accident. If there is an accident, the loss before application of the deductible is exponentially distributed with mean 3000.
Calculate the 95th percentile of the insurance company payout on this policy.

(A) 3466
(B) 3659
(C) 4159
(D) 8487
(E) 8987

Problem 16 ‡
An auto insurance policy has a deductible of 1 and a maximum claim payment of 5. Auto loss amounts follow an exponential distribution with mean 2. Calculate the expected claim payment made for an auto loss.

(A) $0.5e^{-2} - 0.5e^{-12}$
(B) $2e^{-\frac{1}{2}} - 7e^{-3}$
(C) $2e^{-\frac{1}{2}} - 2e^{-3}$
(D) $2e^{-\frac{1}{2}}$
(E) $3e^{-\frac{1}{2}} - 2e^{-3}$

Problem 17 ‡
An insurance agent meets twelve potential customers independently, each of whom is equally likely to purchase an insurance product. Six are interested only in auto insurance, four are interested only in homeowners insurance, and two are interested only in life insurance.
The agent makes six sales. Calculate the probability that two are for auto insurance, two are for homeowners insurance, and two are for life insurance.

(A) 0.001
(B) 0.024
(C) 0.069
(D) 0.097
(E) 0.500

Problem 18 ‡
An insurer offers a travelers insurance policy. Losses under the policy are

uniformly distributed on the interval [0,5]. The insurer reimburses a policyholder for a loss up to a maximum of 4.
Determine the cumulative distribution function, F, of the benefit that the insurer pays a policyholder who experiences exactly one loss under the policy.

(A)
$$F(x) = \begin{cases} 0, & x < 0 \\ 0.20x, & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

(B)
$$F(x) = \begin{cases} 0, & x < 0 \\ 0.20x, & 0 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

(C)
$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25x, & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

(D)
$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5x, & 0 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

(E)
$$F(x) = \begin{cases} 0, & x < 1 \\ 0.25x, & 1 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

Problem 19 ‡
A certain brand of refrigerator has a useful life that is normally distributed with mean 10 years and standard deviation 3 years. The useful lives of these refrigerators are independent. Calculate the probability that the total useful life of two randomly selected refrigerators will exceed 1.9 times the useful life of a third randomly selected refrigerator.

(A) 0.407
(B) 0.444

(C) 0.556
(D) 0.593
(E) 0.604

Problem 20 ‡
In a shipment of 20 packages, 7 packages are damaged. The packages are randomly inspected, one at a time, without replacement, until the fourth damaged package is discovered. Calculate the probability that exactly 12 packages are inspected.

(A) 0.079
(B) 0.119
(C) 0.237
(D) 0.243
(E) 0.358

Problem 21 ‡
On a block of ten houses, k are not insured. A tornado randomly damages three houses on the block. The probability that none of the damaged houses are insured is 1/120. Calculate the probability that at most one of the damaged houses is insured.

(A) 1/5
(B) 7/40
(C) 11/60
(D) 49/60
(E) 119/120

Problem 22 ‡
The distribution of values of the retirement package offered by a company to new employees is modeled by the probability density function

$$f_X(x) = \begin{cases} \frac{1}{5}e^{-\frac{(x-5)}{5}}, & x \geq 5, \\ 0 & \text{otherwise} \end{cases}$$

Calculate the variance of the retirement package value for a new employee, given that the value is at least 10.

(A) 15

(B) 20
(C) 25
(D) 30
(E) 35

Problem 23 ‡
Let X denote the proportion of employees at a large firm who will choose to be covered under the firm's medical plan, and let Y denote the proportion who will choose to be covered under both the firm's medical and dental plans.
Suppose that for $0 \leq y \leq x \leq 1$, X and Y have the joint cumulative distribution function

$$F_{XY}(x, y) = y(x^2 + xy - y^2).$$

Calculate the expected proportion of employees who will choose to be covered under both plans.

(A) 0.06
(B) 0.33
(C) 0.42
(D) 0.50
(E) 0.75

Problem 24 ‡
An insurance policy covers losses incurred by a policyholder, subject to a deductible of 10,000. Incurred losses follow a normal distribution with mean 12,000 and standard deviation c. The probability that a loss is less than k is 0.9582, where k is a constant. Given that the loss exceeds the deductible, there is a probability of 0.9500 that it is less than k.
Calculate c.

(A) 2045
(B) 2267
(C) 2393
(D) 2505
(E) 2840

Problem 25 ‡
A company sells two types of life insurance policies (P and Q) and one type

of health insurance policy. A survey of potential customers revealed the following:
i) No survey participant wanted to purchase both life policies.
ii) Twice as many survey participants wanted to purchase life policy P as life policy Q.
iii) 45% of survey participants wanted to purchase the health policy.
iv) 18% of survey participants wanted to purchase only the health policy.
v) The event that a survey participant wanted to purchase the health policy was independent of the event that a survey participant wanted to purchase a life policy.
Calculate the probability that a randomly selected survey participant wanted to purchase exactly one policy.

(A) 0.51
(B) 0.60
(C) 0.69
(D) 0.73
(E) 0.78

Problem 26 ‡
At a mortgage company, 60% of calls are answered by an attendant. The remaining 40% of callers leave their phone numbers. Of these 40%, 75% receive a return phone call the same day. The remaining 25% receive a return call the next day.
Of those who initially spoke to an attendant, 80% will apply for a mortgage. Of those who received a return call the same day, 60% will apply. Of those who received a return call the next day, 40% will apply.
Calculate the probability that a person initially spoke to an attendant, given that he or she applied for a mortgage.

(A) 0.06
(B) 0.26
(C) 0.48
(D) 0.60
(E) 0.69

Problem 27 ‡
The lifetime of a machine part is exponentially distributed with a mean of

five years. Calculate the mean lifetime of the part, given that it survives less than ten years.

(A) 0.865
(B) 1.157
(C) 2.568
(D) 2.970
(E) 3.435

Problem 28 ‡
The number of burglaries occurring on Burlington Street during a one-year period is Poisson distributed with mean 1. Calculate the expected number of burglaries on Burlington Street in a one-year period, given that there are at least two burglaries.

(A) 0.63
(B) 2.39
(C) 2.54
(D) 3.00
(E) 3.78

Problem 29 ‡
The number of tornadoes in a given year follows a Poisson distribution with mean 3. Calculate the variance of the number of tornadoes in a year given that at least one tornado occurs.

(A) 1.63
(B) 1.73
(C) 2.66
(D) 3.00
(E) 3.16

Problem 30 ‡
Batteries A and B have lifetimes that are independent and exponentially distributed with a common mean of m years. The probability that battery B outlasts battery A by more than one year is 0.33. Calculate m.

(A) 0.42

(B) 0.59
(C) 0.90
(D) 1.80
(E) 2.41

Problem 31 ‡
A couple takes out a medical insurance policy that reimburses them for days of work missed due to illness. Let X and Y denote the number of days missed during a given month by the wife and husband, respectively. The policy pays a monthly benefit of 50 times the maximum of X and Y, subject to a benefit limit of 100. X and Y are independent, each with a discrete uniform distribution on the set $\{0, 1, 2, 3, 4\}$. Calculate the expected monthly benefit for missed days of work that is paid to the couple.

(A) 70
(B) 90
(C) 92
(D) 95
(E) 140

Problem 32 ‡
Let X be a random variable that takes on the values $-1, 0$, and 1 with equal probabilities. Let $Y = X^2$. Which of the following is true?
(A) $\text{Cov}(X, Y) > 0$; the random variables X and Y are dependent.
(B) $\text{Cov}(X, Y) > 0$; the random variables X and Y are independent.
(C) $\text{Cov}(X, Y) = 0$; the random variables X and Y are dependent.
(D) $\text{Cov}(X, Y) = 0$; the random variables X and Y are independent.
(E) $\text{Cov}(X, Y) < 0$; the random variables X and Y are dependent.

Answers

1. A
2. C
3. B
4. B
5. D
6. B
7. D
8. C
9. C
10. B
11. C
12. A
13. C
14. B
15. B
16. C
17. D
18. A
19. C
20. B
21. C
22. C
23. C
24. A
25. A
26. E
27. E
28. B
29. C
30. E
31. B
32. C

Sample Exam 5

Duration: 3 hours

Problem 1 ‡
An auto insurance has 10,000 policyholders. Each policyholder is classified as

(i) young or old;
(ii) male or female;
(iii) married or single.

Of these policyholders, 3,000 are young, 4,600 are male, and 7,000 are married. The policyholders can also be classified as 1,320 young males, 3,010 married males, and 1,400 young married persons. Finally, 600 of the policyholders are young married males.
How many of the company's policyholders are young, female, and single?

(A) 280
(B) 423
(C) 486
(D) 880
(E) 896 6.

Problem 2 ‡
An insurance company determines that N, the number of claims received in a week, is a random variable with $P[N = n] = \frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week

is independent of the number of claims received in any other week.
Determine the probability that exactly seven claims will be received during a given two-week period.

(A) 1/256
(B) 1/128
(C) 7/512
(D) 1/64
(E) 1/32

Problem 3 ‡
The probability that a randomly chosen male has a circulation problem is 0.25 . Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem.
What is the conditional probability that a male has a circulation problem, given that he is a smoker?

(A) 1/4
(B) 1/3
(C) 2/5
(D) 1/2
(E) 2/3

Problem 4 ‡
The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year.
If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

(A) 6,321
(B) 7,358
(C) 7,869
(D) 10,256
(E) 12,642

Problem 5 ‡
A piece of equipment is being insured against early failure. The time from

purchase until failure of the equipment has density function

$$f(t) = \begin{cases} \frac{1}{10}e^{-\frac{t}{10}} & 0 < t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made.
Calculate x such that the expected payment made under this insurance is 1000.

(A) 3858
(B) 4449
(C) 5382
(D) 5644
(E) 7235

Problem 6 ‡
A company insures homes in three cities, J, K, and L . Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent.
The moment generating functions for the loss distributions of the cities are:

$$\begin{aligned} M_J(t) =& (1-2t)^{-3} \\ M_K(t) =& (1-2t)^{-2.5} \\ M_L(t) =& (1-2t)^{-4.5} \end{aligned}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

(A) 1,320
(B) 2,082
(C) 5,760
(D) 8,000
(E) 10,560

Problem 7 ‡
Claims filed under auto insurance policies follow a normal distribution with mean 19,400 and standard deviation 5,000.
Calculate the probability that the average of 25 randomly selected claims

exceeds 20,000.

(A) 0.01
(B) 0.15
(C) 0.27
(D) 0.33
(E) 0.45

Problem 8 ‡
An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{2x+2-y}{4} & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the total loss is at least 1?

(A) 0.33
(B) 0.38
(C) 0.41
(D) 0.71
(E) 0.75

Problem 9 ‡
The profit for a new product is given by $Z = 3X - Y - 5$ where X and Y are independent random variables with $\text{Var}(X) = 1$ and $\text{Var}(Y) = 2$.
Calculate $\text{Var}(Z)$.

(A) 1
(B) 5
(C) 7
(D) 11
(E) 16

Problem 10 ‡
Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss.

When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)} & x > 1, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3 .

(A) 1/9
(B) 2/9
(C) 1/3
(D) 2/3
(E) 8/9

Problem 11 ‡
Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{64}(10 - xy^2) & 2 \le x \le 10, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

(A) 4.9
(B) 5.2
(C) 5.8
(D) 6.0
(E) 6.4

Problem 12 ‡
Let N_1 and N_2 represent the numbers of claims submitted to a life insurance company in April and May, respectively. The joint probability function of N_1 and N_2 is

$$P(n_1,n_2) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^{n_1-1} e^{-n_1}(1-e^{-n_1})^{n_2-1}, & \text{for } n_1 = 1,2,3,\cdots \text{ and } n_2 = 1,2,3,\cdots \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected number of claims that will be submitted to the company in May if exactly 2 claims were submitted in April.

(A) $\frac{3}{16}(e^2-1)$
(B) $\frac{3}{16}e^2$
(C) $\frac{3e}{4-e}$
(D) e^2-1
(E) e^2

Problem 13 ‡
Thirty items are arranged in a 6-by-5 array as shown.

A_1	A_2	A_3	A_4	A_5
A_6	A_7	A_8	A_9	A_{10}
A_{11}	A_{12}	A_{13}	A_{14}	A_{15}
A_{16}	A_{17}	A_{18}	A_{19}	A_{20}
A_{21}	A_{22}	A_{23}	A_{24}	A_{25}
A_{26}	A_{27}	A_{28}	A_{29}	A_{30}

Calculate the number of ways to form a set of three distinct items such that no two of the selected items are in the same row or same column.

(A) 200
(B) 760
(C) 1200
(D) 4560
(E) 7200

Problem 14 ‡
From 27 pieces of luggage, an airline luggage handler damages a random sample of four. The probability that exactly one of the damaged pieces of luggage is insured is twice the probability that none of the damaged pieces are insured. Calculate the probability that exactly two of the four damaged pieces are insured.

(A) 0.06
(B) 0.13
(C) 0.27
(D) 0.30
(E) 0.31

Problem 15 ‡
An actuary analyzes a company's annual personal auto claims, M, and annual commercial auto claims, N. The analysis reveals that $\text{Var}(M) = 1600$, $\text{Var}(N) = 900$, and the correlation between M and N is 0.64. Calculate $\text{Var}(M + N)$.

(A) 768
(B) 2500
(C) 3268
(D) 4036
(E) 4420

Problem 16 ‡
The return on two investments, X and Y, follows the joint probability density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}, & 0 < |x| + |y| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(X)$.

(A) 1/6
(B) 1/3
(C) 1/2
(D) 2/3
(E) 5/6

Problem 17 ‡
A company issues auto insurance policies. There are 900 insured individuals. Fifty-four percent of them are male. If a female is randomly selected from the 900, the probability she is over 25 years old is 0.43. There are 395 total insured individuals over 25 years old.
A person under 25 years old is randomly selected. Calculate the probability that the person selected is male.

(A) 0.47
(B) 0.53
(C) 0.54
(D) 0.55
(E) 0.56

Problem 18 ‡
A fire in an apartment building results in a loss, X, to the owner and a loss, Y, to the tenants. The variables X and Y have a bivariate normal distribution with $E(X) = 40$, $\text{Var}(X) = 76$, $E(Y) = 30$, $\text{Var}(Y) = 32$, and $\text{Var}(X|Y = 28.5) = 57$. Calculate $\text{Var}(Y|X = 25)$.

(A) 13
(B) 24
(C) 32
(D) 50
(E) 57

Problem 19 ‡
A theme park conducts a study of families that visit the park during a year. The fraction of such families of size m is $\frac{8-m}{28}$, $m = 1, 2, 3, 4, 5, 6$, and 7.
For a family of size m that visits the park, the number of members of the family that ride the roller coaster follows a discrete uniform distribution on the set $\{1, \cdots, m\}$.
Calculate the probability that a family visiting the park has exactly six members, given that exactly five members of the family ride the roller coaster

(A) 0.17
(B) 0.21
(C) 0.24
(D) 0.28
(E) 0.31

Problem 20 ‡
In a casino game, a gambler selects four different numbers from the first twelve positive integers. The casino then randomly draws nine numbers without replacement from the first twelve positive integers. The gambler wins the jackpot if the casino draws all four of the gambler's selected numbers.
Calculate the probability that the gambler wins the jackpot.

(A) 0.002
(B) 0.255
(C) 0.296

(D) 0.573
(E) 0.625

Problem 21 ‡
Insurance companies A and B each earn an annual profit that is normally distributed with the same positive mean. The standard deviation of company A's annual profit is one half of its mean.
In a given year, the probability that company B has a loss (negative profit) is 0.9 times the probability that company A has a loss.
Calculate the ratio of the standard deviation of company B's annual profit to the standard deviation of company A's annual profit.

(A) 0.49
(B) 0.90
(C) 0.98
(D) 1.11
(E) 1.71

Problem 22 ‡
Let N denote the number of accidents occurring during one month on the northbound side of a highway and let S denote the number occurring on the southbound side.
Suppose that N and S are jointly distributed as indicated in the table.

$N \backslash S$	0	1	2	3 or more
0	0.04	0.06	0.10	0.04
1	0.10	0.18	0.08	0.03
2	0.12	0.06	0.05	0.02
3 or more	0.05	0.04	0.02	0.01

Calculate $P(N|N+S=2)$.

(A) 0.48
(B) 0.55
(C) 0.67
(D) 0.91
(E) 1.25

Problem 23 ‡
Losses covered by an insurance policy have the density function

$$f(x) = \begin{cases} 0.001 & 0 \leq x \leq 1000 \\ 0 & \text{otherwise.} \end{cases}$$

An insurance company reimburses losses in excess of a deductible of 250. Calculate the difference between the median and the 20^{th} percentile of the insurance company reimbursement, over all losses.

(A) 225
(B) 250
(C) 300
(D) 375
(E) 500

Problem 24 ‡
A state is starting a lottery game. To enter this lottery, a player uses a machine that randomly selects six distinct numbers from among the first 30 positive integers. The lottery randomly selects six distinct numbers from the same 30 positive integers. A winning entry must match the same set of six numbers that the lottery selected. The entry fee is 1, each winning entry receives a prize amount of 500,000, and all other entries receive no prize. Calculate the probability that the state will lose money, given that 800,000 entries are purchased.

(A) 0.33
(B) 0.39
(C) 0.61
(D) 0.67
(E) 0.74

Problem 25 ‡
An insurance company studies back injury claims from a manufacturing company. The insurance company finds that 40% of workers do no lifting on the job, 50% do moderate lifting and 10% do heavy lifting. During a given year, the probability of filing a claim is 0.05 for a worker who does no lifting, 0.08 for a worker who does moderate lifting and 0.20 for a worker who does heavy lifting. A worker is chosen randomly from among those who have filed a back

injury claim.
Calculate the probability that the worker's job involves moderate or heavy lifting.

(A) 0.75
(B) 0.81
(C) 0.85
(D) 0.86
(E) 0.89

Problem 26 ‡
Let X be a random variable with density function

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $P(X \leq 0.5 | X \leq 1.0)$.

(A) 0.433
(B) 0.547
(C) 0.632
(D) 0.731
(E) 0.865

Problem 27 ‡
Losses incurred by a policyholder follow a normal distribution with mean 20,000 and standard deviation 4,500. The policy covers losses, subject to a deductible of 15,000. Calculate the 95^{th} percentile of losses that exceed the deductible. Round your answer to the nearest hundreds.

(A) 27,400
(B) 27,700
(C) 28,100
(D) 28,400
(E) 28,800

Problem 28 ‡
A government employee's yearly dental expense follows a uniform distribution on the interval from 200 to 1200. The government's primary dental plan

reimburses an employee for up to 400 of dental expense incurred in a year, while a supplemental plan pays up to 500 of any remaining dental expense. Let Y represent the yearly benefit paid by the supplemental plan to a government employee. Calculate $\text{Var}(Y)$.

(A) 20,833
(B) 26,042
(C) 41,042
(D) 53,333
(E) 83,333

Problem 29 ‡
A car and a bus arrive at a railroad crossing at times independently and uniformly distributed between 7:15 and 7:30. A train arrives at the crossing at 7:20 and halts traffic at the crossing for five minutes. Calculate the probability that the waiting time of the car or the bus at the crossing exceeds three minutes.

(A) 0.25
(B) 0.27
(C) 0.36
(D) 0.40
(E) 0.56

Problem 30 ‡
An individual experiences a loss due to property damage and a loss due to bodily injury. Losses are independent and uniformly distributed on the interval [0,3]. Calculate the expected loss due to bodily injury, given that at least one of the losses is less than 1.

(A) 0.50
(B) 1.00
(C) 1.10
(D) 1.25
(E) 1.50

Problem 31 ‡
Losses follow an exponential distribution with mean 1. Two independent

losses are observed.
Calculate the expected value of the smaller loss.

(A) 0.25
(B) 0.50
(C) 0.75
(D) 1.00
(E) 1.50

Answers

1. D
2. D
3. C
4. D
5. D
6. E
7. C
8. D
9. D
10. E
11. C
12. E
13. C
14. C
15. D
16. A
17. B
18. B
19. E
20. B
21. C
22. B
23. B
24. B
25. A
26. D
27. B
28. C
29. A
30. C
31. B

Sample Exam 6

Duration: 3 hours

Problem 1 ‡
A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.
Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.
(A) 0.115
(B) 0.173
(C) 0.224
(D) 0.327
(E) 0.514

Problem 2 ‡
An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims.
Calculate the probability that a claim submitted to the insurance company

includes operating room charges.

(A) 0.10
(B) 0.20
(C) 0.25
(D) 0.40
(E) 0.80

Problem 3 ‡
A study of automobile accidents produced the following data:

Model year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

(A) 0.22
(B) 0.30
(C) 0.33
(D) 0.45
(E) 0.50

Problem 4 ‡
An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2 ?

(A) 0.578

(B) 0.684
(C) 0.704
(D) 0.829
(E) 0.875

Problem 5 ‡
An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0 . If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4. What is the expected benefit under this policy?

(A) 2234
(B) 2400
(C) 2500
(D) 2667
(E) 2694

Problem 6 ‡
A loss random variable X has the density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}} & x > 200 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30^{th} and 70^{th} percentiles of X.

(A) 35
(B) 93
(C) 124
(D) 231
(E) 298

Problem 7 ‡
The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1 - \left(\frac{2}{t}\right)^2 & t > 2 \\ 0 & \text{otherwise} \end{cases}$$

The resulting cost to the company is $Y = T^2$. Determine the density function of Y, for $y > 4$.

(A) $\frac{4}{y^2}$
(B) $\frac{8}{y^{\frac{3}{2}}}$
(C) $\frac{8}{y^3}$
(D) $\frac{16}{y}$
(E) $\frac{1024}{y^5}$

Problem 8 ‡
An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2. Assume the numbers of claims filed by distinct policyholders are independent of one another.
What is the approximate probability that there is a total of between 2450 and 2600 claims during a one-year period?

(A) 0.68
(B) 0.82
(C) 0.87
(D) 0.95
(E) 1.00

Problem 9 ‡
Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200 . The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200.
Determine the probability that the company considers the two bids further.

(A) 0.10
(B) 0.19
(C) 0.20
(D) 0.41
(E) 0.60

Problem 10 ‡
A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails.
Calculate the variance of the total time that the generators produce electricity.

(A) 10
(B) 20
(C) 50
(D) 100
(E) 200

Problem 11 ‡
A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy.
Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f_{XY}(x, y) = 2(x + y)$ on the region where the density is positive.
Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

(A) 0.010
(B) 0.013
(C) 0.108
(D) 0.417
(E) 0.500

Problem 12 ‡
A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails.
Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f_{XY}(x, y) = \begin{cases} 6e^{-x}e^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time at which the device fails?

(A) 0.33
(B) 0.50
(C) 0.67
(D) 0.83
(E) 1.50

Problem 13 ‡
A store has 80 modems in its inventory, 30 coming from Source A and the remainder from Source B. Of the modems from Source A, 20% are defective. Of the modems from Source B, 8% are defective.
Calculate the probability that exactly two out of a random sample of five modems from the store's inventory are defective.

(A) 0.010
(B) 0.078
(C) 0.102
(D) 0.105
(E) 0.125

Problem 14 ‡
An auto insurance company is implementing a new bonus system. In each month, if a policyholder does not have an accident, he or she will receive a 5.00 cash-back bonus from the insurer.
Among the 1,000 policyholders of the auto insurance company, 400 are classified as low-risk drivers and 600 are classified as high-risk drivers.
In each month, the probability of zero accidents for high-risk drivers is 0.80 and the probability of zero accidents for low-risk drivers is 0.90.
Calculate the expected bonus payment from the insurer to the 1000 policyholders in one year.

(A) 48,000
(B) 50,400
(C) 51,000
(D) 54,000
(E) 60,000

Problem 15 ‡
Let X represent the number of customers arriving during the morning hours and let Y represent the number of customers arriving during the afternoon hours at a diner. You are given:
(i) X and Y are Poisson distributed.
(ii) The first moment of X is less than the first moment of Y by 8.
(iii) The second moment of X is 60% of the second moment of Y.
Calculate the variance of Y.

(A) 4
(B) 12
(C) 16
(D) 27
(E) 35

Problem 16 ‡
The joint probability density function of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{x+y}{8} & 0 < x < 2, 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\text{Var}(0.5(X+Y))$.

(A) 10/72
(B) 11/72
(C) 12/72
(D) 20/72
(E) 22/72

Problem 17 ‡
In a given region, the number of tornadoes in a one-week period is modeled by a Poisson distribution with mean 2. The numbers of tornadoes in different weeks are mutually independent.
Calculate the probability that fewer than four tornadoes occur in a three-week period.

(A) 0.13
(B) 0.15
(C) 0.29

(D) 0.43
(E) 0.86

Problem 18 ‡
An insurance company insures red and green cars. An actuary compiles the following data:

Color of car	Red	Green
Number insured	300	700
Probability an accident occurs	0.10	0.05
Probability that the claim exceeds the deductible, given an accident occurs from this group	0.90	0.80

The actuary randomly picks a claim from all claims that exceed the deductible. Calculate the probability that the claim is on a red car.

(A) 0.300
(B) 0.462
(C) 0.491
(D) 0.667
(E) 0.692

Problem 19 ‡
Losses covered by a flood insurance policy are uniformly distributed on the interval [0,2]. The insurer pays the amount of the loss in excess of a deductible d. The probability that the insurer pays at least 1.20 on a random loss is 0.30.
Calculate the probability that the insurer pays at least 1.44 on a random loss.

(A) 0.06
(B) 0.16
(C) 0.18
(D) 0.20
(E) 0.28

Problem 20 ‡
The following information is given about a group of high-risk borrowers.
i) Of all these borrowers, 30% defaulted on at least one student loan.

ii) Of the borrowers who defaulted on at least one car loan, 40% defaulted on at least one student loan.
iii) Of the borrowers who did not default on any student loans, 28% defaulted on at least one car loan.
A statistician randomly selects a borrower from this group and observes that the selected borrower defaulted on at least one student loan.
Calculate the probability that the selected borrower defaulted on at least one car loan.

(A) 0.33
(B) 0.40
(C) 0.44
(D) 0.65
(E) 0.72

Problem 21 ‡
The number of days an employee is sick each month is modeled by a Poisson distribution with mean 1. The numbers of sick days in different months are mutually independent.
Calculate the probability that an employee is sick more than two days in a three-month period.

(A) 0.199
(B) 0.224
(C) 0.423
(D) 0.577
(E) 0.801

Problem 22 ‡
Let X represent the number of policies sold by an agent in a day. The moment generating function of X is

$$M_X(t) = 0.45e^t + 0.35e^{2t} + 0.15e^{3t} + 0.05e^{4t}, \ -\infty < t < \infty.$$

Calculate the standard deviation of X.

(A) 0.76
(B) 0.87
(C) 1.48

(D) 1.80
(E) 4.00

Problem 23 ‡
An insurance company sells automobile liability and collision insurance. Let X denote the percentage of liability policies that will be renewed at the end of their terms and Y the percentage of collision policies that will be renewed at the end of their terms. X and Y have the joint cumulative distribution function

$$F_{XY}(x,y) = \frac{xy(x+y)}{2,000,000}, \quad 0 \le x \le 100, \ 0 \le y \le 100.$$

Calculate $\mathrm{Var}(X)$.

(A) 764
(B) 833
(C) 3402
(D) 4108
(E) 4167

Problem 24 ‡
The annual profits that company A and company B earn follow a bivariate normal distribution. Company A's annual profit has mean 2000 and standard deviation 1000. Company B's annual profit has mean 3000 and standard deviation 500. The correlation coefficient between these annual profits is 0.80. Calculate the probability that company B's annual profit is less than 3900, given that company A's annual profit is 2300.

(A) 0.8531
(B) 0.9192
(C) 0.9641
(D) 0.9744
(E) 0.9953

Problem 25 ‡
A life insurance company has found there is a 3% probability that a randomly selected application contains an error. Assume applications are mutually independent in this respect. An auditor randomly selects 100 applications.

Calculate the probability that 95% or less of the selected applications are error-free.

(A) 0.08
(B) 0.10
(C) 0.13
(D) 0.15
(E) 0.18

Problem 26 ‡
The number of traffic accidents occurring on any given day in Coralville is Poisson distributed with mean 5. The probability that any such accident involves an uninsured driver is 0.25, independent of all other such accidents. Calculate the probability that on a given day in Coralville there are no traffic accidents that involve an uninsured driver.

(A) 0.007
(B) 0.010
(C) 0.124
(D) 0.237
(E) 0.287

Problem 27 ‡
Events E and F are independent with $P(E) = 0.84$ and $P(F) = 0.65$. Calculate the probability that exactly one of the two events occurs.

(A) 0.056
(B) 0.398
(C) 0.546
(D) 0.650
(E) 0.944

Problem 28 ‡
For a certain health insurance policy, losses are uniformly distributed on the interval [0,450]. The policy has a deductible of d and the expected value of the unreimbursed portion of a loss is 56. Calculate d.

(A) 60

(B) 87
(C) 112
(D) 169
(E) 224

Problem 29 ‡
Under a liability insurance policy, losses are uniformly distributed on $[0, b]$, where b is a positive constant. There is a deductible of $0.5b$. Calculate the ratio of the variance of the claim payment (greater than or equal to zero) from a given loss to the variance of the loss.

(A) 1:8
(B) 3:16
(C) 1:4
(D) 5:16
(E) 1:2

Problem 30 ‡
An insurance company sells automobile liability and collision insurance. Let X denote the percentage of liability policies that will be renewed at the end of their terms and Y the percentage of collision policies that will be renewed at the end of their terms. X and Y have the joint cumulative distribution function

$$F_{XY}(x, y) = \frac{xy(x + y)}{2,000,000}, \quad 0 \leq x \leq 100, \ 0 \leq y \leq 100.$$

Calculate $\text{Var}(X)$.

(A) 0.072
(B) 0.200
(C) 0.280
(D) 0.360
(E) 0.488

Problem 31 ‡
The table below shows the joint probability function of a sailor's number of boating accidents X and number of hospitalizations from these accidents Y for this year.

$Y \backslash X$	0	1	2	3
0	0.70	0	0	0
1	0.15	0.05	0	0
2	0.06	0.02	0.01	0
3	0.005	0.002	0.002	0.001

Calculate the sailor's expected number of hospitalizations from boating accidents this year.

(A) 0.085
(B) 0.099
(C) 0.410
(D) 1.000
(E) 1.500

Problem 32 ‡
A delivery service owns two cars that consume 15 and 30 miles per gallon. Fuel costs 3 per gallon. On any given business day, each car travels a number of miles that is independent of the other and is normally distributed with mean 25 miles and standard deviation 3 miles. Calculate the probability that on any given business day, the total fuel cost to the delivery service will be less than 7.

(A) 0.13
(B) 0.23
(C) 0.29
(D) 0.38
(E) 0.47

Answers

1. B
2. D
3. D
4. A
5. E
6. B
7. A
8. B
9. B
10. E
11. D
12. D
13. C
14. B
15. E
16. A
17. B
18. C
19. C
20. C
21. D
22. B
23. A
24. E
25. E
26. E
27. B
28. A
29. D
30. A
31. B
32. B

Sample Exam 7

Duration: 3 hours

Problem 1 ‡
An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

(A) 20%
(B) 29%
(C) 41%
(D) 53%
(E) 70%

Problem 2 ‡
Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$. The errors from the two instruments are independent of each other.

Calculate the probability that the average value of the two measurements is within $0.005h$ of the height of the tower.

(A) 0.38
(B) 0.47
(C) 0.68
(D) 0.84
(E) 0.90

Problem 3 ‡
A hospital receives 1/5 of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials.
For Company X shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective.
What is the probability that this shipment came from Company X?

(A) 0.10
(B) 0.14
(C) 0.37
(D) 0.63
(E) 0.86

Problem 4 ‡
A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05 .
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

(A) 0.06
(B) 0.19
(C) 0.38

(D) 0.62
(E) 0.92

Problem 5 ‡
A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and $10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5 .
What is the expected amount paid to the company under this policy during a one-year period?

(A) 2,769
(B) 5,000
(C) 7,231
(D) 8,347
(E) 10,578

Problem 6 ‡
A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260. A tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive).
Calculate the variance of the annual cost of maintaining and repairing a car after the tax is introduced.

(A) 208
(B) 260
(C) 270
(D) 312
(E) 374

Problem 7 ‡
An investment account earns an annual interest rate R that follows a uniform distribution on the interval $(0.04, 0.08)$. The value of a 10,000 initial investment in this account after one year is given by $V = 10,000e^R$.
Determine the cumulative distribution function, $F_V(v)$ of V.

(A) $\frac{10{,}000v^{\frac{v}{10{,}000}}-10{,}408}{425}$
(B) $25v^{\frac{v}{10{,}000}}-0.04$
(C) $\frac{v-10{,}408}{10{,}833-10{,}408}$
(D) $\frac{25}{v}$
(E) $25\left[\ln\left(\frac{v}{10{,}000}\right)-0.04\right]$

Problem 8 ‡
A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1 . A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes.
What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772?

(A) 14
(B) 16
(C) 20
(D) 40
(E) 55

Problem 9 ‡
A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy has a deductible of 1 and the other has a deductible of 2. The family experiences exactly one loss under each policy.
Calculate the probability that the total benefit paid to the family does not exceed 5.

(A) 0.13
(B) 0.25
(C) 0.30
(D) 0.32
(E) 0.42

Problem 10 ‡
In a small metropolitan area, annual losses due to storm, fire, and theft are

assumed to be independent, exponentially distributed random variables with respective means 1.0, 1.5, and 2.4 .
Determine the probability that the maximum of these losses exceeds 3.

(A) 0.002
(B) 0.050
(C) 0.159
(D) 0.287
(E) 0.414

Problem 11 ‡
Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96 .
What is the expected excess of premiums over claims, given that the husband survives at least ten years?

(A) 350
(B) 385
(C) 397
(D) 870
(E) 897

Problem 12 ‡
You are given the following information about N, the annual number of claims for a randomly selected insured:

$$\begin{aligned} \mathrm{P}(N=0) &= \frac{1}{2} \\ \mathrm{P}(N=1) &= \frac{1}{3} \\ \mathrm{P}(N>1) &= \frac{1}{6} \end{aligned}$$

Let S denote the total annual claim amount for an insured. When $N=1, S$ is exponentially distributed with mean 5 . When $N>1, S$ is exponentially

distributed with mean 8 . Determine P$(4 < S < 8)$.

(A) 0.04
(B) 0.08
(C) 0.12
(D) 0.24
(E) 0.25

Problem 13 ‡
A man purchases a life insurance policy on his 40^{th} birthday. The policy will pay 5000 if he dies before his 50^{th} birthday and will pay 0 otherwise. The length of lifetime, in years from birth, of a male born the same year as the insured has the cumulative distribution function

$$F(t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{\frac{1-1.1^t}{1000}}, & t > 0. \end{cases}$$

Calculate the expected payment under this policy.

(A) 333
(B) 348
(C) 421
(D) 549
(E) 574

Problem 14 ‡
The probability that a member of a certain class of homeowners with liability and property coverage will file a liability claim is 0.04, and the probability that a member of this class will file a property claim is 0.10. The probability that a member of this class will file a liability claim but not a property claim is 0.01.
Calculate the probability that a randomly selected member of this class of homeowners will not file a claim of either type.

(A) 0.850
(B) 0.860
(C) 0.864
(D) 0.870
(E) 0.890

Problem 15 ‡
Automobile policies are separated into two groups: low-risk and high-risk. Actuary Rahul examines low-risk policies, continuing until a policy with a claim is found and then stopping. Actuary Toby follows the same procedure with high-risk policies. Each low-risk policy has a 10% probability of having a claim. Each high-risk policy has a 20% probability of having a claim. The claim statuses of policies are mutually independent.
Calculate the probability that Actuary Rahul examines fewer policies than Actuary Toby.

(A) 0.2857
(B) 0.3214
(C) 0.3333
(D) 0.3571
(E) 0.4000

Problem 16 ‡
A student takes a multiple-choice test with 40 questions. The probability that the student answers a given question correctly is 0.5, independent of all other questions. The probability that the student answers more than N questions correctly is greater than 0.10. The probability that the student answers more than $N + 1$ questions correctly is less than 0.10. Calculate N using a normal approximation with the continuity correction.

(A) 23
(B) 25
(C) 32
(D) 33
(E) 35

Problem 17 ‡
A policyholder has probability 0.7 of having no claims, 0.2 of having exactly one claim, and 0.1 of having exactly two claims. Claim amounts are uniformly distributed on the interval [0,60] and are independent. The insurer covers 100% of each claim. Calculate the probability that the total benefit paid to the policyholder is 48 or less.

(A) 0.320

(B) 0.400
(C) 0.800
(D) 0.892
(E) 0.924

Problem 18 ‡
X is a random variable with probability density function

$$f_X(x) = \begin{cases} e^{-2x}, & x \geq 0 \\ 2e^{4x}, & x < 0. \end{cases}$$

Determine the probability density function for $T = X^2$ for positive values of t.

(A) $f(t) = \frac{e^{-2\sqrt{t}}}{2\sqrt{t}} + \frac{e^{-4\sqrt{t}}}{\sqrt{t}}$
(B) $f(t) = \frac{e^{-2\sqrt{t}}}{2\sqrt{t}}$
(C) $f(t) = e^{-2t} + 2e^{-4t}$
(D) $f(t) = 2te^{-2t^2} + 4te^{-4t^2}$
(E) $f(t) = 2te^{-2t^2}$

Problem 19 ‡
The lifespan, in years, of a certain computer is exponentially distributed. The probability that its lifespan exceeds four years is 0.30.
Let $f(x)$ represent the density function of the computer's lifespan, in years, for $x > 0$. Determine the formula for $f(x)$.

(A) $1 - (0.3)^{-\frac{x}{4}}$
(B) $1 - (0.7)^{\frac{x}{4}}$
(C) $1 - (0.3)^{\frac{x}{4}}$
(D) $-\frac{\ln 0.7}{4}(0.7)^{\frac{x}{4}}$
(E) $-\frac{\ln 0.3}{4}(0.3)^{\frac{x}{4}}$

Problem 20 ‡
A machine has two components and fails when both components fail. The number of years from now until the first component fails, X, and the number of years from now until the machine fails, Y, are random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{18}e^{-\frac{(x+y)}{6}} & \text{for } 0 < x < y \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Var}(Y|X = 2)$.

(A) 6
(B) 8
(C) 36
(D) 45
(E) 64

Problem 21 ‡
The number of traffic accidents per week at intersection Q has a Poisson distribution with mean 3. The number of traffic accidents per week at intersection R has a Poisson distribution with mean 1.5.
Let A be the probability that the number of accidents at intersection Q exceeds its mean. Let B be the corresponding probability for intersection R. Calculate $B - A$.

(A) 0.00
(B) 0.09
(C) 0.13
(D) 0.19
(E) 0.31

Problem 22 ‡
Claim amounts at an insurance company are independent of one another. In year one, claim amounts are modeled by a normal random variable X with mean 100 and standard deviation 25. In year two, claim amounts are modeled by the random variable $Y = 1.04X + 5$.
Calculate the probability that a random sample of 25 claim amounts in year two average between 100 and 110.

(A) 0.48
(B) 0.53
(C) 0.54
(D) 0.67
(E) 0.68

Problem 23 ‡
A hurricane policy covers both water damage, X, and wind damage, Y, where

X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} 0.13e^{-0.5x-0.2y} - 0.06e^{-x-0.2y} - 0.06e^{-0.5x-0.4y} + 0.12e^{-x-0.4y}, & x > 0, y > 0, \\ 0, \text{otherwise.} \end{cases}$$

Calculate the standard deviation of X.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Problem 24 ‡
An insurance agent's files reveal the following facts about his policyholders:
i) 243 own auto insurance.
ii) 207 own homeowner insurance.
iii) 55 own life insurance and homeowner insurance.
iv) 96 own auto insurance and homeowner insurance.
v) 32 own life insurance, auto insurance and homeowner insurance.
vi) 76 more clients own only auto insurance than only life insurance.
vii) 270 own only one of these three insurance products.
Calculate the total number of the agent's policyholders who own at least one of these three insurance products.

(A) 389
(B) 407
(C) 423
(D) 448
(E) 483

Problem 25 ‡
Let A, B, and C be events such that $P(A) = 0.2,\ P(B) = 0.1$, and $P(C) = 0.3$. The events A and B are independent, the events B and C are independent, and the events A and C are mutually exclusive. Calculate $P(A \cup B \cup C)$.

(A) 0.496
(B) 0.540
(C) 0.544

(D) 0.550
(E) 0.600

Problem 26 ‡
A group of 100 patients is tested, one patient at a time, for three risk factors for a certain disease until either all patients have been tested or a patient tests positive for more than one of these three risk factors. For each risk factor, a patient tests positive with probability p, where $0 < p < 1$. The outcomes of the tests across all patients and all risk factors are mutually independent.
Determine an expression for the probability that exactly n patients are tested, where n is a positive integer less than 100.

(A) $[3p^2(1-p)][1-3p^2(1-p)]^{n-1}$
(B) $[3p^2(1-p)+p^3][1-3p^2(1-p)-p^3]^{n-1}$
(C) $[3p^2(1-p)+p^3]^{n-1}[1-3p^2(1-p)-p^3]^{n-1}$
(D) $n[3p^2(1-p)+p^3][1-3p^2(1-p)-p^3]^{n-1}$
(E) $3[(1-p)^{n-1}p]^2[1-(1-p)^{n-1}p]+[(1-p)^{n-1}p]^3$

Problem 27 ‡
A flood insurance company determines that N, the number of claims received in a month, is a random variable with $P(N=n)=\frac{2}{3^{n+1}}$, for $n=0,1,2,\cdots$ The numbers of claims received in different months are mutually independent. Calculate the probability that more than three claims will be received during a consecutive two-month period, given that fewer than two claims were received in the first of the two months.

(A) 0.0062
(B) 0.0123
(C) 0.0139
(D) 0.0165
(E) 0.0185

Problem 28 ‡
A motorist just had an accident. The accident is minor with probability 0.75 and is otherwise major. Let b be a positive constant. If the accident is minor, then the loss amount follows a uniform distribution on the interval $[0,b]$. If the accident is major, then the loss amount follows a uniform distribution

on the interval $[b, 3b]$. The median loss amount due to this accident is 672. Calculate the mean loss amount due to this accident.

(A) 392
(B) 512
(C) 672
(D) 882
(E) 1008

Problem 29 ‡
A company's annual profit, in billions, has a normal distribution with variance equal to the cube of its mean. The probability of an annual loss is 5%. Calculate the company's expected annual profit.

(A) 370 million
(B) 520 million
(C) 780 million
(D) 950 million
(E) 1645 million

Problem 30 ‡
Skateboarders A and B practice one difficult stunt until becoming injured while attempting the stunt. On each attempt, the probability of becoming injured is p, independent of the outcomes of all previous attempts.
Let $F(x, y)$ represent the probability that skateboarders A and B make no more than x and y attempts, respectively, where x and y are positive integers. It is given that $F(2, 2) = 0.0441$. Calculate $F(1, 5)$.

(A) 0.0093
(B) 0.0216
(C) 0.0495
(D) 0.0551
(E) 0.1112

Problem 31 ‡
Every day, the 30 employees at an auto plant each have probability 0.03 of having one accident and zero probability of having more than one accident. Given there was an accident, the probability of it being major is 0.01. All

other accidents are minor. The numbers and severities of employee accidents are mutually independent. Let X and Y represent the numbers of major accidents and minor accidents, respectively, occurring in the plant today. Determine the joint moment generating function $M_{XY}(s,t)$.

(A) $(0.01e^s + 0.02e^t + 0.97)^{30}$
(B) $(0.0003e^s + 0.0297e^t + 0.97)^{30}$
(C) $(0.01e^s + 0.99)^{30}(0.02e^t + 0.98)^{30}$
(D) $(0.01e^s + 0.99)^{30} + (0.02e^t + 0.98)^{30}$
(E) $(0.0003e^s + 0.9997)^{30}(0.0297e^t + 0.9703)^{30}$

Problem 32 ‡
Two independent estimates are to be made on a building damaged by fire. Each estimate is normally distributed with mean $10b$ and variance b^2 .
Calculate the probability that the first estimate is at least 20 percent higher than the second.

(A) 0.023
(B) 0.100
(C) 0.115
(D) 0.221
(E) 0.444

Answers

1. D
2. D
3. A
4. E
5. C
6. E
7. E
8. B
9. C
10. E
11. E
12. C
13. B
14. E
15. A
16. A
17. D
18. A
19. E
20. C
21. B
22. B
23. B
24. B
25. D
26. B
27. E
28. D
29. A
30. C
31. B
32. B

Sample Exam 8

Duration: 3 hours

Problem 1 ‡
Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 14% the probability that a patient visits a physical therapist.
Determine the probability that a randomly chosen member of this group visits a physical therapist.

(A) 0.26
(B) 0.38
(C) 0.40
(D) 0.48
(E) 0.62

Problem 2 ‡
An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16 - 20	0.06	0.08
21 - 30	0.03	0.15
31 - 65	0.02	0.49
66 - 99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16-20.

(A) 0.13
(B) 0.16
(C) 0.19
(D) 0.23
(E) 0.40

Problem 3 ‡
The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.
What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

(A) 0.15
(B) 0.34
(C) 0.43
(D) 0.57
(E) 0.66

Problem 4 ‡
An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64 . Calculate C.

(A) 0.1
(B) 0.3
(C) 0.4
(D) 0.6
(E) 0.8

Problem 5 ‡
A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \frac{2.5(0.6)^{2.5}}{x^{3.5}} & x > 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2.
What is the mean of the manufacturer's annual losses not paid by the insurance policy?

(A) 0.84
(B) 0.88
(C) 0.93
(D) 0.95
(E) 1.00

Problem 6 ‡
A random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2-2x+2}{2} & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

Calculate the variance of X.

(A) 7/72
(B) 1/8
(C) 5/36
(D) 4/3
(E) 23/12

Problem 7 ‡
An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1 year.
Determine the probability density function $f_Y(y)$, for $y > 0$, of the random variable Y.

(A) $10y^{0.8}e^{-8y^{-0.2}}$

(B) $8y^{-0.2}e^{-10y^{0.8}}$
(C) $8y^{-0.2}e^{-(0.1y)^{1.25}}$
(D) $(0.1y)^{1.25}e^{-0.125(0.1y)^{0.25}}$
(E) $0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}$

Problem 8 ‡
Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y :

E(X) =	50
E(Y) =	20
Var(X) =	50
Var(Y) =	30
Cov (X,Y) =	10

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period.
Approximate the value of P($T < 7100$).

(A) 0.62
(B) 0.84
(C) 0.87
(D) 0.92
(E) 0.97

Problem 9 ‡
Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6, 0 < t_2 < 6, t_1 + t_2 < 10$, and zero otherwise.
Determine $E[T_1+T_2]$, the expected time between a car accident and payment of the claim.

(A) 4.9
(B) 5.0
(C) 5.7

(D) 6.0
(E) 6.7

Problem 10 ‡
A joint density function is given by

$$f_{XY}(x,y) = \begin{cases} kx & 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X, Y)$

(A) $-\frac{1}{6}$
(B) 0
(C) 1/9
(D) 1/6
(E) 2/3

Problem 11 ‡
A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$\begin{aligned} \text{P}(X = 0, Y = 0) &= 0.800 \\ \text{P}(X = 1, Y = 0) &= 0.050 \\ \text{P}(X = 0, Y = 1) &= 0.025 \\ \text{P}(X = 1, Y = 1) &= 0.125. \end{aligned}$$

Calculate $\text{Var}(Y|X = 1)$.

(A) 0.13
(B) 0.15
(C) 0.20
(D) 0.51
(E) 0.71

Problem 12 ‡
The joint probability density for X and Y is

$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+2y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the variance of Y given that $X > 3$ and $Y > 3$.

(A) 0.25
(B) 0.50
(C) 1.00
(D) 3.25
(E) 3.50

Problem 13 ‡
A mattress store sells only king, queen and twin-size mattresses. Sales records at the store indicate that one-fourth as many queen-size mattresses are sold as king and twin-size mattresses combined. Records also indicate that three times as many king-size mattresses are sold as twin-size mattresses.
Calculate the probability that the next mattress sold is either king or queen-size.

(A) 0.12
(B) 0.15
(C) 0.80
(D) 0.85
(E) 0.95

Problem 14 ‡
A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent. The joint density function of X and Y can be modeled by

$$f(x,y) = \begin{cases} \frac{1}{800} e^{-\frac{x}{40} - \frac{y}{20}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that a client spends less than 60 minutes at the agent's office. You do NOT have to evaluate the integrals.

(A) $\frac{1}{800} \int_0^{40} \int_0^{20} e^{-\frac{x}{40}} e^{-\frac{-y}{20}} dydx$
(B) $\frac{1}{800} \int_0^{40} \int_0^{20-x} e^{-\frac{x}{40}} e^{-\frac{-y}{20}} dydx$
(C) $\frac{1}{800} \int_0^{20} \int_0^{40-x} e^{-\frac{x}{40}} e^{-\frac{-y}{20}} dydx$
(D) $\frac{1}{800} \int_0^{60} \int_0^{60} e^{-\frac{x}{40}} e^{-\frac{-y}{20}} dydx$
(E) $\frac{1}{800} \int_0^{60} \int_0^{60-x} e^{-\frac{x}{40}} e^{-\frac{-y}{20}} dydx$

Problem 15 ‡
In a certain game of chance, a square board with area 1 is colored with sectors of either red or blue. A player, who cannot see the board, must specify a point on the board by giving an $x-$coordinate and a $y-$coordinate. The player wins the game if the specified point is in a blue sector. The game can be arranged with any number of red sectors, and the red sectors are designed so that

$$R_i = \left(\frac{9}{20}\right)^i$$

where R_i is the area of the i^{th} red sector.
Calculate the minimum number of red sectors that makes the chance of a player winning less than 20%.

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

Problem 16 ‡
In each of the months June, July, and August, the number of accidents occurring in that month is modeled by a Poisson random variable with mean 1. In each of the other 9 months of the year, the number of accidents occurring is modeled by a Poisson random variable with mean 0.5. Assume that these 12 random variables are mutually independent.
Calculate the probability that exactly two accidents occur in July through November.

(A) 0.084
(B) 0.185
(C) 0.251
(D) 0.257
(E) 0.271

Problem 17 ‡
An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail.

Calculate the probability that the system will overheat.

(A) 0.007
(B) 0.045
(C) 0.098
(D) 0.135
(E) 0.143

Problem 18 ‡
George and Paul play a betting game. Each chooses an integer from 1 to 20 (inclusive) at random. If the two numbers differ by more than 3, George wins the bet. Otherwise, Paul wins the bet. Calculate the probability that Paul wins the bet.

(A) 0.27
(B) 0.32
(C) 0.40
(D) 0.48
(E) 0.66

Problem 19 ‡
The lifetime of a light bulb has density function

$$f(x) = \begin{cases} \frac{Kx^2}{1+x^3} & 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mode of X.

(A) 0.00
(B) 0.79
(C) 1.26
(D) 4.42
(E) 5.00

Problem 20 ‡
The elapsed time, T, between the occurrence and the reporting of an accident has probability density function

$$f_T(t) = \begin{cases} \frac{8t-t^2}{72} & \text{for } 0 < t < 6 \\ 0 & \text{otherwise.} \end{cases}$$

Given that $T = t$, the elapsed time between the reporting of the accident and payment by the insurer is uniformly distributed on $[2 + t, 10]$.
Calculate the probability that the elapsed time between the occurrence of the accident and payment by the insurer is less than 4.

(A) 0.005
(B) 0.023
(C) 0.033
(D) 0.035
(E) 0.133

Problem 21 ‡
Losses due to accidents at an amusement park are exponentially distributed. An insurance company offers the park owner two different policies, with different premiums, to insure against losses due to accidents at the park.
Policy A has a deductible of 1.44. For a random loss, the probability is 0.640 that under this policy, the insurer will pay some money to the park owner.
Policy B has a deductible of d. For a random loss, the probability is 0.512 that under this policy, the insurer will pay some money to the park owner.
Calculate d.

(A) 0.960
(B) 1.152
(C) 1.728
(D) 1.800
(E) 2.160

Problem 22 ‡
Random variables $X \geq 0$ and $Y \geq 0$ are uniformly distributed on the region bounded by the x and y axes, and the curve $y = 1 - x^2$. Calculate $E(XY)$.

(A) 0.083
(B) 0.125
(C) 0.150
(D) 0.267
(E) 0.400

Problem 23 ‡
An insurance company has an equal number of claims in each of three terri-

tories. In each territory, only three claim amounts are possible: 100, 500, and 1000. Based on the companys data, the probabilities of each claim amount are:

	Claim Amount		
	100	500	1000
Territory 1	0.90	0.08	0.02
Territory 2	0.80	0.11	0.09
Territory 3	0.70	0.20	0.10

Calculate the standard deviation of a randomly selected claim amount.

(A) 254
(B) 291
(C) 332
(D) 368
(E) 396

Problem 24 ‡
A profile of the investments owned by an agent's clients follows:
i) 228 own annuities.
ii) 220 own mutual funds.
iii) 98 own life insurance and mutual funds.
iv) 93 own annuities and mutual funds.
v) 16 own annuities, mutual funds, and life insurance.
vi) 45 more clients own only life insurance than own only annuities.
vii) 290 own only one type of investment (i.e., annuity, mutual fund, or life insurance).
Calculate the agent's total number of clients.

(A) 455
(B) 495
(C) 496
(D) 500
(E) 516

Problem 25 ‡
The annual numbers of thefts a homeowners insurance policyholder experiences are analyzed over three years. Define the following events:

i) A = the event that the policyholder experiences no thefts in the three years.
ii) B = the event that the policyholder experiences at least one theft in the second year.
iii) C = the event that the policyholder experiences exactly one theft in the first year.
iv) D = the event that the policyholder experiences no thefts in the third year.
v) E = the event that the policyholder experiences no thefts in the second year, and at least one theft in the third year.
Determine which three events satisfy the condition that the probability of their union equals the sum of their probabilities.

(A) Events i, ii, and v
(B) Events i, iii, and v
(C) Events i, iv, and v
(D) Events ii, iii, and iv
(E) Events ii, iii, and v

Problem 26 ‡
A representative of a market research firm contacts consumers by phone in order to conduct surveys. The specific consumer contacted by each phone call is randomly determined. The probability that a phone call produces a completed survey is 0.25.
Calculate the probability that more than three phone calls are required to produce one completed survey.

(A) 0.32
(B) 0.42
(C) 0.44
(D) 0.56
(E) 0.58

Problem 27 ‡
Patients in a study are tested for sleep apnea, one at a time, until a patient is found to have this disease. Each patient independently has the same probability of having sleep apnea. Let r represent the probability that at least four patients are tested.

Determine the probability that at least twelve patients are tested given that at least four patients are tested.

(A) $r^{\frac{11}{3}}$
(B) r^3
(C) $r^{\frac{8}{3}}$
(D) r^2
(E) $r^{\frac{1}{3}}$

Problem 28 ‡
An insurance policy will reimburse only one claim per year. For a random policyholder, there is a 20% probability of no loss in the next year, in which case the claim amount is 0. If a loss occurs in the next year, the claim amount is normally distributed with mean 1000 and standard deviation 400. Calculate the median claim amount in the next year for a random policyholder.

(A) 663
(B) 790
(C) 873
(D) 994
(E) 1000

Problem 29 ‡
The number of claims X on a health insurance policy is a random variable with $E(X^2) = 61$ and $E[(X-1)^2] = 47$. Calculate the standard deviation of the number of claims.

(A) 2.18
(B) 2.40
(C) 7.31
(D) 7.50
(E) 7.81

Problem 30 ‡
The number of minor surgeries, X, and the number of major surgeries, Y, for a policyholder, this decade, has joint cumulative distribution function

$$F(x,y) = [1-(0.5)^{x+1}][1-(0.2)^{y+1}]$$

for non-negative integers x and y. Calculate the probability that the policyholder experiences exactly three minor surgeries and exactly three major surgeries this decade.

(A) 0.00004
(B) 0.00040
(C) 0.03244
(D) 0.06800
(E) 0.12440

Problem 31 ‡
The returns on two investments, X and Y, follow the joint probability density function

$$f_{XY}(x,y) = \begin{cases} k, & 0 < |x| + |y| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the maximum value of $\text{Var}(Y|X = x), \ -1 < x < 1$.

(A) 1/12
(B) 1/6
(C) 1/3
(D) 2/3
(E) 1

Problem 32 ‡
Two independent random variables X and Y have the same mean. The coefficients of variation of X and Y are 3 and 4 respectively. Calculate the coefficient of variation of $0.5(X + Y)$.

(A) 5/4
(B) 7/4
(C) 5/2
(D) 7/2
(E) 7

Answers

1. D
2. B
3. C
4. B
5. C
6. C
7. E
8. B
9. C
10. B
11. C
12. A
13. C
14. E
15. C
16. B
17. A
18. B
19. C
20. A
21. E
22. B
23. A
24. D
25. A
26. B
27. C
28. C
29. A
30. B
31. C
32. C

Sample Exam 9

Duration: 3 hours

Problem 1 ‡
An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

(A) 0.13
(B) 0.21
(C) 0.24
(D) 0.25
(E) 0.30

Problem 2 ‡
An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year,

each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year.
A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

(A) 0.0001
(B) 0.0010
(C) 0.0071
(D) 0.0141
(E) 0.2817

Problem 3 ‡
An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims.
If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

(A) $\frac{1}{\sqrt{3}}$
(B) 1
(C) $\sqrt{2}$
(D) 2
(E) 4

Problem 4 ‡
A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants).
What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

(A) 0.096
(B) 0.192
(C) 0.235
(D) 0.376
(E) 0.469

Problem 5 ‡
An insurance company sells a one-year automobile policy with a deductible of 2 . The probability that the insured will incur a loss is 0.05 . If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \cdots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur.
Calculate the expected payment for this policy.

(A) 0.031
(B) 0.066
(C) 0.072
(D) 0.110
(E) 0.150

Problem 6 ‡
The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

(A) 1.3
(B) 1.4
(C) 1.7
(D) 2.1
(E) 7.5

Problem 7 ‡
Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries.
Find the density function $f_R(r)$ of R.

(A) $\frac{12}{5}$
(B) $3 - \frac{5}{2r}$

(C) $3r - \frac{5 \ln r}{2}$
(D) $\frac{10}{r^2}$
(E) $\frac{5}{2r^2}$

Problem 8 ‡
The total claim amount for a health insurance policy follows a distribution with density function

$$f(x) = \begin{cases} \frac{1}{1000} e^{-\frac{x}{1000}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The premium for the policy is set at 100 over the expected total claim amount. If 100 policies are sold, what is the approximate probability that the insurance company will have claims exceeding the premiums collected?

(A) 0.001
(B) 0.159
(C) 0.333
(D) 0.407
(E) 0.460

Problem 9 ‡
X and Y are independent random variables with common moment generating function $M(t) = e^{\frac{t^2}{2}}$. Let $W = X + Y$ and $Z = X - Y$. Determine the joint moment generating function, $M(t_1, t_2)$ of W and Z.

(A) $e^{2t_1^2 + 2t_2^2}$
(B) $e^{(t_1^2 - t_2)^2}$
(C) $e^{(t_1^2 + t_2)^2}$
(D) $e^{2t_1 t_2}$
(E) $e^{t_1^2 + t_2^2}$

Problem 10 ‡
Let X and Y be continuous random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} \frac{8}{3} xy & 0 \le x \le 1, x \le y \le 2x \\ 0 & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X, Y)$.

(A) 0.04
(B) 0.25
(C) 0.67
(D) 0.80
(E) 1.24

Problem 11 ‡
The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f_{XY}(x,y) = \begin{cases} 2x & 0 < x < 1, x < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

(A) 1/12
(B) 7/6
(C) $x + \frac{1}{2}$
(D) $x^2 - \frac{1}{6}$
(E) $x^2 + x + \frac{1}{3}$

Problem 12 ‡
The distribution of Y, given X, is uniform on the interval $[0, X]$. The marginal density of X is

$$f_X(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the conditional density of X, given $Y = y > 0$.

(A) 1
(B) 2
(C) $2x$
(D) $1/y$
(E) $1/(1-y)$

Problem 13 ‡
The number of workplace injuries, N, occurring in a factory on any given day is Poisson distributed with mean λ. The parameter λ is a random variable that is determined by the level of activity in the factory, and is uniformly distributed on the interval $[0, 3]$.

Calculate $\text{Var}(N)$.

(A) λ
(B) 2λ
(C) 0.75
(D) 1.50
(E) 2.25

Problem 14 ‡
New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y) = \begin{cases} 0.50 & \text{for } 0 < x, y < 0.5 \\ 1.25 & \text{for } 0 < x < 0.5, 0.5 < y < 1 \\ 1.50 & \text{for } 0.5 < x < 1, 0 < y < 0.5 \\ 0.75 & \text{for } 0.5 < x < 1, 0.5 < y < 1. \end{cases}$$

Calculate $\text{Var}(Y|X = 0.75)$.

(A) 0.000
(B) 0.061
(C) 0.076
(D) 0.083
(E) 0.141

Problem 15 ‡
Automobile claim amounts are modeled by a uniform distribution on the interval $[0, 10,000]$. Actuary A reports X, the claim amount divided by 1000. Actuary B reports Y, which is X rounded to the nearest integer from 0 to 10.
Calculate the absolute value of the difference between the 4^{th} moment of X and the 4^{th} moment of Y.

(A) 0
(B) 33
(C) 296

(D) 303
(E) 533

Problem 16 ‡
Two claimants place calls simultaneously to an insurer's claims call center. The times X and Y, in minutes, that elapse before the respective claimants get to speak with call center representatives are independently and identically distributed. The moment generating function of each random variable is

$$M(t) = \left(\frac{1}{1-1.5t}\right)^2, \ t < \frac{2}{3}.$$

Find the standard deviation of $X + Y$.

(A) 2.1
(B) 3.0
(C) 4.5
(D) 6.7
(E) 9.0

Problem 17 ‡
An insurance company's annual profit is normally distributed with mean 100 and variance 400. Let Z be normally distributed with mean 0 and variance 1 and let Φ be the cumulative distribution function of Z.
Determine, in terms of $\Phi(x)$, the probability that the company's profit in a year is at most 60, given that the profit in the year is positive.

(A) $1 - F(2)$
(B) $F(2)/F(5)$
(C) $[1 - F(2)]/F(5)$
(D) $[F(0.25) - F(0.1)]/F(0.25)$
(E) $[F(5) - F(2)]/F(5)$

Problem 18 ‡
A student takes an examination consisting of 20 true-false questions. The student knows the answer to N of the questions, which are answered correctly, and guesses the answers to the rest. The conditional probability that the student knows the answer to a question, given that the student answered it correctly, is 0.824. Calculate N.

(A) 8
(B) 10
(C) 14
(D) 16
(E) 18

Problem 19 ‡
An insurer's medical reimbursements have density function

$$f(x) = \begin{cases} Kxe^{-x^2} & 0 < x < 1,\ K > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mode of X.

(A) 0.00
(B) 0.50
(C) 0.71
(D) 0.84
(E) 1.00

Problem 20 ‡
The time until failure, T, of a product is modeled by a uniform distribution on [0,10]. An extended warranty pays a benefit of 100 if failure occurs between time $t = 1.5$ and $t = 8$.
The present value, W, of this benefit is

$$W = \begin{cases} 0, & 0 \le T < 1.5, \\ 100e^{-0.04T}, & 1.5 \le T < 8, \\ 0, & 8 \le T \le 10. \end{cases}$$

Calculate $P(W < 79)$.

(A) 0.21
(B) 0.41
(C) 0.44
(D) 0.56
(E) 0.59

Problem 21 ‡
The distribution of the size of claims paid under an insurance policy has probability density function

$$f(x) = \begin{cases} cx^a, & 0 < x < 5 \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0$ and $c > 0$.
For a randomly selected claim, the probability that the size of the claim is less than 3.75 is 0.4871. Calculate the probability that the size of a randomly selected claim is greater than 4.

(A) 0.404
(B) 0.428
(C) 0.500
(D) 0.572
(E) 0.596

Problem 22 ‡
An insurance company will cover losses incurred from tornadoes in a single calendar year. However, the insurer will only cover losses for a maximum of three separate tornadoes during this time frame. Let X be the number of tornadoes that result in at least 50 million in losses, and let Y be the total number of tornadoes. The joint probability function for X and Y is

$$p_{XY}(x,y) = \begin{cases} c(x+2y), & x = 0,1,2,3,\ y = 0,1,2,3,\ x \leq y \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant. Calculate the expected number of tornadoes that result in fewer than 50 million in losses.

(A) 0.19
(B) 0.28
(C) 0.76
(D) 1.00
(E) 1.10

Problem 23 ‡
At the start of a week, a coal mine has a high-capacity storage bin that is

half full. During the week, 20 loads of coal are added to the storage bin. Each load of coal has a volume that is normally distributed with mean 1.50 cubic yards and standard deviation 0.25 cubic yards.
During the same week, coal is removed from the storage bin and loaded into 4 railroad cars. The amount of coal loaded into each railroad car is normally distributed with mean 7.25 cubic yards and standard deviation 0.50 cubic yards.
The amounts added to the storage bin or removed from the storage bin are mutually independent. Calculate the probability that the storage bin contains more coal at the end of the week than it had at the beginning of the week.

(A) 0.56
(B) 0.63
(C) 0.67
(D) 0.75
(E) 0.98

Problem 24 ‡
An insurance policy pays 100 per day for up to 3 days of hospitalization and 50 per day for each day of hospitalization thereafter.
The number of days of hospitalization, X, is a discrete random variable with probability function

$$p(k) = \begin{cases} \frac{6-k}{15} & k = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the expected payment for hospitalization under this policy.

(A) 123
(B) 210
(C) 220
(D) 270
(E) 367

Problem 25 ‡
Four letters to different insureds are prepared along with accompanying envelopes. The letters are put into the envelopes randomly. Calculate the probability that at least one letter ends up in its accompanying envelope.

(A) 27/256
(B) 1/4
(C) 11/24
(D) 5/8
(E) 3/4

Problem 26 ‡
Four distinct integers are chosen randomly and without replacement from the first twelve positive integers. Let X be the random variable representing the second smallest of the four selected integers, and let $p(x)$ be the probability mass function of X. Determine $p(x)$, for $x = 2, 3, 4, \cdots$

(A) $\frac{(x-1)(11-x)(12-x)}{495}$
(B) $\frac{(x-1)(11-x)(12-x)}{990}$
(C) $\frac{(x-1)(x-2)(12-x)}{990}$
(D) $\frac{(x-1)(x-2)(12-x)}{495}$
(E) $\frac{(10-x)(11-x)(12-x)}{495}$

Problem 27 ‡
A factory tests 100 light bulbs for defects. The probability that a bulb is defective is 0.02. The occurrences of defects among the light bulbs are mutually independent events. Calculate the probability that exactly two are defective given that the number of defective bulbs is two or fewer.

(A) 0.133
(B) 0.271
(C) 0.273
(D) 0.404
(E) 0.677

Problem 28 ‡
A gun shop sells gunpowder. Monthly demand for gunpowder is normally distributed, averages 20 pounds, and has a standard deviation of 2 pounds. The shop manager wishes to stock gunpowder inventory at the beginning of each month so that there is only a 2% chance that the shop will run out of gunpowder (i.e., that demand will exceed inventory) in any given month. Calculate the amount of gunpowder to stock in inventory, in pounds.

(A) 16
(B) 23
(C) 24
(D) 32
(E) 43

Problem 29 ‡
Ten cards from a deck of playing cards are in a box: two diamonds, three spades, and five hearts. Two cards are randomly selected without replacement.
Calculate the variance of the number of diamonds selected, given that no spade is selected.

(A) 0.24
(B) 0.28
(C) 0.32
(D) 0.34
(E) 0.41

Problem 30 ‡
A company provides a death benefit of 50,000 for each of its 1000 employees. There is a 1.4% chance that any one employee will die next year, independent of all other employees. The company establishes a fund such that the probability is at least 0.99 that the fund will cover next year's death benefits. Calculate, using the Central Limit Theorem, the smallest amount of money, rounded to the nearest 50 thousand, that the company must put into the fund.

(A) 750,000
(B) 850,000
(C) 1,050,000
(D) 1,150,000
(E) 1,400,000

Problem 31 ‡
On Main Street, a driver's speed just before an accident is uniformly distributed on $[5, 20]$. Given the speed, the resulting loss from the accident is

exponentially distributed with mean equal to three times the speed. Calculate the variance of a loss due to an accident on Main Street.

(A) 525
(B) 1463
(C) 1575
(D) 1632
(E) 1744

Problem 32 ‡
Points scored by a game participant can be modeled by $Z = 3X + 2Y - 5$. X and Y are independent random variables with $\text{Var}(X) = 3$ and $\text{Var}(Y) = 4$. Calculate $\text{Var}(Z)$.

(A) 12
(B) 17
(C) 38
(D) 43
(E) 68

Answers

1. B
2. D
3. D
4. E
5. A
6. C
7. E
8. B
9. E
10. A
11. A
12. E
13. E
14. C
15. B
16. B
17. E
18. C
19. C
20. D
21. B
22. E
23. D
24. C
25. D
26. B
27. D
28. C
29. D
30. D
31. E
32. D

Sample Exam 10

Duration: 3 hours

Problem 1 ‡
An actuary studying the insurance preferences of automobile owners makes the following conclusions:
(i) An automobile owner is twice as likely to purchase a collision coverage as opposed to a disability coverage.
(ii) The event that an automobile owner purchases a collision coverage is independent of the event that he or she purchases a disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.
What is the probability that an automobile owner purchases neither collision nor disability coverage?

(A) 0.18
(B) 0.33
(C) 0.48
(D) 0.67
(E) 0.82

Problem 2 ‡
Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) the rest of the emergency room patients were stable;
(iv) 40% of the critical patients died;
(v) 10% of the serious patients died; and
(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

(A) 0.06
(B) 0.29
(C) 0.30
(D) 0.39
(E) 0.64

Problem 3 ‡
A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee.
Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

(A) 24
(B) 30
(C) 40
(D) 60
(E) 120

Problem 4 ‡
For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000.
For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000.
The total claim amounts of the two companies are independent. Calculate the probability that, in the coming year, Company B's total claim amount

will exceed Company A's total claim amount.

(A) 0.180
(B) 0.185
(C) 0.217
(D) 0.223
(E) 0.240

Problem 5 ‡
An insurance policy reimburses a loss up to a benefit limit of 10 . The policyholder's loss, X, follows a distribution with density function:

$$f(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

(A) 1.0
(B) 1.3
(C) 1.8
(D) 1.9
(E) 2.0

Problem 6 ‡
A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

(A) 45%
(B) 55%
(C) 68%
(D) 85%
(E) 100%

Problem 7 ‡
The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is twice that of Company I.
Let g be the density function for the distribution of the monthly profit of Company II.
Determine $g(y)$ where it is not zero.

(A) $\frac{1}{2}f\left(\frac{y}{2}\right)$
(B) $f\left(\frac{y}{2}\right)$
(C) $2f\left(\frac{y}{2}\right)$
(D) $2f(y)$
(E) $2f(2y)$

Problem 8 ‡
A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

(i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.
(ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.
(iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.
Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

(A) 0.60

(B) 0.67
(C) 0.75
(D) 0.93
(E) 0.99

Problem 9 ‡
A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists.
Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist.
What is the expected revenue of the tour operator?

(A) 955
(B) 962
(C) 967
(D) 976
(E) 985

Problem 10 ‡
Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval (0, 12) . Given $X = x$, Y is uniformly distributed on the interval $(0, x)$.
Determine $\mathrm{Cov}(X, Y)$ according to this model.

(A) 0
(B) 4
(C) 6
(D) 12
(E) 24

Problem 11 ‡
An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

$X\backslash Y$	0	1	2	$P_X(x)$
0	0.12	0.13	0.05	0.30
1	0.06	0.15	0.15	0.36
2	0.05	0.12	0.10	0.27
3	0.02	0.03	0.02	0.07
$p_Y(y)$	0.25	0.43	0.32	1

where X is the number of tornadoes in county Q and Y that of county P. Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P.

(A) 0.51
(B) 0.84
(C) 0.88
(D) 0.99
(E) 1.76

Problem 12 ‡
Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:
(i) $p_n \geq p_{n+1}$ for $0 \leq n \leq 4$.
(ii) The difference between p_n and p_{n+1} is the same for $0 \leq n \leq 4$.
(iii) Exactly 40% of policyholders file fewer than two claims during a given year.
Calculate the probability that a random policyholder will file more than three claims during a given year.

(A) 0.14
(B) 0.16
(C) 0.27
(D) 0.29
(E) 0.33

Problem 13 ‡
A fair die is rolled repeatedly. Let X be the number of rolls needed to obtain a 5 and Y the number of rolls needed to obtain a 6. Calculate $E(X|Y = 2)$.

(A) 5.0
(B) 5.2
(C) 6.0
(D) 6.6
(E) 6.8

Problem 14 ‡
A survey of 100 TV watchers revealed that over the last year:
i) 34 watched CBS.
ii) 15 watched NBC.
iii) 10 watched ABC.
iv) 7 watched CBS and NBC.
v) 6 watched CBS and ABC.
vi) 5 watched NBC and ABC.
vii) 4 watched CBS, NBC, and ABC.
viii) 18 watched HGTV and of these, none watched CBS, NBC, or ABC.
Calculate how many of the 100 TV watchers did not watch any of the four channels (CBS, NBC, ABC or HGTV).

(A) 1
(B) 37
(C) 45
(D) 55
(E) 82

Problem 15 ‡
The probability of x losses occurring in year 1 is $(0.5)^{x+1},\ x = 0, 1, 2, \cdots$.
The probability of y losses in year 2 given x losses in year 1 is given by the table:

$X \backslash Y$	0	1	2	3	4^+
0	0.60	0.25	0.05	0.05	0.05
1	0.45	0.30	0.10	0.10	0.05
2	0.25	0.30	0.20	0.20	0.05
3	0.15	0.20	0.20	0.30	0.15
4^+	0.05	0.15	0.25	0.35	0.20

Calculate the probability of exactly 2 losses in 2 years.

(A) 0.025
(B) 0.031
(C) 0.075
(D) 0.100
(E) 0.131

Problem 16 ‡
An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. For every full ten inches of snow in excess of 40 inches during the winter season, the insurer pays the airport 300 up to a policy maximum of 700. The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	$[90,\infty)$
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

(A) 134
(B) 235
(C) 271
(D) 313
(E) 352

Problem 17 ‡
In a group of health insurance policyholders, 20% have high blood pressure and 30% have high cholesterol. Of the policyholders with high blood pressure, 25% have high cholesterol. A policyholder is randomly selected from the group. Calculate the probability that a policyholder has high blood pressure, given that the policyholder has high cholesterol.

(A) 1/6
(B) 1/5
(C) 1/4
(D) 2/3
(E) 5/6

Problem 18 ‡
The minimum force required to break a particular type of cable is normally distributed with mean 12,432 and standard deviation 25. A random sample of 400 cables of this type is selected.
Calculate the probability that at least 349 of the selected cables will not break under a force of 12,400.

(A) 0.62
(B) 0.67
(C) 0.92
(D) 0.97
(E) 1.00

Problem 19 ‡
A company has five employees on its health insurance plan. Each year, each employee independently has an 80% probability of no hospital admissions. If an employee requires one or more hospital admissions, the number of admissions is modeled by a geometric distribution with a mean of 1.50. The numbers of hospital admissions of different employees are mutually independent. Each hospital admission costs 20,000.
Calculate the probability that the company's total hospital costs in a year are less than 50,000.

(A) 0.41
(B) 0.46
(C) 0.58
(D) 0.69
(E) 0.78

Problem 20 ‡
An insurance company issues policies covering damage to automobiles. The amount of damage is modeled by a uniform distribution on $[0, b]$. The policy payout is subject to a deductible of $0.1b$. A policyholder experiences automobile damage.
Calculate the ratio of the standard deviation of the policy payout to the standard deviation of the amount of the damage.

(A) 0.8100

(B) 0.9000
(C) 0.9477
(D) 0.9487
(E) 0.9735

Problem 21 ‡
Company XYZ provides a warranty on a product that it produces. Each year, the number of warranty claims follows a Poisson distribution with mean c. The probability that no warranty claims are received in any given year is 0.60.
Company XYZ purchases an insurance policy that will reduce its overall warranty claim payment costs. The insurance policy will pay nothing for the first warranty claim received and 5000 for each claim thereafter until the end of the year.
Calculate the expected amount of annual insurance policy payments to Company XYZ.

(A) 554
(B) 872
(C) 1022
(D) 1354
(E) 1612

Problem 22 ‡
At a polling booth, ballots are cast by ten voters, of whom three are Republicans, two are Democrats, and five are Independents. A local journalist interviews two of these voters, chosen randomly.
Calculate the expectation of the absolute value of the difference between the number of Republicans interviewed and the number of Democrats interviewed.

(A) 1/5
(B) 7/15
(C) 3/5
(D) 11/15
(E) 1

Problem 23 ‡
An insurance company insures a good driver and a bad driver on the same

policy. The table below gives the probability of a given number of claims occurring for each of these drivers in the next ten years.

Number of claims	Probability for the good driver	Probability for for the bad driver
0	0.5	0.2
1	0.3	0.3
2	0.2	0.4
3	0.0	0.1

The number of claims occurring for the two drivers are independent. Calculate the mode of the distribution of the total number of claims occurring on this policy in the next ten years.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

Problem 24 ‡
An actuary compiles the following information from a portfolio of 1000 homeowners insurance policies:
i) 130 policies insure three-bedroom homes.
ii) 280 policies insure one-story homes.
iii) 150 policies insure two-bath homes.
iv) 30 policies insure three-bedroom, two-bath homes.
v) 50 policies insure one-story, two-bath homes.
vi) 40 policies insure three-bedroom, one-story homes.
vii) 10 policies insure three-bedroom, one-story, two-bath homes.
Calculate the number of homeowners policies in the portfolio that insure neither one-story nor two-bath nor three-bedroom homes.

(A) 310
(B) 450
(C) 530
(D) 550
(E) 570

Problem 25 ‡
A health insurance policy covers visits to a doctors office. Each visit costs 100. The annual deductible on the policy is 350. For a policy, the number of visits per year has the following probability distribution:

Number of Visits	0	1	2	3	4	5	6
Probability	0.6	0.15	0.10	0.08	0.04	0.02	0.01

A policy is selected at random from those where costs exceed the deductible. Calculate the probability that this policyholder had exactly five office visits.

(A) 0.050
(B) 0.133
(C) 0.286
(D) 0.333
(E) 0.429

Problem 26 ‡
Losses due to burglary are exponentially distributed with mean 100. The probability that a loss is between 40 and 50 equals the probability that a loss is between 60 and r, with $r > 60$. Calculate r.

(A) 68.26
(B) 70.00
(C) 70.51
(D) 72.36
(E) 75.00

Problem 27 ‡
A certain town experiences an average of 5 tornadoes in any four year period. The number of years from now until the town experiences its next tornado as well as the number of years between tornadoes have identical exponential distributions and all such times are mutually independent Calculate the median number of years from now until the town experiences its next tornado.

(A) 0.55
(B) 0.73
(C) 0.80
(D) 0.87
(E) 1.25

Problem 28 ‡
A large university will begin a 13-day period during which students may register for that semester's courses. Of those 13 days, the number of elapsed days before a randomly selected student registers has a continuous distribution with density function $f(t)$ that is symmetric about $t = 6.5$ and proportional to $\frac{1}{t+1}$ between days 0 and 6.5.
A student registers at the 60^{th} percentile of this distribution. Calculate the number of elapsed days in the registration period for this student.

(A) 4.01
(B) 7.80
(C) 8.99
(D) 10.22
(E) 10.51

Problem 29 ‡
A homeowners insurance policy covers losses due to theft, with a deductible of 3. Theft losses are uniformly distributed on $[0, 10]$. Determine the moment generating function, $M(t)$, for $t \neq 0$, of the claim payment on a theft.

(A) $\frac{3}{10} + \frac{e^{7t}-1}{10t}$
(B) $\frac{e^{10t}-1}{10t} - 3$
(C) $\frac{e^{7t}-e^{-3t}}{10t}$
(D) $\min\{0, \frac{e^{10t}-1}{10t} - 3\}$
(E) $\max\{0, \frac{e^{10t}-1}{10t} - 3\}$

Problem 30 ‡
An investor invests 100 dollars in a stock. Each month, the investment has probability 0.5 of increasing by 1.10 dollars and probability 0.5 of decreasing by 0.90 dollars. The changes in price in different months are mutually independent. Calculate the probability that the investment has a value greater than 91 dollars at the end of month 100.

(A) 0.63
(B) 0.75
(C) 0.82
(D) 0.94
(E) 0.97

Problem 31 ‡
Let X be the annual number of hurricanes hitting Florida, and let Y be the annual number of hurricanes hitting Texas. X and Y are independent Poisson variables with respective means 1.70 and 2.30.
Calculate $\text{Var}(X - Y|X + Y = 3)$.

(A) 1.71
(B) 1.77
(C) 2.93
(D) 3.14
(E) 4.00

Problem 32 ‡
An actuary is studying hurricane models. A year is classified as a high, medium, or low hurricane year with probabilities 0.1, 0.3, and 0.6, respectively. The numbers of hurricanes in high, medium, and low years follow Poisson distributions with means 20, 15, and 10, respectively. Calculate the variance of the number of hurricanes in a randomly selected year.

(A) 11.25
(B) 12.50
(C) 12.94
(D) 13.42
(E) 23.75

Problem 33 ‡
A dental insurance company pays 100% of the cost of fillings and 70% of the cost of root canals. Fillings and root canals cost 50 and 500 each, respectively. The tables below show the probability distributions of the annual number of fillings and annual number of root canals for each of the company's policyholders.

# of fillings	0	1	2	3
Probability	0.60	0.20	0.15	0.05

# of root canals	0	1
Probability	0.80	0.20

Calculate the expected annual payment per policyholder for fillings and root canals.

(A) 90.00

(B) 102.50
(C) 132.50
(D) 250.00
(E) 400.00

Answers

1. B
2. B
3. D
4. D
5. D
6. A
7. A
8. E
9. E
10. C
11. D
12. C
13. D
14. B
15. E
16. B
17. A
18. D
19. E
20. E
21. A
22. D
23. C
24. D
25. C
26. D
27. A
28. C
29. A
30. E
31. C
32. E
33. B

Answer Keys

Section 1

1.1 $A = \{2, 3, 5\}$
1.2 (a)$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$
(b)$E = \{TTT, TTH, HTT, THT\}$
(c)$F = \{x | x$ is an element of S with more than one head$\}$
1.3 $F \subset E$
1.4 $E = \emptyset$
1.5 (a) Since every element of A is in A, $A \subseteq A$.
(b) Since every element in A is in B and every element in B is in A, $A = B$.
(c) If x is in A then x is in B since $A \subseteq B$. But $B \subseteq C$ and this implies that x is in C. Hence, every element of A is also in C. This shows that $A \subseteq C$
1.6 The result is true for $n = 1$ since $1 = \frac{1(1+1)}{2}$. Assume that the equality is true for $1, 2, \cdots, n$. Then

$$\begin{aligned} 1 + 2 + \cdots + n + 1 =& (1 + 2 + \cdots + n) + n + 1 \\ =& \frac{n(n+1)}{2} + n + 1 = (n+1)[\frac{n}{2} + 1] \\ =& \frac{(n+1)(n+2)}{2} \end{aligned}$$

1.7 Let $S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$. For $n = 1$, we have $S_1 = 1 = \frac{1(1+1)(2+1)}{6}$. Suppose that $S_n = \frac{n(n+1)(2n+1)}{6}$. We next want to show that $S_{n+1} = \frac{(n+1)(n+2)(2n+3)}{6}$. Indeed, $S_{n+1} = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left[\frac{n(2n+1)}{6} + n + 1\right] = \frac{(n+1)(n+2)(2n+3)}{6}$

1.8 The result is true for $n = 1$. Suppose true up to n. Then

$$\begin{aligned}(1+x)^{n+1} =&(1+x)(1+x)^n\\ \geq&(1+x)(1+nx), \quad since\ 1+x>0\\ =&1+nx+x+nx^2\\ =&1+nx^2+(n+1)x \geq 1+(n+1)x\end{aligned}$$

1.9 The identity is valid for $n = 1$. Assume true for $1, 2, \cdots, n$. Then

$$\begin{aligned}1+a+a^2+\cdots+a^{n+1} =&[1+a+a^2+\cdots+a^n]+a^{n+1}\\ =&\frac{1-a^{n+1}}{1-a}+a^{n+1}=\frac{1-a^{n+2}}{1-a}\end{aligned}$$

1.10 (a) 55 sandwiches with tomatoes or onions.
(b) There are 40 sandwiches with onions.
(c) There are 10 sandwiches with onions but not tomatoes

1.11 (a) 20 (b) 5 (c) 11 (d) 42 (e) 46 (f) 46
1.12 Since We have

$$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,H),(T,T)\}$$

and $\#(S) = 8$.
1.13 Suppose that $f(a) = f(b)$. Then $3a + 5 = 3b + 5 \Longrightarrow 3a + 5 - 5 = 3b + 5 - 5 \Longrightarrow 3a = 3b \Longrightarrow \frac{3a}{3} = \frac{3b}{3} \Longrightarrow a = b$. That is, f is one-to-one.
Let $y \in \mathbb{R}$. From the equation $y = 3x + 5$ we find $x = \frac{y-5}{3} \in \mathbb{R}$ and $f(x) = f\left(\frac{y-5}{3}\right) = y$. That is, f is onto.
1.14 5
1.15 (a) The condition $f(n) = f(m)$ with n even and m odd leads to $n + m = 1$ with $n, m \in \mathbb{N}$ which cannot happen.
(b) Suppose that $f(n) = f(m)$. If n and m are even, we have $\frac{n}{2} = \frac{m}{2} \Longrightarrow n = m$. If n and m are odd then $-\frac{n-1}{2} = -\frac{m-1}{2} \Longrightarrow n = m$. Thus, f is one-to-one. Now, if $m = 0$ then $n = 1$ and $f(n) = m$. If $m \in \mathbb{N} = \mathbb{Z}^+$ then $n = 2m$ and $f(n) = m$. If $n \in \mathbb{Z}^-$ then $n = 2|m| + 1$ and $f(n) = m$. Thus, f is onto. If follows that $\mathbb{Z}$ is countable.
1.16 Suppose the contrary. That is, there is a $b \in A$ such that $f(b) = B$. Since $B \subseteq A$, either $b \in B$ or $b \notin B$. If $b \in B$ then $b \notin f(b)$. But $B = f(b)$ so

$b \in B$ implies $b \in f(b)$, a contradiction. If $b \notin B$ then $b \in f(b) = B$ which is again a contradiction. Hence, we conclude that there is no onto map from A to its power set.

1.17 By the previous problem there is no onto map from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ so that $\mathcal{P}(\mathbb{N})$ is uncountable.

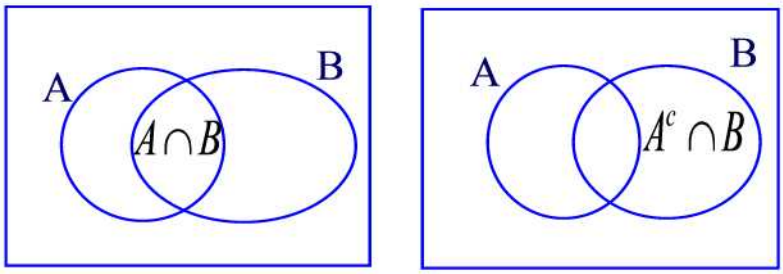
1.18 The function $f : \mathbb{R} \longmapsto (0, \infty)$ defined by $f(x) = e^x$ is one-to-one (increasing function) and onto (for every $x \in (0, \infty),\ y = \ln x,\ f(y) = x$.

1.19 Suppose that $f(n, m) = f(k, \ell)$. Then $2^n 3^m = 2^k 3^\ell$. If $n < k$ then $3^m = 2^{k-n} 3^\ell$. The left-hand side is odd whereas the right-hand side is even. Hence, $n \geq k$. If $n > k$ then $3^\ell = 2^{n-k} 3^m$. Again, the left-hand side is odd and the right-hand side is even. It follows that $k = n$ and this implies that $3^m = 3^\ell$. Hence, $m = \ell$ and $(n, m) = (k, \ell)$.

1.20 50%.

Section 2

2.1



2.2 Since $A \subseteq B$, we have $A \cup B = B$. Now the result follows from the previous problem.

2.3 Let

G = event that a viewer watched gymnastics
B = event that a viewer watched baseball
S = event that a viewer watched soccer

Then the event "the group that watched none of the three sports during the last year" is the set $(G \cup B \cup S)^c$

2.4 The events $R_1 \cap R_2$ and $B_1 \cap B_2$ represent the events that both ball are the same color and therefore as sets they are disjoint

2.5 880

2.6 50%
2.7 5%
2.8 60
2.9 53%
2.10 Using Theorem 2.4, we find

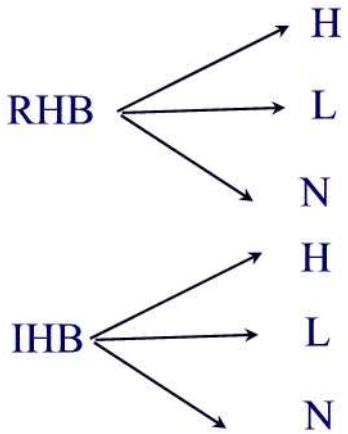
$$\begin{aligned}\#(A\cup B\cup C) =&\#(A\cup(B\cup C))\\ =&\#(A)+\#(B\cup C)-\#(A\cap(B\cup C))\\ =&\#(A)+(\#(B)+\#(C)-\#(B\cap C))\\ &-\#((A\cap B)\cup(A\cap C))\\ =&\#(A)+(\#(B)+\#(C)-\#(B\cap C))\\ &-(\#(A\cap B)+\#(A\cap C)-\#(A\cap B\cap C))\\ =&\#(A)+\#(B)+\#(C)-\#(A\cap B)-\#(A\cap C)\\ &-\#(B\cap C)+\#(A\cap B\cap C)\end{aligned}$$

2.11 50
2.12 10
2.13 (a) 3 (b) 6
2.14 20%
2.15 (a) Let $x\in A\cap(B\cup C)$. Then $x\in A$ and $x\in B\cup C$. Thus, $x\in A$ and ($x\in B$ or $x\in C$). This implies that ($x\in A$ and $x\in B$) or ($x\in A$ and $x\in C$). Hence, $x\in A\cap B$ or $x\in A\cap C$, i.e. $x\in(A\cap B)\cup(A\cap C)$. The converse is similar.
(b) Let $x\in A\cup(B\cap C)$. Then $x\in A$ or $x\in B\cap C$. Thus, $x\in A$ or ($x\in B$ and $x\in C$). This implies that ($x\in A$ or $x\in B$) and ($x\in A$ or $x\in C$). Hence, $x\in A\cup B$ and $x\in A\cup C$, i.e. $x\in(A\cup B)\cap(A\cup C)$. The converse is similar.
2.16 (a) $B\subseteq A$
(b) $A\cap B=\emptyset$ or $A\subseteq B^c$.
(c) $A\cup B-A\cap B$
(d) $(A\cup B)^c$
2.17 37
2.18 Suppose that $f(s_1,s_2,\cdots,s_n)=f(t_1,t_2,\cdots,t_n)$. Then $((s_1,s_2,\cdots,s_{n-1}),s_n)=((t_1,t_2,\cdots,t_{n-1}),t_n)$. By Example 2.14, we have $s_i=t_i$ for $i=1,2,\cdots,n$. Hence, $(s_1,s_2,\cdots,s_n)=(t_1,t_2,\cdots,t_n)$ and f is one-to-one. Now, for any $((s_1,s_2,\cdots,s_{n-1}),s_n)\in(S_1\times S_2\times\cdots\times S_{n-1})\times S_n$, we have $(s_1,s_2,\cdots,s_n)\in S_1\times S_2\times\cdots\times S_n$ and $f(s_1,s_2,\cdots,s_n)=((s_1,s_2,\cdots,s_{n-1}),s_n)$. That is, f

is onto.
2.19 365^n
2.20 10%

Section 3

3.1 (a) 100 (b) 900 (c) 5,040 (d) 90,000
3.2 (a) 336 (b) 6
3.3 6
3.4 90
3.5



3.6 48 ways
3.7 380
3.8 255,024
3.9 5,040
3.10 384
3.11 96
3.12 There are n successive steps to create a subset of A. For the first step one can pick or not pick a_1, for the second step one can pick or not pick $a_2, \cdots$, and so on. One each step there are two options, to pick or not to pick. By the Fundamental Principle of Counting, the number of subsets that can be selected is

$$\underbrace{2 \cdot 2 \cdots 2}_{n \ factors} = 2^n$$

3.13 For each of the n values in A, the function can take on m possible values. Hence, by the Fundamental Principle of Counting, the number of functions is

$$\underbrace{m \cdot m \cdots m}_{n \ factors} = m^n$$

3.14 The first element of A can be assigned to any of the m elements of B. The second element of A can be assigned to any of the remaining $m-1$ elements of $B, \cdots$, the n^{th} element of A can be assigned to any of the remaining $m-n+1$ of B. Thus, by the Fundamental Principle of Counting, the number of one-to-one functions is

$$m \cdot (m-1) \cdots (m-n+1)$$

3.15 Yes
3.16 4,500
3.17 676
3.18 32
3.19 98,176
3.20 2^{50}

Section 4

4.1 $m = 9$ and $n = 3$
4.2 (a) 456,976 (b) 358,800
4.3 (a) 15,600,000 (b) 11,232,000
4.4 (a) 64,000 (b) 59,280
4.5 (a) 479,001,600 (b) 604,800
4.6 (a) 5 (b) 20 (c) 60 (d) 120
4.7 60
4.8 15,600
4.9 (a) 17,576 (b) 15,600 (c) 1,976.
4.10 $n = 3$.
4.11 36.
4.12 20.
4.13 n^r
4.14 $n = 5$

4.15

$$\begin{aligned} {}_{n-1}P_r + r \cdot_{n-1} P_{r-1} =& \frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r)!} \\ =& \frac{(n-1)!(n-r) + r(n-1)!}{(n-r)!} \\ =& \frac{n!}{(n-r)!} =_n P_r. \end{aligned}$$

4.16

$$\begin{aligned} \frac{{}_nP_r}{{}_nP_{r-1}} =& \frac{\frac{n!}{(n-r)!}}{\frac{n!}{(n-r+1)!}} \\ =& \frac{n!}{(n-r)!}\frac{(n-r+1)!}{n!} = \cdot n - r + 1. \end{aligned}$$

4.17 10^5
4.18 720
4.19 10,080
4.20 36

Section 5

5.1 $m = 13$ and $n = 1$ or $n = 12$
5.2 11,480
5.3 300
5.4 10
5.5 28
5.6 4,060
5.7 Recall that ${}_mP_n = \frac{m!}{(m-n)!} = n!{}_mC_n$. Since $n! \geq 1$, we can multiply both sides by ${}_mC_n$ to obtain ${}_mP_n = n!{}_mC_n \geq_m C_n$.
5.8 (a) Combination (b) Permutation
5.9 $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
5.10 $22,680a^3b^4$
5.11 1,200
5.12 $a_1a_2a_2a_2a_3a_3a_5$.
5.13 5,151

5.14 90,720
5.15 4
5.16 27,720
5.17 $m = 1365,\ n = 1364.$
5.18 3,116,960
5.19 2,463,300
5.20 196
5.21 140

Section 6

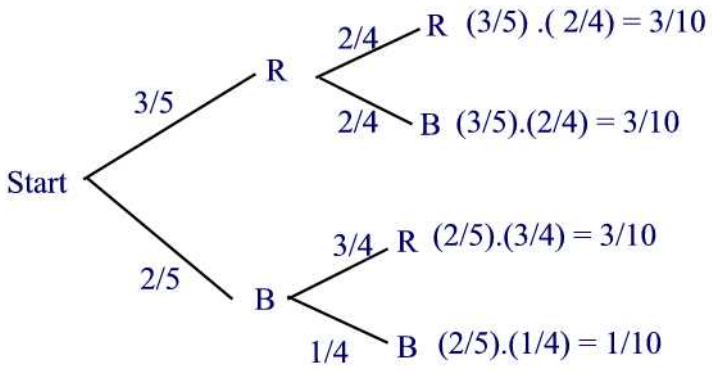
6.1 (a) $S = \{1, 2, 3, 4, 5, 6\}$ (b) $\{2, 4, 6\}$
6.2 $S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, H), (T, T)\}$
6.3 50%
6.4 (a) $(i, j), i, j = 1, \cdots, 6$ (b) $E^c = \{(5, 6), (6, 5), (6, 6)\}$ (c) $\frac{11}{12}$ (d) $\frac{5}{6}$ (e) $\frac{7}{9}$
6.5 (a) 0.5 (b) 0 (c) 1 (d) 0.4 (e) 0.3
6.6 (a) 0.75 (b) 0.25 (c) 0.5 (d) 0 (e) 0.375 (f) 0.125
6.7 25%
6.8 $\frac{6}{128}$
6.9 0.9849
6.10 (a) 10 (b) 40%
6.11 (a) $S = \{D_1D_2, D_1N_1, D_1N_2, D_1N_3, D_2N_1, D_2N_2, D_2N_3, N_1N_2, N_1N_3, N_2N_3\}$ (b) 10%
6.12 ${}_{36}C_{12}$
6.13 0.102
6.14 0.27
6.15 $\frac{36}{65}$
6.16 $\frac{625}{937}$
6.17 0.40
6.18 (a) 0.19 (b) 0.60 (c) 0.31
6.19 0.60
6.20 0.60
6.21 0.533
6.22 0.32
6.23 0.35

Section 7

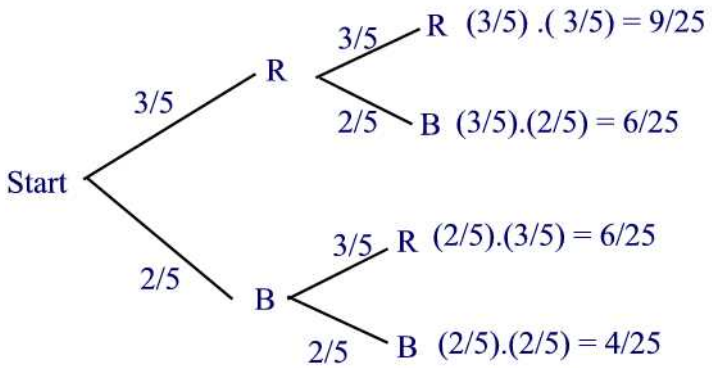
7.1 (a) 0.78 (b) 0.57 (c) 0
7.2 0.32
7.3 0.308
7.4 0.555
7.5 Since $P(A \cup B) \leq 1$, we have $-P(A \cup B) \geq -1$. Add $P(A) + P(B)$ to both sides to obtain $P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$. But the left hand side is just $P(A \cap B)$.
7.6 (a) 0.181 (b) 0.818 (c) 0.545
7.7 0.889
7.8 No
7.9 0.52
7.10 0.05
7.11 0.6
7.12 0.48
7.13 0.04
7.14 0.5
7.15 10%
7.16 80%
7.17 0.89
7.18 $\frac{2}{3}$
7.19 $\frac{29}{30}$
7.20 We know that $P(A \cap B) \geq 0$. Multiplying both sides by -1 to obtain $-P(A \cap B) \leq 0$. Adding $P(A) + P(B)$ to both sides to obtain $P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. But the left-hand side is just $P(A \cup B)$. Hence, $P(A \cup B) \leq P(A) + P(B)$.
7.21 0.06
7.22 2 events (iii and iv)
7.23 407
7.24 500
7.25 550
7.26 A, B, E

Section 8

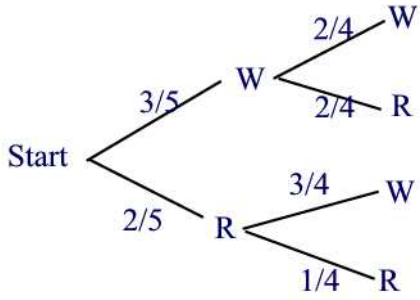
8.1



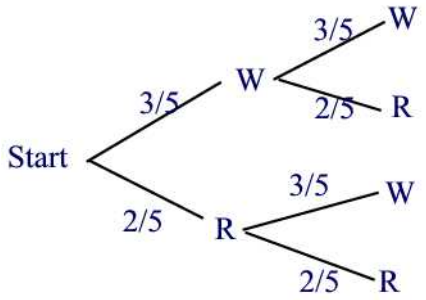
8.2



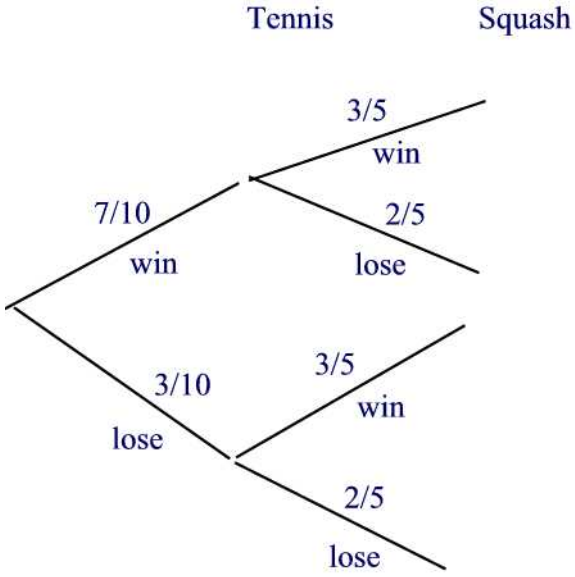
8.3 $P(A) = 0.6, P(B) = 0.3, P(C) = 0.1$
8.4 0.1875
8.5 0.444
8.6 0.167
8.7 The probability is $\frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5} = 0.6$



8.8 The probability is $\frac{3}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{25} = 0.48$



8.9 0.14
8.10 $\frac{36}{65}$
8.11 0.102
8.12 0.27
8.13 (a) The probability tree diagram is shown below.



(b) $\frac{7}{10} \cdot \frac{3}{5} = \frac{21}{50}$.
8.14 0.691
8.15 (a) $\frac{1}{21}$ (b) 0.380
8.16 0.584
8.17 0.311
8.18 $\frac{1}{720}$

8.19 7.06×10^{-4}
8.20 0.4060
8.21 0.097
8.22 0.245
8.23 $\frac{8}{35}$

Section 9

9.1 0.173
9.2 0.205
9.3 0.467
9.4 0.5
9.5 (a) 0.19 (b) 0.60 (c) 0.31 (d) 0.317 (e) 0.613
9.6 0.151
9.7 0.133
9.8 0.978
9.9 $\frac{7}{1912}$
9.10 (a) $\frac{1}{221}$ (b) $\frac{1}{169}$
9.11 $\frac{1}{114}$
9.12 80.2%
9.13 (a) 0.021 (b) 0.2381, 0.2857, 0.476
9.14 (a) 0.57 (b) 0.211 (c) 0.651
9.15 (a) Since $S = B \cup B^c$, we have

$$A = A \cap S = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Also, $(A \cap B) \cap (A \cap B^c) = A \cap (B \cap B^c) = A \cap \emptyset = \emptyset$.
(b) We have

$$\begin{aligned} P(A) =& P[(A \cap B) \cup (A \cap B^c)] \\ =& P(A \cap B) + P(A \cap B^c) \\ =& P(A|B)P(B) + P(A|B^c)P(B^c). \end{aligned}$$

9.16 Using the previous problem, We have

$$\begin{aligned} P(B|A) =& \frac{P(A|B)P(B)}{P(A)} \\ =& P(A|B)P(B) + P(A|B^c)P(B^c). \end{aligned}$$

9.17 We have

$$\begin{aligned}
P(A|B) > P(A) \Leftrightarrow & \frac{P(A \cap B)}{P(B)} > P(A) \\
\Leftrightarrow & P(A \cap B) > P(A)P(B) \\
\Leftrightarrow & P(B) - P(A \cap B) < P(B) - P(A)P(B) \\
\Leftrightarrow & P(A^c \cap B) < P(B)(1 - P(A)) \\
\Leftrightarrow & P(A^c \cap B) < P(B)P(A^c) \\
\Leftrightarrow & \frac{P(A^c \cap B)}{P(B)} < P(A^c) \\
\Leftrightarrow & P(A^c|B) < P(A^c).
\end{aligned}$$

9.18 A and B are independent if and only if $P(A|B) = P(A)$. But $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Thus, A and B are independent if and only if $\frac{P(A \cap B)}{P(B)} = P(A)$. This is equivalent to $P(A \cap B) = P(A)P(B)$.

9.19 First note that A can be written as the union of two mutually exclusive events: $A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$. Thus, $P(A) = P(A \cap B) + P(A \cap B^c)$. It follows that

$$\begin{aligned}
P(A \cap B^c) = & P(A) - P(A \cap B) \\
= & P(A) - P(A)P(B) \\
= & P(A)(1 - P(B)) = P(A)P(B^c).
\end{aligned}$$

9.20 Using De Morgan's formula we have

$$\begin{aligned}
P(A^c \cap B^c) = & 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\
= & [1 - P(A)] - P(B) + P(A)P(B) \\
= & P(A^c) - P(B)[1 - P(A)] = P(A^c) - P(B)P(A^c) \\
= & P(A^c)[1 - P(B)] = P(A^c)P(B^c).
\end{aligned}$$

9.21 0.119
9.22 $\frac{70}{143}$
9.23 0.625

Section 10

10.1 (a) 0.26 (b) $\frac{6}{13}$

10.2 0.1584
10.3 0.0141
10.4 0.29
10.5 0.42
10.6 0.22
10.7 0.657
10.8 0.4
10.9 0.45
10.10 0.66
10.11 (a) 0.22 (b) $\frac{15}{22}$
10.12 (a) 0.56 (b) $\frac{4}{7}$
10.13 0.36
10.14 $\frac{1}{3}$
10.15 (a) $\frac{17}{140}$ (b) $\frac{7}{17}$
10.16 $\frac{15}{74}$.
10.17 $\frac{72}{73}$.
10.18 0.264
10.19 0.93
10.20 0.362
10.21 $\frac{7}{24}$
10.22 $\frac{1}{6}$
10.23 0.491
10.24 14
10.25 0.6
10.26 0.4356
10.27 0.02
10.28 $\frac{24}{35}$
10.29 0.75

Section 11

11.1 (a) Dependent (b) Independent
11.2 0.02
11.3 (a) 21.3% (b) 21.7%
11.4 0.72
11.5 4
11.6 0.328
11.7 0.4

11.8 We have

$$\begin{aligned} P(A \cap B) =& P(\{1\}) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)P(B) \\ P(A \cap C) =& P(\{1\}) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)P(C) \\ P(B \cap C) =& P(\{1\}) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(B)P(C) \end{aligned}$$

It follows that the events A, B, and C are pairwise independent. However,

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C).$$

Thus, the events A, B, and C are not independent
11.9 $P(C) = 0.56,\ P(D) = 0.38$
11.10 0.93
11.11 0.43
11.12 (a) We have $P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A)P(B)P(C)}{P(B)P(C)} = P(A)$. Thus, A and $B \cap C$ are independent.
(b) We have $P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{P((A \cap B) \cup (A \cap C))}{P(B \cup C)} = \frac{P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)}{P(B) + P(C) - P(B \cap C)} = \frac{P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)}{P(B) + P(C) - P(B)P(C)} = \frac{P(A)P(B)[1 - P(C)] + P(A)P(C)}{P(B) + P(C) - P(B)P(C)} = \frac{P(A)P(B)P(C^c) + P(A)P(C)}{P(B)P(C^c) + P(C)} = \frac{P(A)[P(B)P(C^c) + P(C)]}{P(B)P(C^c) + P(C)} = P(A)$. Hence, A and $B \cup C$ are independent
11.13 (a) We have

$$\begin{aligned} S =& \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \\ A =& \{HHH, HHT, HTH, THH, TTH\} \\ B =& \{HHH, THH, HTH, TTH\} \\ C =& \{HHH, HTH, THT, TTT\} \end{aligned}$$

(b) $P(A) = \frac{5}{8},\ P(B) = 0.5,\ P(C) = \frac{1}{2}$ (c) $\frac{4}{5}$ (d) We have $B \cap C = \{HHH, HTH\}$, so $P(B \cap C) = \frac{1}{4}$. That is equal to $P(B)P(C)$, so B and C are independent
11.14 0.65
11.15 (a) 0.70 (b) 0.06 (c) 0.24 (d) 0.72 (e) 0.4615
11.16 0.892
11.17 Dependent
11.18 We have $P(A) = P(A \cap S) = P(A)P(S)$ since $P(S) = 1$.

11.19 Thus,

$$\begin{aligned} P(A \cap B) =& \frac{1}{36} = \frac{6}{36} \cdot \frac{6}{36} = P(A)P(B) \\ P(A \cap C) =& \frac{1}{36} = \frac{6}{36} \cdot \frac{6}{36} = P(A)P(C) \\ P(B \cap C) =& \frac{1}{36} = \frac{6}{36} \cdot \frac{6}{36} = P(B)P(C) \\ P(A \cap B \cap C) =& \frac{1}{36} \neq \frac{6}{36} \cdot \frac{6}{36} = P(A)P(B)P(C). \end{aligned}$$

11.20

$$\begin{aligned} A \cap B =& \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\} \\ A \cap C =& \{(3,6)\} \\ B \cap C =& \{(3,6),(4,5),(5,4)\} \\ A \cap B \cap C =& \{(3,6)\} \\ P(A \cap B) =& \frac{1}{6} \neq \frac{1}{4} = P(A)P(B) \\ P(A \cap C) =& \frac{1}{36} \neq \frac{1}{18} = P(A)P(C) \\ P(B \cap C) =& \frac{1}{12} \neq \frac{1}{18} = P(B)P(C) \\ P(A \cap B \cap C) =& \frac{1}{36} = P(A)P(B)P(C). \end{aligned}$$

11.21 5
11.22 A
11.23 0.51
11.24 0.55
11.25 0.398

Section 12

12.1 15:1
12.2 62.5%
12.3 1:1
12.4 4:6
12.5 4%

12.6 (a) 1:5 (b) 1:1 (c) 1:0 (d) 0:1
12.7 1:3
12.8 (a) 43% (b) 0.3
12.9 10:3
12.10 No

Section 13

13.1 (a) Continuous (b) Discrete (c) Discrete (d) Continuous (e) mixed.
13.2 If G and R stand for golden and red, the probabilities for GG, GR, RG, and RR are, respectively $\frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14}, \frac{5}{8} \cdot \frac{3}{7} = \frac{15}{56}, \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56}$, and $\frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$. The results are shown in the following table.

Element of sample space	Probability	x
GG	$\frac{5}{14}$	2
GR	$\frac{15}{56}$	1
RG	$\frac{15}{56}$	1
RR	$\frac{3}{28}$	0

13.3 0.139
13.4 0.85
13.5 $\left(\frac{1}{2}\right)^n$
13.6 $\frac{1}{2}$
13.7 0.4
13.8 0.9722
13.9 (a)

$$X(s) = \begin{cases} 0 & s \in \{(NS, NS, NS)\} \\ 1 & s \in \{(S, NS, NS), (NS, S, NS), (NS, NS, S)\} \\ 2 & s \in \{(S, S, NS), (S, NS, S), (NS, S, S)\} \\ 3 & s \in \{(S, S, S)\} \end{cases}$$

(b) 0.09 (c) 0.36 (d) 0.41 (e) 0.14
13.10 $\frac{1}{1+e}$
13.11 0.267
13.12

x	0	1	2	3	4
$P(X = x)$	$\frac{5}{210}$	$\frac{50}{210}$	$\frac{100}{210}$	$\frac{50}{210}$	$\frac{5}{210}$

13.13

$$P(X=x)=\begin{cases}\frac{2x-1}{36} & \text{if } x=1,2,3,4,5,6\\ 0 & \text{otherwise.}\end{cases}$$

13.14 $F(n)=1-\left(\frac{2}{3}\right)^{n+1}$.

13.15

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

13.16

X	Event	$P(X=x)$
1	R	$\frac{1}{6}$
2	(NR,R)	$\frac{5}{6}\cdot\frac{1}{5}=\frac{1}{6}$
3	(NR,NR,R)	$\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{1}{4}=\frac{1}{6}$
4	(NR,NR,NR,R)	$\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{6}$
5	(NR,NR,NR,NR,R)	$\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{6}$
6	(NR,NR,NR,NR,NR,R)	$\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{6}$

13.17

$$X(s)=\begin{cases}0 & \text{if } s\in\{(1,1),(1,3),(3,1),(1,5),(5,1),(3,3),(3,5),(5,3),(5,5)\}\\ 1 & \text{if } s\in\left\{\begin{array}{cccccc}(2,1)&(2,3)&(2,5)&(1,2)&(3,2)&(5,2)\\(4,1)&(4,3)&(4,5)&(1,4)&(3,4)&(5,4)\\(6,1)&(6,3)&(6,5)&(1,6)&(3,6)&(5,6)\end{array}\right\}\\ 2 & \text{if } s\in\{(2,2),(2,4),(4,2),(2,6),(6,2),(4,4),(4,6),(6,4),(6,6)\}\end{cases}$$

13.18

x	0	1	2
$P(X=x)$	1/4	1/2	1/4

13.19

$$P(X=n)=\frac{95}{100}\cdot\frac{94}{99}\cdots\frac{95-n+2}{100-n+2}\frac{5}{100-n+1}$$

and $P(X=1)=\frac{5}{100}$.

13.20 $P(X=2)=\frac{3}{7}$ and $P(X=-1)=\frac{4}{7}$.

13.21 $\frac{2}{3}$

13.22 $\frac{11}{60}$

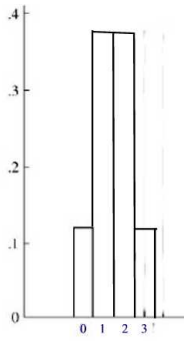
13.23 $\frac{5}{8}$

Section 14

14.1 (a)

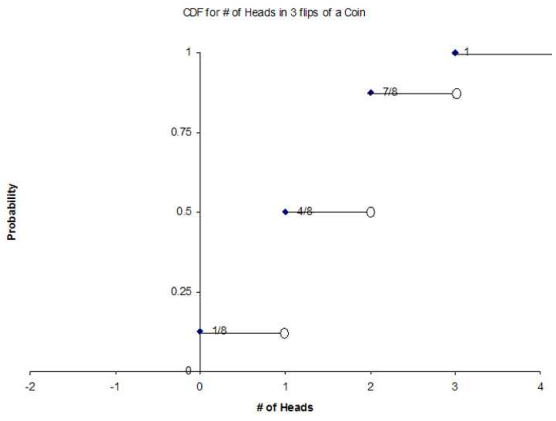
x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

and 0 otherwise.
(b)



14.2

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}$$



14.3

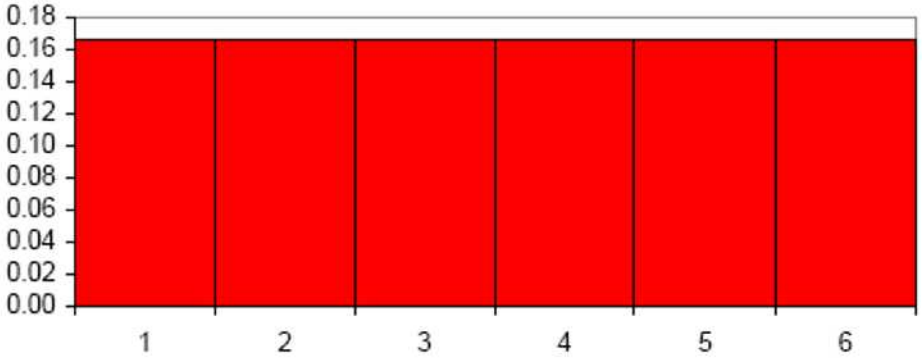
x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

and 0 otherwise.

14.4

X	Event	$p(x)$
1	R	$\frac{1}{6}$
2	(NR,R)	$\frac{5}{6} \cdot \frac{1}{5} = \frac{1}{6}$
3	(NR,NR,R)	$\frac{5}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{6}$
4	(NR,NR,NR,R)	$\frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{6}$
5	(NR,NR,NR,NR,R)	$\frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$
6	(NR,NR,NR,NR,NR,R)	$\frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$

and 0 otherwise.



14.5

x	0	1	2
$p(x)$	1/4	1/2	1/4

and 0 otherwise.

14.6

$$\begin{aligned} F(n) =& P(X \le n) = \sum_{k=0}^{n} P(X = k) \\ =& \sum_{k=0}^{n} \frac{1}{3} \left(\frac{2}{3}\right)^k \\ =& \frac{1}{3} \frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}} \\ =& 1 - \left(\frac{2}{3}\right)^{n+1} \end{aligned}$$

14.7 (a) For $n = 2, 3, \cdots, 96$ we have

$$P(X = n) = \frac{95}{100} \cdot \frac{94}{99} \cdots \frac{95 - n + 2}{100 - n + 2} \frac{5}{100 - n + 1}$$

$P(X = 1) = \frac{5}{100}$ and 0 otherwise.

(b)

$$P(Y = n) = \frac{\begin{pmatrix} 5 \\ n \end{pmatrix} \begin{pmatrix} 95 \\ 10 - n \end{pmatrix}}{\begin{pmatrix} 100 \\ 10 \end{pmatrix}}, \quad n = 0, 1, 2, 3, 4, 5$$

and 0 otherwise.

14.8

$$p(x) = \begin{cases} \frac{3}{10} & x = -4 \\ \frac{4}{10} & x = 1 \\ \frac{3}{10} & x = 4 \end{cases}$$

and 0 otherwise.

14.9

$$\begin{aligned} P(X = 2) =& P(RR) + P(BB) = \frac{{}_3C_2}{{}_7C_2} + \frac{{}_4C_2}{{}_7C_2} \\ =& \frac{3}{21} + \frac{6}{21} = \frac{9}{21} = \frac{3}{7} \end{aligned}$$

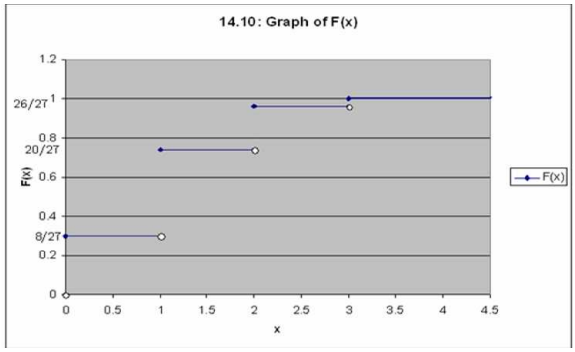
and

$$P(X = -1) = 1 - P(X = 2) = 1 - \frac{3}{7} = \frac{4}{7}$$

and 0 otherwise.
14.10 (a)

$$\begin{aligned} p(0) &= \left(\frac{2}{3}\right)^3 \\ p(1) &= 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \\ p(2) &= 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) \\ p(3) &= \left(\frac{1}{3}\right)^3 \end{aligned}$$

and 0 otherwise.
(b)



14.11

$$\begin{aligned} p(0) &= \frac{220}{455} \\ p(1) &= \frac{198}{455} \\ p(2) &= \frac{36}{455} \\ p(3) &= \frac{1}{455} \end{aligned}$$

and 0 otherwise.
14.12 $p(2) = \frac{1}{36}, p(3) = \frac{2}{36}, p(4) = \frac{3}{36}, p(5) = \frac{4}{36}, p(6) = \frac{5}{36}, p(7) = \frac{6}{36}, p(8) = \frac{5}{36}, p(9) = \frac{4}{36}, p(10) = \frac{3}{36}, p(11) = \frac{2}{36}$, and $p(12) = \frac{1}{36}$ and 0 otherwise

14.13 (a) 0.267 (b) 0.449 (c) $p(x) = \dfrac{\begin{pmatrix} 8 \\ x \end{pmatrix}\begin{pmatrix} 22 \\ 4-x \end{pmatrix}}{\begin{pmatrix} 30 \\ 4 \end{pmatrix}}$ and 0 otherwise.

14.14 (a) 0.3 (b) 0.7 (c) $p(0) = 0.3,\ p(1) = 0,\ p(P(X \leq 0) = 0$
(d)

$$p(x) = \begin{cases} 0.2 & x = -2 \\ 0.3 & x = 0 \\ 0.1 & x = 2.2 \\ 0.3 & x = 3 \\ 0.1 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

14.15 (a)

$$\begin{aligned} P(1) = &P(X = 1) = \frac{{}_3C_3 \cdot {}_7C_1}{210} = \frac{7}{210} \\ P(2) = &P(X = 2) = \frac{{}_3C_2 \cdot {}_7C_2}{210} = \frac{63}{210} \\ P(3) = &P(X = 3) = \frac{{}_3C_1 \cdot {}_7C_3}{210} = \frac{105}{210} \\ P(4) = &P(X = 4) = \frac{{}_3C_0 \cdot {}_7C_4}{210} = \frac{35}{210}. \end{aligned}$$

and 0 otherwise.
(b)

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{7}{210} & 1 \leq x < 2 \\ \frac{70}{210} & 2 \leq x < 3 \\ \frac{175}{210} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

14.16 $\frac{1}{30}$.
14.17 (a) $\frac{1}{9}$ (b) $p(-1) = \frac{2}{9}, p(1) = \frac{3}{9}$, and $p(2) = \frac{4}{9}$.
14.18 (a) $k = \frac{1}{21}$ (b) $\frac{4}{7}$
14.19 (a) $K = \frac{60}{137}$ (b) $p(n) = \frac{3}{137n}, \quad n = 1, \cdots, 5$ and 0 otherwise.
14.20 $P(X = 2) = \frac{3}{7},\ P(X = -1) = \frac{4}{7}$ and 0 otherwise
14.21 $\frac{(x-1)(x-2)(12-x)}{990}$
14.22 0.286

14.23 $\frac{(x-1)(11-x)(12-x)}{495}$

Section 15

15.1 7
15.2 \$ 16.67
15.3 $E(X) = 10 \times \frac{1}{6} - 2 \times \frac{5}{6} = 0$ Therefore, you should come out about even if you play for a long time
15.4 -1
15.5 \$26
15.6 $E(X) = -\$0.125$ So the owner will make on average 12.5 cents per spin
15.7 \$110
15.8 -0.54
15.9 897
15.10 (a) 0.267 (b) 0.449 (c) 1.067
15.11 (a) 0.3 (b) 0.7 (c) $p(0) = 0.3,\ p(1) = 0,\ p(P(X \leq 0)) = 0$ (d)

$$P(x) = \begin{cases} 0.2 & x = -2 \\ 0.3 & x = 0 \\ 0.1 & x = 2.2 \\ 0.3 & x = 3 \\ 0.1 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

(e) 1.12
15.12 (a) 390 (b) Since $E(V) < 400$ the answer is no.
15.13 (a)

$$\begin{aligned} p(1) = & P(X = 1) = \frac{{}_3C_3 \cdot {}_7C_1}{210} = \frac{7}{210} \\ p(2) = & P(X = 2) = \frac{{}_3C_2 \cdot {}_7C_2}{210} = \frac{63}{210} \\ p(3) = & P(X = 3) = \frac{{}_3C_1 \cdot {}_7C_3}{210} = \frac{105}{210} \\ p(4) = & P(X = 4) = \frac{{}_3C_0 \cdot {}_7C_4}{210} = \frac{35}{210} \end{aligned}$$

and 0 otherwise.
(b)

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{7}{210} & 1 \leq x < 2 \\ \frac{70}{210} & 2 \leq x < 3 \\ \frac{175}{210} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

(c) 2.8
15.14 \$50,400
15.15 $c = \frac{1}{30}$ (b) $E(X) = 3.333$.
15.16 (a) $c = \frac{1}{9}$ (b) $p(-1) = \frac{2}{9}, p(1) = \frac{3}{9}$, and $p(2) = \frac{4}{9}$ (c) $E(X) = 1$.
15.17 (a) $k = \frac{1}{21}$ (b) $\frac{4}{7}$ (c) 4.333.
15.18 1.07
15.19 (a) $p(2) = \frac{3}{7},\ p(-1) = \frac{4}{7}$ and 0 otherwise (b) $\frac{2}{7}$.
15.20 55
15.21 $\frac{11}{15}$
15.22 2.18

Section 16

16.1 (a) $c = \frac{1}{30}$ (b) 3.333 (c) 8.467
16.2 (a) $c = \frac{1}{9}$ (b) $p(-1) = \frac{2}{9}, p(1) = \frac{3}{9}, p(2) = \frac{4}{9}$ (c) $E(X) = 1$ and $E(X^2) = \frac{7}{3}$
16.3 (a)

$$p(x) = \begin{cases} \frac{x}{21} & x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

(b) $\frac{4}{7}$ (c) $E(X) = 4.333$
16.4 Let D denote the range of X. Then

$$\begin{aligned} E(aX^2 + bX + c) &= \sum_{x \in D} (ax^2 + bx + c)p(x) \\ &= \sum_{x \in D} ax^2 p(x) + \sum_{x \in D} bxp(x) + \sum_{x \in D} cp(x) \\ &= a \sum_{x \in D} x^2 p(x) + b \sum_{x \in D} xp(x) + c \sum_{x \in D} p(x) \\ &= aE(X^2) + bE(X) + c \end{aligned}$$

16.5 0.62
16.6 \$220
16.7 0.0314
16.8 0.24
16.9 (a)

$$\begin{aligned} P(X=2) =& P(RR)+P(BB) = \frac{{}_3C_2}{{}_7C_2} + \frac{{}_4C_2}{{}_7C_2} \\ =& \frac{3}{21} + \frac{6}{21} = \frac{9}{21} = \frac{3}{7} \end{aligned}$$

$$P(X=-1) = 1 - P(X=2) = 1 - \frac{3}{7} = \frac{4}{7}$$

and 0 otherwise.
(b) $E(2^X) = 2$
16.10 (a) $P = 3C + 8A + 5S - 300$ (b) \$1,101
16.11 475
16.12 (a) $c = \frac{1}{55}$ (b) 1.064
16.13 (a) The range of X consists of the numbers 1,2,3, and 4. Thus,

$$\begin{aligned} p(1) =& \frac{{}_3C_3 \cdot {}_7C_1}{{}_{10}C_4} = \frac{1}{30} \\ p(2) =& \frac{{}_3C_2 \cdot {}_7C_2}{{}_{10}C_4} = \frac{3}{10} \\ p(3) =& \frac{{}_3C_1 \cdot {}_7C_3}{{}_{10}C_4} = \frac{1}{2} \\ p(4) =& \frac{{}_3C_0 \cdot {}_7C_4}{{}_{10}C_4} = \frac{1}{6} \end{aligned}$$

and 0 otherwise.
(b) 0.56.
16.14 (a) $c = \frac{1}{30}$ (b) $E(X) = 3.33$ and $E(X(X-1)) = 8.467$ (c) 0.689.
16.15 144.
16.16 (a) $P(X=2) = \frac{3}{7}$ $P(X=-1) = \frac{4}{7}$ and 0 otherwise (b) $\frac{108}{49}$.
16.17 9.84
16.18 $p(1-p)$.
16.19 $E(X) = np,\ E(X(X-1)) = n(n-1)p^2,\ E(X^2) = n(n-1)p^2 + np.$

16.20 We have

$$\begin{aligned} E[(X-E(X))^2] =& E[X^2 - 2XE(X) + (E(X))^2] \\ =& E(X^2) - 2E(X)E(X) + (E(X))^2 \\ =& E(X^2) - (E(X))^2. \end{aligned}$$

16.21 166.67

Section 17

17.1 0.45
17.2 374
17.3 (a) $c = \frac{1}{55}$ (b) $E(X) = -1.09$ $\text{Var}(X) = 1.064$
17.4 (a)

$$\begin{aligned} p(1) =& \frac{{}_3C_3 \cdot {}_7C_1}{{}_{10}C_4} = \frac{1}{30} \\ p(2) =& \frac{{}_3C_2 \cdot {}_7C_2}{{}_{10}C_4} = \frac{3}{10} \\ p(3) =& \frac{{}_3C_1 \cdot {}_7C_3}{{}_{10}C_4} = \frac{1}{2} \\ p(4) =& \frac{{}_3C_0 \cdot {}_7C_4}{{}_{10}C_4} = \frac{1}{6} \end{aligned}$$

and 0 otherwise.
(b) $E(X) = 2.8$ $\text{Var}(X) = 0.56$
17.5 (a) $c = \frac{1}{30}$ (b) $E(X) = 3.33$ and $E(X(X-1)) = 8.467$ (c) $E(X^2) = 11.8$ and $\text{Var}(X) = 0.6889$
17.6 $E(Y) = 21$ and $\text{Var}(Y) = 144$
17.7 (a)

$$\begin{aligned} P(X=2) =& P(RR) + P(BB) = \frac{{}_3C_2}{{}_7C_2} + \frac{{}_4C_2}{{}_7C_2} \\ =& \frac{3}{21} + \frac{6}{21} = \frac{9}{21} = \frac{3}{7} \end{aligned}$$

and

$$P(X=-1) = 1 - P(X=2) = 1 - \frac{3}{7} = \frac{4}{7}$$

and 0 otherwise.
(b) $E(X) = \frac{2}{7}$ and $E(X^2) = \frac{16}{7}$ (c) $\text{Var}(X) = \frac{108}{49}$
17.8 $E(X) = \frac{4}{10}$; $\text{Var}(X) = 9.84$; $\sigma_X = 3.137$
17.9 $E(X) = p$ $\text{Var}(X) == p(1-p)$
17.10 $np(1-p)$.
17.11 (a) $E(X) = \lambda,\ E(X(X-1)) = \lambda^2$ and $E(X^2) = \lambda^2 + \lambda$ (b) λ.
17.12 (a) $f'(1-p) = p^{-2}$ and $f''(1-p) = 2p^{-3}$ (b) $E(X) = \frac{1}{p},\ E(X(X-1)) = 2p^{-2}(1-p)$ and $E(X^2) = (2-p)p^{-2}$ (c) $\frac{1-p}{p^2}$.
17.13 (a) (a) The probability mass function is given by

$$p(x) = \begin{cases} 0.3 & x = -4 \\ 0.4 & x = 1 \\ 0.3 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

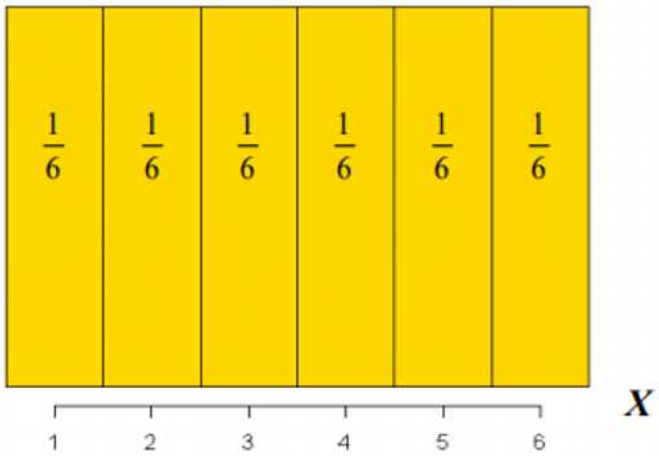
(b) $\text{Var}(X) = 9.84$ and $\sigma_X = 3.137$.
17.14 2.36
17.15 9.44
17.16 $E(X) = \frac{n+1}{2}$ and $\text{Var}(X) = \frac{n^2-1}{12}$.
17.17 $E(X) = 3.5$ and $\text{Var}(X) = 2.92$.
17.18 $E(X) = \frac{a+b}{2},\ E(X^2) = ab + \frac{(b-a)(2b-2a+1)}{6}$ and $\text{Var}(X) = \frac{(b-a+1)^2-1}{12}$.
17.19 We have

$$\begin{aligned} \text{Var}(ag(X)+b) =& E[(ag(X)+b)^2] - [E(ag(X)+b)]^2 \\ =& E(a^2 g(X)^2 + 2ag(X) + b^2) - [aE(g(X)) + b]^2 \\ =& a^2 E(g(X)^2) + 2abE(g(X)) + b^2 - a^2[E(g(X))]^2 - 2abE(g(X)) - b^2 \\ =& a^2[E(g(X)^2) - [E(g(X))]^2] \\ =& a^2 \text{Var}(g(X)). \end{aligned}$$

17.20 $E(Z) = 0$ and $\text{Var}(Z) = 1$.
17.21 235.465
17.22 253.53

Section 18

18.1 No.
18.2 Yes.
18.3 The graph is shown below.



18.4 $E(X) = 6.5,\ \text{Var}(X) = 5.25.$
18.5

$$F(x) = \begin{cases} 0 & x < a \\ \frac{\lfloor x \rfloor - a + 1}{b-a+1} & a \le x \le b \\ 1 & x > b \end{cases}$$

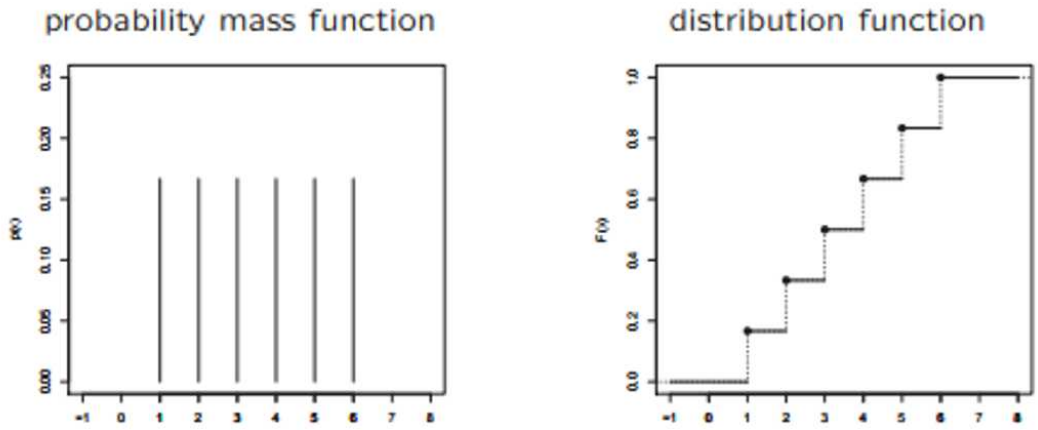
18.6 $E(X) = \frac{n+1}{2}$ and $\text{Var}(X) = \frac{n^2-1}{12}$.
18.7 $E(X) = 11,\ \text{Var}(X) = 40.$
18.8 $\frac{1}{12}$.
18.9 $\frac{3}{7}$.
18.10 We have

$$\begin{aligned} M_X(t) =& E(e^{tX}) = \sum_{x=0}^{b-a} e^{t(x+a)} p(x+a) \\ =& \frac{1}{b-a+1} \sum_{x=0}^{b-a} e^{t(x+a)} \\ =& \frac{e^{ta}}{b-a+1} \sum_{x=0}^{b-a} e^{tx} \\ =& \frac{e^{ta}}{b-a+1} \left(\frac{1 - e^{t(b-a+1)}}{1 - e^t} \right) \\ =& \frac{e^{ta} - e^{(b+1)t}}{(b-a+1)(1-e^t)}. \end{aligned}$$

18.11 $E(X) = 4.5$ and $\text{Var}(X) = 8.25$.
18.12 $a = 3,\ b = 10$
18.13 We have $p(x) = \frac{1}{9}$ for $x = -4, -3, \cdots, 3, 4$ and 0 otherwise. The cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & x < -4 \\ \frac{1}{9}(\lfloor x \rfloor + 5) & -4 \leq x \leq 4 \\ 1 & x > 4 \end{cases}$$

18.14



18.15

$$p_Y(0) = \frac{1}{5}$$
$$p_Y(1) = \frac{2}{5}$$
$$p_Y(2) = \frac{2}{5}$$

and 0 otherwise.
18.16

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{\lfloor x \rfloor}{n} & 1 \leq x \leq n \\ 1 & x > n \end{cases}$$

18.17 $E(Y) = 30.5$ and $\text{Var}(Y) = 206.25$
18.18 $P(Z = 2) \neq P(Z = 3)$
18.19 5

18.20 120
18.21 0.3097

Section 19

19.1 0.3826
19.2 0.211
19.3

$$p(x) = \begin{cases} \frac{1}{8}, & \text{if } x = 0, 3 \\ \frac{3}{8}, & \text{if } x = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

19.4 0.096
19.5 \$60
19.6 0.925
19.7 0.144
19.8 (a) 0.1875 (b) 0.5
19.9 0.1198
19.10 0.6242
19.11 0.784
19.12 (a) 0.2321 (b) 0.2649 (c) 0.1238
19.13 0.2639
19.14 0.104
19.15 $E(X) = np$ and $E(X(X-1)) = n(n-1)p^2$.
19.16 $np(1-p)$.
19.17 $E(X) = 4.8$ and $\sigma_X = 0.9798$.
19.18 2.
19.19 (a) 0.6242 (b) 2
19.20 We have

$$\begin{aligned} \frac{p(k)}{p(k-1)} =& \frac{{}_nC_k p^k (1-p)^{n-k}}{{}_nC_{k-1} p^{k-1} (1-p)^{n-k+1}} \\ =& \frac{\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}}{\frac{n!}{(k-1)!(n-k+1)!} p^{k-1} (1-p)^{n-k+1}} \\ =& \frac{(n-k+1)p}{k(1-p)} = \frac{p}{1-p} \frac{n-k+1}{k} \end{aligned}$$

19.21 0.417
19.22 0.404

Section 20

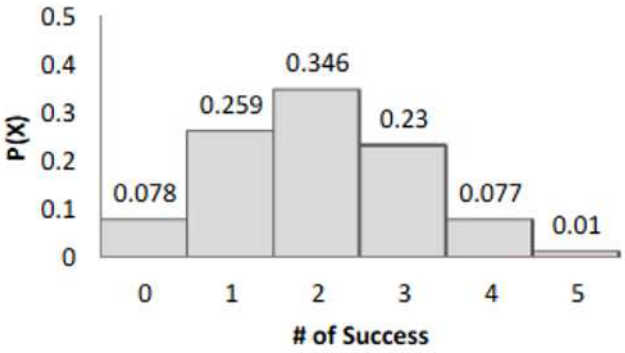
20.1 154
20.2 $985
20.3

n	$E(Y)$	$E(Y^2)$	$E(S)$
1	0.20	0.20	100+10-2 = 108
2	0.40	0.48	100+20-4.8 = 115.2
3	0.60	0.84	100+30-8.4 = 121.6

20.4 $E(X) = 400$ and $\sigma_X = 15.492$
20.5 (a) $p(x) = {}_5C_x(0.4)^x(0.6)^{5-x},\ x = 0, 1, 2, 3, 4, 5.$
(b)

$$\begin{aligned} p(0) &= {}_5C_0(0.4)^0(0.6)^{5-0} = 0.078 \\ p(1) &= {}_5C_1(0.4)^1(0.6)^{5-1} = 0.259 \\ p(2) &= {}_5C_2(0.4)^2(0.6)^{5-2} = 0.346 \\ p(3) &= {}_5C_3(0.4)^3(0.6)^{5-3} = 0.230 \\ p(4) &= {}_5C_4(0.4)^4(0.6)^{5-4} = 0.077 \\ p(5) &= {}_5C_5(0.4)^5(0.6)^{5-5} = 0.01. \end{aligned}$$

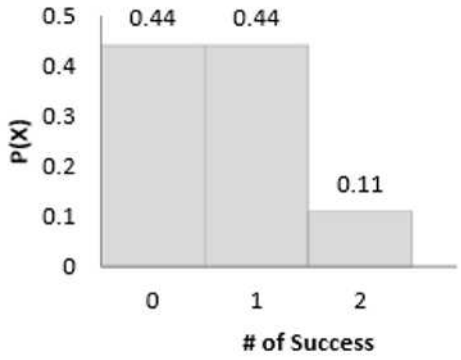
(c)



(d) $E(X) = 2$ and $\sigma_X = 1.095$.
20.6 (a) $p(x) = {}_2C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{2-x}$.

(b)

$$p(0) = {}_2C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{2-0} = 0.44$$
$$p(1) = {}_2C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{2-1} = 0.44$$
$$p(2) = {}_2C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{2-2} = 0.11.$$



(c)

(d) $E(X) = 0.667$ and $\sigma_X = 0.667$.
20.7 $E(N) = 0.75$ and $\text{Var}(N) = 0.5625$.
20.8 $E(C) = 20$ and $\text{Var}(C) = 10$.
20.9 (a) $E(F) = 0.15$ and $\text{Var}(F) = 0.1425$ (b) 0.00725.
20.10 (a) $E(X) = 13.5$ and $\text{Var}(X) = 1.35$ (b) $E(Y) = 6$ and $\text{Var}(Y) = 3.6$
20.11 We have

$$\begin{aligned} E(e^{tX}) &= \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} \frac{n!}{x!(n-x)!} p^x q^{n-x} \\ &= \sum_{x=0}^{n} (pe^t)^x \frac{n!}{x!(n-x)!} q^{n-x} \\ &= (q + pe^t)^n. \end{aligned}$$

20.12 $M'_X(t) = npe^t(q + pe^t)^{n-1}$ and $M'_X(0) = np = E(X)$.

20.13 Taking the second derivative with respect to t and using the chain rule, we find

$$M_X''(t) = npe^t(q + pe^t)^{n-2}(q + npe^t).$$

Letting $t = 0$ in the above formula, we find

$$M_X''(0) = np(q + np) = np(1 - p) + n^2p^2 = n(n - 1)p^2 + np = E(X^2).$$

20.14 0.6664.
20.15 (a) $E(X) = 11.25$ and $\text{Var}(X) = 2.8125$.
20.16 2
20.17 1
20.18 8
20.19 2
20.20 40

Section 21

21.1 0.0183 and 0.0027
21.2 0.1251
21.3 3.06×10^{-7}
21.4 (a) 0.577 (b) 0.05
21.5 (a) 0.947 (b) 0.762 (c) 0.161
21.6 0.761897
21.7 (a) 0.5654 (b) 0.4963
21.8 2
21.9 \$7231.30
21.10 699
21.11 0.1550
21.12 0.7586
21.13 4
21.14 (a) 0.2873 (b) mean = 20 and standard deviation = 4.47
21.15 35.

21.16 We have

$$\begin{aligned}
E[\lambda g(X+1) - Xg(X)] &= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=1}^{\infty} k g(k) e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=1}^{\infty} g(k) e^{-\lambda} \frac{\lambda^k}{(k-1)!} \\
&= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=1}^{\infty} \lambda g(k) e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} \\
&= 0.
\end{aligned}$$

21.17 235.264
21.18 No.
21.19 $M_X(t) = e^{\lambda\omega(e^t-1)}$
21.20 0.8009
21.21 $n \geq 20$ and $p \leq 0.05$. $P(X \geq 2) \approx 0.9084$
21.22 0.3293
21.23 0.0144
21.24 0.3679
21.25 (a) 0.177 (b) 0.876
21.26 0.3230
21.27 np
21.28 0.6063.
21.29 0.9596.
21.30 0.0894
21.31 554
21.32 0.18
21.33 0.287
21.34 104
21.35 192
21.36 $p(0) = 1.1e^{-0.1}$ and $p(y) = e^{-0.1}\frac{(0.1)^{y+1}}{(y+1)!}$ for $y > 0$

Section 22

22.1 (a) 0.1 (b) 0.09 (c) $\left(\frac{9}{10}\right)^{n-1}\left(\frac{1}{10}\right)$

22.2 0.387
22.3 0.916
22.4 (a) 0.001999 (b) 1000
22.5 (a) 0.1406 (b) 0.3164
22.6 (a) 0.1481 and 7.842×10^{-10} (b) 3
22.7 (a) $p_X(x) = (0.85)(0.15)^{x-1}$, $x = 1, 2, \cdots$ and 0 otherwise (b) $p_Y(y) = (0.85)(0.15)^{y-1}$, $y = 1, 2, 3, \cdots$ and 0 otherwise. Thus, Y is a geometric random variable with parameter 0.85
22.8 (a) 0.1198 (b) 0.3999
22.9 We have

$$\begin{aligned} P(X > i + j | X > i) =& \frac{P(X > i + j, X > i)}{P(X > i)} \\ =& \frac{P(X > i + j)}{P(X > i)} = \frac{(1-p)^{i+j}}{(1-p)^i} \\ =& (1-p)^j = P(X > j) \end{aligned}$$

22.10 0.053
22.11 (a) 10 (b) 0.81
22.12 (a) X is a geometric distribution with pmf $p(x) = 0.4(0.6)^{x-1}$, $x = 1, 2, \cdots$ (b) X is a binomial random variable with pmf $p(x) = {}_{20}C_x(0.60)^x(0.40)^{20-x}$ where $x = 0, 1, \cdots, 20$
22.13 0.07776
22.14 5
22.15 0.984375
22.16 $\frac{MN^{n-1}}{(N+M)^n}$
22.17 $M_X(t) = \frac{pe^t}{1-e^t(1-p)}$, $t < -\ln(1-p)$
22.18 $E(Y) = \frac{1}{p} - 1$ and $\text{Var}(Y) = \frac{1-p}{p^2}$
22.19 4
22.20 $E(X) = 25$ and $\sigma_X \approx 24.4949$.
22.21 2694
22.22 0.78279
22.23 $[1 - 3p^2(1-p) - p^3]^{n-1}[3p^2(1-p) + p^3]$
22.24 0.42
22.25 $r^{\frac{8}{3}}$

Section 23

23.1 (a) 0.0103 (b) $E(X) = 80; \sigma_X = 26.833$
23.2 0.0307
23.3 (a) X is negative binomial distribution with $r = 3$ and $p = \frac{4}{52} = \frac{1}{13}$. So $p(n) = {}_{k-1}C_2 \left(\frac{1}{13}\right)^3 \left(\frac{12}{13}\right)^{n-3}$ (b) 0.01793
23.4 $E(X) = 24$ and $\text{Var}(X) = 120$
23.5 0.109375
23.6 0.1875
23.7 0.2898
23.8 0.022
23.9 (a) 0.1198 (b) 0.0254
23.10 $E(X) = \frac{r}{p} = 20$ and $\sigma_X = \sqrt{\frac{r(1-p}{p^2}} = 13.416$
23.11 0.0645
23.12 ${}_{n-1}C_2 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{n-3}$
23.13 3
23.14 0.0344
23.15 $E(Y) = 6.6667,\ \text{Var}(Y) = 11.1111$
23.16 0.049
23.17 $E(X) = 15,\ \text{Var}(X) = 60$
23.18 $\frac{(pe^t)^r}{[1-(1-p)e^t]^r}$
23.19 We have $E(X) = E(Xe^{tX})\big|_{t=0} = M'_X(0)$. But

$$M'_X(t) = r(pe^t)^r[1-(1-p)e^t]^{-r} + r(pe^t)^r[1-(1-p)e^t]^{-r-1}(1-p)e^t.$$

Thus,

$$E(X) = M'_X(0) = rp^r p^{-r} + rp^r p^{-r-1}(1-p) = \frac{r}{p}$$

23.20 We have $\text{Var}(X) = M''_X(0) - [M'_X(0)]^2$. The second derivative of $M_X(t)$ is

$$\begin{aligned} M''_X(t) =& r^2(pe^t)^r[1-(1-p)e^t]^{-r} + r^2(pe^t)^r[1-(1-p)e^t]^{-r-1}(1-p)e^t \\ &+ r^2(pe^t)^r[1-(1-p)e^t]^{-r-1}(1-p)e^t + r(pe^t)^r[1-(1-p)e^t]^{-r-1}(1-p)e^t \\ &+ r(r+1)(pe^t)^r[1-(1-p)e^t]^{-r-2}(1-p)^2e^{2t}. \end{aligned}$$

Hence,

$$\begin{aligned} M_X''(0) =& r^2p^rp^{-r} + 2r^2p^rp^{-r-1}(1-p) + rp^rp^{-r-1}(1-p) \\ &+r(r+1)p^rp^{-r-2}(1-p)^2 \\ =& r^2 + \frac{2r^2(1-p)}{p} + \frac{r(1-p)}{p} + r(r+1)\frac{(1-p)^2}{p^2} \\ =& rp^{-2}(r+1-p). \end{aligned}$$

Hence,

$$\text{Var}(X) = rp^{-2}(r+1-p) - r^2p^{-2} = \frac{r(1-p)}{p^2}.$$

23.21 0.1477

Section 24

24.1 0.32513
24.2 0.1988
24.3 We have

k	0	1	2	3	4	5	6
$P(X=k)$	0.468	0.401	0.117	0.014	7.06×10^{-4}	1.22×10^{-5}	4.36×10^{-8}

24.4 0.247678
24.5 0.073
24.6 (a) 0.214 (b) $E(X) = 3$ and $\text{Var}(X) = 0.429$
24.7 0.793
24.8 $\frac{{}_{2477}C_3 \cdot {}_{121373}C_{97}}{{}_{123850}C_{100}}$
24.9 0.033
24.10 0.2880
24.11 0.375
24.12 0.956
24.13 0.028
24.14 $p(k) = P(X=k) = \frac{({}_5C_k)({}_{45}C_{4-k})}{{}_{50}C_4},\ k = 0,1,2,3,4$ and 0 otherwise
24.15 (a) 7.574×10^{-5} (b) 0.9999
24.16 0.63
24.17 $E(X) = 2,\ \sigma_X = 1$
24.18 0.45596

24.19 0.3333
24.20 0.2545
24.21 0.39

Section 25

25.1 (a)

$$\begin{aligned} P(2) =& \frac{C(2,2)}{C(5,2)} = 0.1 \\ P(6) =& \frac{C(2,1)C(2,1)}{C(5,2)} = 0.4 \\ P(10) =& \frac{C(2,2)}{C(5,2)} = 0.1 \\ P(11) =& \frac{C(1,1)C(2,1)}{C(5,2)} = 0.2 \\ P(15) =& \frac{C(1,1)C(2,1)}{C(5,2)} = 0.2 \end{aligned}$$

(b)

$$F(x) = \begin{cases} 0 & x < 2 \\ 0.1 & 2 \leq x < 6 \\ 0.5 & 6 \leq x < 10 \\ 0.6 & 10 \leq x < 11 \\ 0.8 & 11 \leq x < 15 \\ 1 & x \geq 15. \end{cases}$$

(c) 8.8
25.2

x	$(-\infty, 0)$	$[0, 1)$	$[1, 2)$	$[2, 3)$	$[3, \infty)$
$P(X \leq x)$	0	0.495	0.909	0.996	1

25.3 (a)

$$\begin{aligned}
P(X=1) =& P(X \leq 1) - P(X < 1) = F(1) - F(1^-) \\
=& \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
P(X=2) =& P(X \leq 2) - P(X < 2) = F(2) - F(2^-) \\
=& \frac{11}{12} - \left(\frac{1}{2} - \frac{2-1}{4}\right) = \frac{1}{6} \\
P(X=3) =& P(X \leq 3) - P(X < 3) = F(3) - F(3^-) \\
=& 1 - \frac{11}{12} = \frac{1}{12}
\end{aligned}$$

(b) 0.5

25.4

$$\begin{aligned}
P(X=0) =& F(1) = \frac{1}{2} \\
P(X=1) =& F(1) - F(1^-) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \\
P(X=2) =& F(2) - F(2^-) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \\
P(X=3) =& F(3) - F(3^-) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \\
P(X=3.5) =& F(3.5) - F(3.5^-) = 1 - \frac{9}{10} = \frac{1}{10}
\end{aligned}$$

and 0 otherwise.

25.5 (a)

$$P(x) = \begin{cases} 0.1 & x = -2 \\ 0.2 & x = 1.1 \\ 0.3 & x = 2 \\ 0.4 & x = 3 \\ 0 & \text{otherwise.} \end{cases}$$

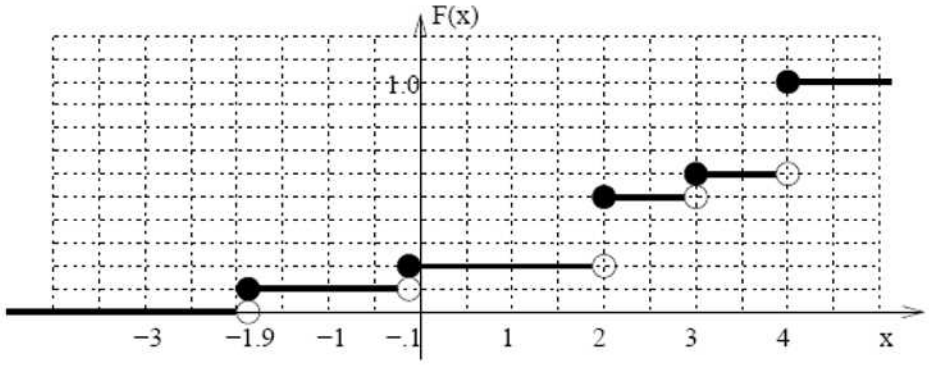
(b) 0 (c) 0.4 (d) 0.444

25.6 (a) 0.1

(b)

$$F(x) = \begin{cases} 0 & x < -1.9 \\ 0.1 & -1.9 \leq x < -0.1 \\ 0.2 & -0.1 \leq x < 2 \\ 0.5 & 2 \leq x < 3 \\ 0.6 & 3 \leq x < 4 \\ 1 & x \geq 4. \end{cases}$$

The graph of $F(x)$ is shown below.



(c) $F(0) = 0.2; F(2) = 0.5; F(F(3.1)) = 0.2$. (d) 0.5 (e) 0.64

25.7 (a)

$$P(x) = \begin{cases} 0.3 & x = -4 \\ 0.4 & x = 1 \\ 0.3 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

(b)$E(X) = 0.4$, $\text{Var}(X) = 9.84$, and $\sigma_X = 3.137$

25.8 (a)

$$P(x) = \begin{cases} \frac{1}{12} & x = 1, 3, 5, 8, 10, 12 \\ \frac{2}{12} & x = 2, 4, 6 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{12} & 1 \le x < 2 \\ \frac{3}{12} & 2 \le x < 3 \\ \frac{4}{12} & 3 \le x < 4 \\ \frac{6}{12} & 4 \le x < 5 \\ \frac{7}{12} & 5 \le x < 6 \\ \frac{9}{12} & 6 \le x < 8 \\ \frac{10}{12} & 8 \le x < 10 \\ \frac{11}{12} & 10 \le x < 12 \\ 1 & x \ge 12. \end{cases}$$

(c) $P(X < 4) = 0.333$. This is not the same as $F(4)$ which is the probability that $X \le 4$. The difference between them is the probability that X is EQUAL to 4.

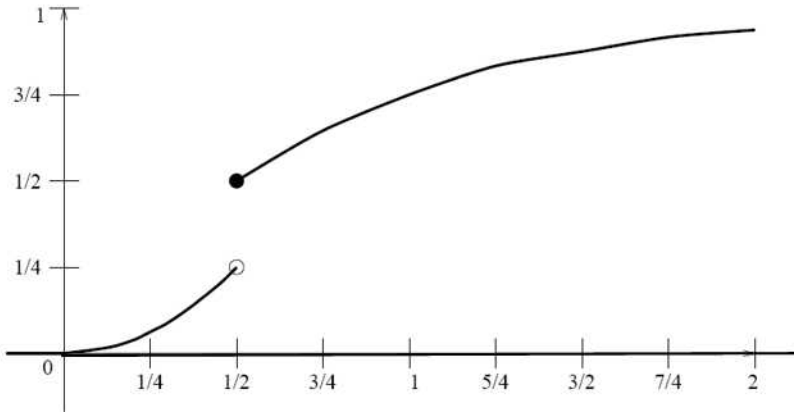
25.9 (a) We have

$$\begin{aligned} P(X = 0) &= (\text{jump in } F(x) \text{ at } x = 0) = 0 \\ P(X = 1) &= (\text{jump in } F(x) \text{ at } x = 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ P(X = 2) &= (\text{jump in } F(x) \text{ at } x = 2) = 1 - \frac{3}{4} = \frac{1}{4} \end{aligned}$$

(b) $\frac{7}{16}$ (c) $\frac{3}{16}$ (d) $\frac{3}{8}$

25.10 (a) 0.125 (b) 0.584 (c) 0.5 (d) 0.25

(e)



25.11 (a) 1/4 (b) 5/8 (c) 1/2

25.12 -1.9282
25.13 No. $F(x)$ is not right-continuous at 0
25.14 $F(\sqrt{a}) - F(\sqrt{a^-})$
25.15

$$\begin{aligned} p(0) &= F(0) - F(0^-) = 0.1 \\ p(1) &= F(1) - F(1^-) = 0.2 \\ p(2) &= F(2) - F(2^-) = 0.2 \\ p(4) &= F(4) - F(4^-) = 0.3 \\ p(5) &= F(5) - F(5^-) = 0.2 \end{aligned}$$

and 0 otherwise
25.16 348
25.17 0.10

Section 26

26.1 (a) We have

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{1}{10}(100 - x)^{\frac{1}{2}}, & 0 \leq x \leq 100 \\ 1, & x > 100. \end{cases}$$

(b) 0.092
26.2 (a) 0.3 (b) 0.3
26.3 Using the fundamental theorem of calculus, we have

$$S'(x) = \left(\int_x^\infty f(t)dt \right)' = \left(- \int_\infty^x f(t)dt \right)' = -f(x)$$

26.4

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \lambda e^{-\lambda x}, & x > 0. \end{cases}$$

26.5

$$S(x) = \begin{cases} 1, & x \leq 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

26.6 $a = -\frac{1}{\omega^2},\ b = 1$
26.7 (a) S satisfies the properties of a survival function (b)

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{1}{10}(100 - x)^{\frac{1}{2}}, & 0 \leq x \leq 100 \\ 1 & x > 100. \end{cases}$$

(c) 0.092
26.8 0.9618
26.9 0.033
26.10 We have
• $S(0) = 1$.
• $S'(x) = -0.34e^{-0.34x} < 0$ for $x \geq 0$.
• $\lim_{x\to\infty} S(x) = 0$.
Hence, $S(x)$ can be a candidate for a survival model
26.11 0.149
26.12

$$F(x) = 1 - S(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{100}, & 0 \leq x \leq 10. \\ 0, & x > 10. \end{cases}$$

26.13 (I) Yes (II) No (III) NO
26.14

$$S(x) = \begin{cases} 1, & x < 0 \\ 1 - \frac{x}{108}, & 0 < x < 108 \\ 0, & x \geq 108. \end{cases}$$

26.15

$$S(x) = \begin{cases} 1, & x < 0 \\ (x+1)e^{-x}, & x \geq 0 \end{cases}$$

26.16 We have

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-0.34x}, & x \geq 0. \end{cases}$$

and

$$f(x) = 0.34e^{-0.34x},\ x \geq 0$$

Section 27

27.1 Divergent

27.2 Divergent
27.3 Divergent
27.4 6
27.5 $\frac{\pi}{2}$
27.6 $\frac{\pi}{4}$
27.7 1
27.8 Divergent
27.9 Divergent
27.10 Divergent
27.11 2π
27.12 1
27.13 $\frac{2}{3}$
27.14 Divergent
27.15 Divergent
27.16 Divergent
27.17 convergent

Section 28

28.1 2
28.2 (a) 0.135 (b) 0.233
(c)

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

28.3 $k = 0.003$, 0.027
28.4 0.938
28.5 (b)

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 < x < 1 \\ 1/6 & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

28.6 (a) 1 (b) 0.736
28.7(a)

$$f(x) = F'(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{e^x}{(e^x+1)^2} & x > 0. \end{cases}$$

(b) 0.231
28.8 About 4 gallons

28.9 $\frac{1}{9}$
28.10 0.469
28.11 0.132
28.12 0.578
28.13 0.3
28.14 2
28.15 1.867
28.16

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{2.5}{y^2} & \frac{10}{12} \leq y \leq \frac{10}{8} \\ 0 & \text{otherwise} \end{cases}$$

28.17 0.5
28.18 $\int_{-\infty}^{\infty} xf(x)dx = 2.4$ and $\int_{-\infty}^{\infty} x^2 f(x)dx \approx 7.4667$
28.19 $g(y) = \frac{4}{y^2}$ for $y > 4$ and 0 otherwise
28.20 $g(y) = \frac{1}{2} f\left(\frac{y}{2}\right)$
28.21 0.2572
28.22 0.42757
28.23 8.99
28.23 $e^{-\lambda k}(e^{\lambda} - 1)$

Section 29

29.1 (a) 1.2
(b) The cdf is given by

$$F(x) = \begin{cases} 0 & x \leq -1 \\ 0.2 + 0.2x & -1 < x \leq 0 \\ 0.2 + 0.2x + 0.6x^2 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

(c) 0.25 (d) 0.4
29.2 (a) $a = \frac{3}{5}$ and $b = \frac{6}{5}$.
(b)

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} \int_{-\infty}^{x} 0du = 0 & x < 0 \\ \int_{-\infty}^{x} \frac{1}{5}(3 + 5u^2)du = \frac{3}{5}x + \frac{2}{5}x^3 & 0 \leq x \leq 1 \\ \int_{0}^{1} \frac{1}{5}(3 + 6u^2)du = 1 & x > 1 \end{cases}$$

29.3 (a) 4 (b) 0 (c) ∞
29.4 $E(X) = \frac{2}{3}$ and $\text{Var}(X) = \frac{2}{9}$

29.5 (a) $E(X) = \frac{1}{3}$ and $\text{Var}(X) = \frac{2}{9}$. $\sigma_X = \approx 0.471$ (b) 0.938
29.6 0.5
29.7 .06
29.8 $E(Y) = \frac{7}{3}$ and $\text{Var}(Y) = 0.756$
29.9 1.867
29.10 0.5
29.11 0.139
29.12 6
29.13 $2 + 3e^{-\frac{2}{3}}$
29.14 $\frac{2}{3}$
29.15 1.9
29.16 0.9343
29.17 5644
29.18 500
29.19 5000
29.20 \$328
29.21 374.40
29.22 3
29.23 1.25
29.24 6

Section 30

30.1 Median = 1; 70^{th}percentile =2
30.2 0.693
30.3 $a + 2\sqrt{\ln 2}$
30.4 $3 \ln 2$
30.5 0.8409
30.6 3
30.7 2
30.8 $-\ln(1-p)$
30.9 $M = \ln(2p)$ for $0 < p \leq 0.5$ and $M = -\ln[2(1-p)]$ for $0.5 < p < 1$
30.10 2
30.11 0.4472
30.12 6299.61
30.13 2.3811
30.14 2.71
30.15 998.72

30.16 1.7726
30.17 250
30.18 0.42
30.19 173
30.20 3
30.21 4
30.22 1.26
30.23 0.71
30.24 3659
30.25 2

Section 31

31.1 (a) The pdf is given by

$$f(x) = \begin{cases} \frac{1}{4} & 3 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

(b) 0 (c) 0.5
31.2 (a) The pdf is given by

$$f(x) = \begin{cases} \frac{1}{10} & 5 \leq x \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

(b) 0.3 (c) $E(X) = 10$ and $\text{Var}(X) = 8.33$
31.3 (a)

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(b) $P(a \leq X \leq a+b) = F(a+b) - F(a) = a + b - a = b$
31.4 $\frac{1}{n+1}$
31.5 0.693
31.6 0.667
31.7 500
31.8 1.707
31.9 403.44
31.10 (a) $\frac{e^2-1}{2e}$ (b) $\frac{1}{2} - \frac{1}{2e^2}$
31.11 3

31.12 $\frac{7}{10}$
31.13 $25(\ln(0.0001v) - 0.04)$
31.14 $\frac{5}{2r^2}$
31.15

$$F(r) = \begin{cases} 0 & r < 0 \\ 0.2r & 0 \leq x < 4 \\ 1 & r \geq 4 \end{cases}$$

31.16 0.18
31.17 0.9734
31.18

$$f(x) = \begin{cases} 0.04 & 0 \leq x \leq 25 \\ 0 & \text{otherwise.} \end{cases}$$

$E(X) = 12.5,\ \mathrm{Var}(X) = 52.08.$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{25} & 0 \leq x \leq 25 \\ 1 & x > 25 \end{cases}$$

31.19 $\frac{e^{tb} - e^{ta}}{t(b-a)}$
31.20 16.4
31.21 33.3
31.22 450
31.23 60
31.24 $\frac{5}{16}$
31.25 $\frac{123{,}125}{3}$
31.26 $\frac{56}{225}$

Section 32

32.1 (a) 0.2389 (b) 0.1423 (c) 0.6188 (d) 88
32.2 (a) 0.7517 (b) 0.8926 (c) 0.0238
32.3 (a) 0.5 (b) 0.9876
32.4 (a) 0.4772 (b) 0.004
32.5 0.0228
32.6 (a) 0.9452 (b) 0.8185
32.7 $\frac{\Phi(5) - \Phi(2)}{\Phi(5)}$
32.8 75
32.9 0.4721 (b) 0.1389 (c) 0.6664 (d) 0.58

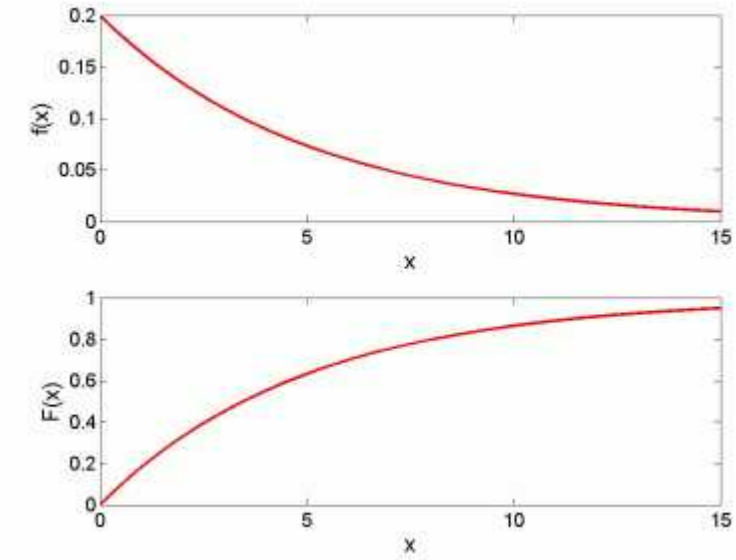
32.10 0.86
32.11 (a) 0.1056 (b) 362.84
32.12 1:18pm
32.13 0.9
32.14 90th percentile
32.15 7.68
32.16 23.6
32.17 0.98
32.18 2045
32.19 0.5478
32.20 24.108
32.21 37,000,000
32.22 27,700
32.23 872

Section 33

33.1 (a) 0.2578 (b) 0.832
33.2 0.1788
33.3 (a) 0.8281 (b) 0.0021
33.4 0.0158
33.5 0.9854
33.6 0.97
33.7 23

Section 34

34.1 0.593
34.2



34.3 0.393
34.4 0.1175
34.5 0.549
34.6 (a) 0.189 (b) 0.250
34.7 (a) 0.1353 (b) 0.167 (c) 0.05
34.8 0.134
34.9 0.186
34.10 0.435
34.11 10256
34.12 3.540
34.13 5644.23
34.14 173.29
34.15 0.420
34.16 $\frac{4}{(\ln 2)^2}$
34.17 0.727
34.18 $f(x) = -0.25 \ln (0.30)(0.3)^{\frac{x}{4}}$ for $x > 0$ and 0 otherwise
34.19 0.205
34.20 2.16
34.21 72.36
34.22 0.1352
34.23 1.77
34.24 90.1

34.25 $100[2e^{-0.1d} - e^{-0.2d}]$
34.26 $2e^{-\frac{1}{2}} - 2e^{-3}$
34.27 0.731
34.28 25
34.29 0.55

Section 35

35.1 We have

$$\begin{aligned} F_Y(y) =& P(Y \leq y) = P\left(X \leq \frac{y}{c}\right) \\ =& \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\frac{y}{c}} t^{\alpha-1} e^{-\lambda t} dt \\ =& \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)} \int_0^{y} z^{\alpha-1} e^{-\lambda \frac{z}{c}} dz. \end{aligned}$$

35.2 $E(X) = 1.5$ and $\text{Var}(X) = 0.75$
35.3 0.0948
35.4 0.014
35.5 480
35.6 For $t \geq 0$ we have

$$F_{X^2}(t) = P(X^2 \leq t) = P(-\sqrt{t} < X < \sqrt{t}) = \Phi(\sqrt{t}) - \Phi(-\sqrt{t})$$

Now, taking the derivative (and using the chain rule) we find

$$\begin{aligned} f_{X^2}(t) =& \frac{1}{2\sqrt{t}} \Phi'(\sqrt{t}) + \frac{1}{2\sqrt{t}} \Phi'(-\sqrt{t}) \\ =& \frac{1}{\sqrt{t}} \Phi'(\sqrt{t}) = \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{t}{2}} \end{aligned}$$

which is the density function of gamma distribution with $\alpha = \lambda = \frac{1}{2}$
35.7 $E(e^{tX}) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \ t < \lambda$
35.8 We have

$$f'(x) = -\frac{\lambda^2 e^{-\lambda x} (\lambda x)^{\alpha-2}}{\Gamma(\alpha)} (\lambda x - \alpha + 1).$$

Thus, the only critical point of $f(x)$ is $x = \frac{1}{\lambda}(\alpha - 1)$. One can easily show that $f''\left(\frac{1}{\lambda}(\alpha - 1)\right) < 0$

35.9 (a) The density function is

$$f(x) = \begin{cases} \frac{x^2}{432} e^{-\frac{x}{6}} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$E(X) = \frac{\alpha}{\lambda} = 18,\ \sigma_X = \sqrt{\frac{\alpha}{\lambda^2}} \approx 10.39$
(b) $E(3X^2 + X + 1) = 3E(X^2) + E(X) - 1 = 1313$
35.10 5,000
35.11 2.12
35.12 $\alpha = 4$ and $\lambda = 0.5$.
35.13 The density function is

$$f(x) = \begin{cases} \frac{2^{-\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{n}{2}\right)} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

The expected value is $E(X) = n$ and the variance is $\text{Var}(X) = 2n$.
35.14 $\alpha = 6, \lambda = \frac{1}{2},\ n = 12$.
35.15 $\frac{4464}{\lambda^6}$
35.16 The cumulative distribution function of X is $F_X(x) = 1 - e^{-x}(1 + x)$ for $x > 0$ and 0 otherwise. Thus,

$$F_Y(y) = P(Y \leq y) = P(X \leq \ln y) = 1 - e^{-\ln y}(1 + \ln y) = 1 - \frac{1 + \ln y}{y}$$

for $y > 1$ and 0 otherwise. The pdf of Y is $f_Y(y) = F'_Y(y) = \frac{\ln y}{y^2}$ for $y > 1$ and 0 otherwise

Section 36

36.1 $f_Y(y) = \frac{1}{|a|\sqrt{2\pi}} e^{-\frac{(\frac{y-b}{a} - \mu)^2}{2}},\ a \neq 0$
36.2 $f_Y(y) = \frac{2(y+1)}{9}$ for $-1 \leq y \leq 2$ and $f_Y(y) = 0$ otherwise
36.3 For $0 \leq y \leq 8$ we have $f_Y(y) = \frac{y^{-\frac{1}{3}}}{6}$ and 0 otherwise
36.4 For $y \geq 1$ we have $f_Y(y) = \lambda y^{-\lambda - 1}$ and 0 otherwise
36.5 For $y > 0$ we have $f_Y(y) = \frac{c}{m}\sqrt{\frac{2y}{m}} e^{-\frac{2\beta y}{m}}$ and 0 otherwise
36.6 For $y > 0$ we have $f_Y(y) = e^{-y}$ and 0 otherwise
36.7 For $-1 < y < 1$ we have $f_Y(y) = \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}}$ and 0 otherwise
36.8 (a) For $0 < y < 1$ we have $f_Y(y) = \frac{1}{\alpha} y^{\frac{1-\alpha}{\alpha}}$ and 0 otherwise, $E(Y) = \frac{1}{\alpha+1}$

(b) For $y < 0$ we have $f_Y(y) = e^y$ and 0 otherwise, $E(Y) = -1$
(c) For $1 < y < e$ we have $f_Y(y) = \frac{1}{y}$ and 0 otherwise, $E(Y) = \int_1^e dy = e - 1$.
(d) For $0 < y < 1$ we have $f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$, $E(Y) = \frac{2}{\pi}$ and 0 otherwise.

36.9 For $y > 4$ $f_Y(y) = 4y^{-2}$ and 0 otherwise

36.10 For $10,000e^{0.04} < v < 10,000e^{0.08}$ $F_V(v) = F_R(g^{-1}(v)) = 25\left(\ln\left(\frac{v}{10,000}\right) - 0.04\right)$ and $F_V(v) = 0$ for $v \leq 10,000e^{0.04}$ and $F_V(v) = 1$ for $v \geq 10,000e^{0.08}$

36.11 For $y > 0$ we have $f_Y(y) = \frac{1}{8}\left(\frac{y}{10}\right)^{\frac{1}{4}} e^{-\left(\frac{y}{10}\right)^{\frac{5}{4}}}$ and 0 otherwise

36.12 $f_R(r) = \frac{5}{2r^2}$ for $\frac{5}{6} < r < \frac{5}{4}$ and 0 otherwise

36.13 $f_Y(y) = \frac{1}{2}f_X\left(\frac{y}{2}\right)$ where X and Y are the monthly profits of Company A and Company B, respectively.

36.14 (a) 0.341 (b) $f_Y(y) = \frac{1}{2y\sqrt{2\pi}} exp\left(-\frac{1}{2\cdot 2^2}(\ln y - 1)^2\right)$ for $y > 0$ and 0 otherwise.

36.15 The cdf is given by

$$F_Y(a) = \begin{cases} 0 & a \leq 0 \\ \sqrt{a} & 0 < a < 1 \\ 1 & a \geq 1 \end{cases}$$

Thus, the density function of Y is

$$f_Y(a) = F_Y'(a) = \begin{cases} \frac{1}{2}a^{-\frac{1}{2}} & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

36.16 (a)

$$f_Y(a) = \begin{cases} \frac{a^2}{18} & -3 < a < 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Z(a) = \begin{cases} \frac{3}{2}(3-a)^2 & 2 < a < 4 \\ 0 & \text{otherwise} \end{cases}$$

36.17 $f_Y(y) = \frac{1}{\sqrt{y}} - 1,\ 0 < y \leq 1$

36.18 $f_Y(y) = e^{2y - \frac{1}{2}e^{2y}}$

36.19 (a) $f_Y(y) = \frac{8-y}{50}$ for $-2 \leq y \leq 8$ and 0 elsewhere.
(b) We have $E(Y) = \frac{4}{3}$ (c) $\frac{9}{25}$

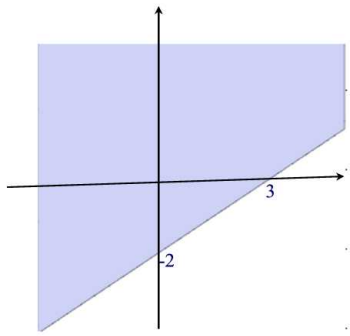
36.20 $f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$

36.21 $\frac{e^{-2\sqrt{t}} + 2e^{-4\sqrt{t}}}{2\sqrt{t}}$

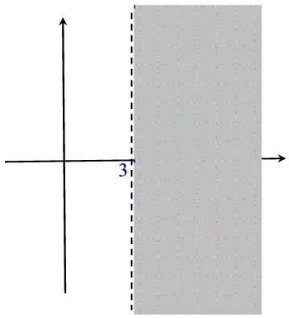
36.22 0.561

Section 37

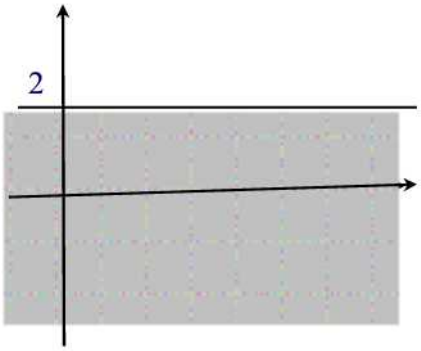
37.1



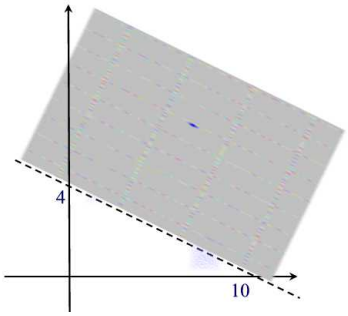
37.2



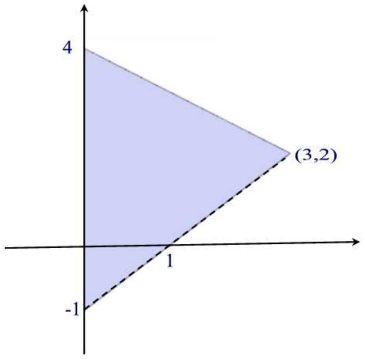
37.3



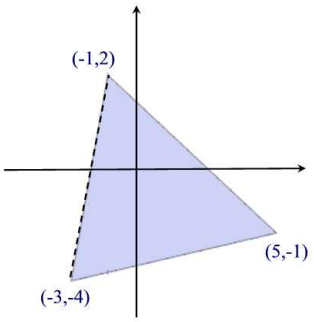
37.4



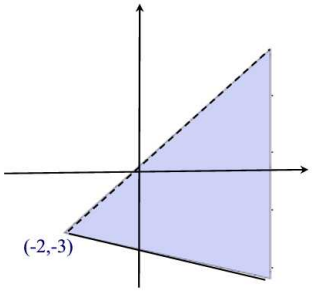
37.5



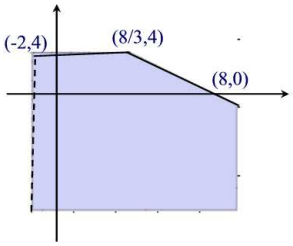
37.6



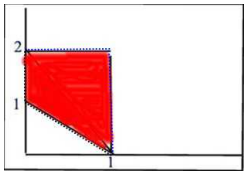
37.7



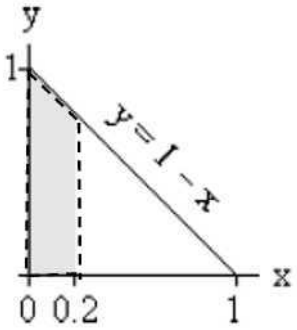
37.8



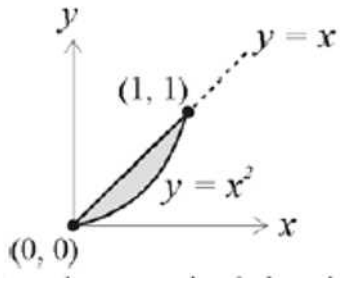
37.9



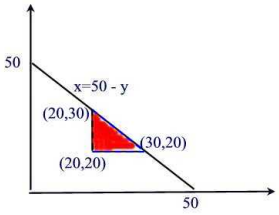
37.10



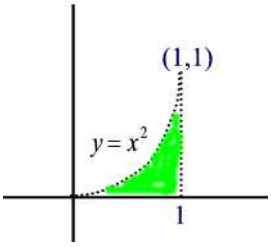
37.11



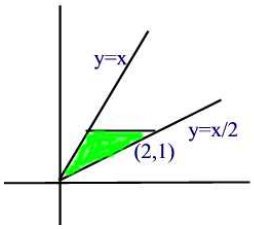
37.12



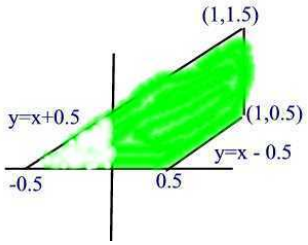
37.13



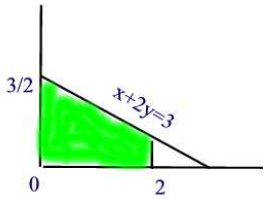
37.14



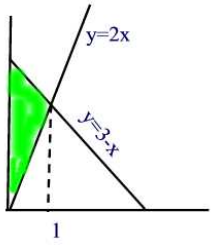
37.15



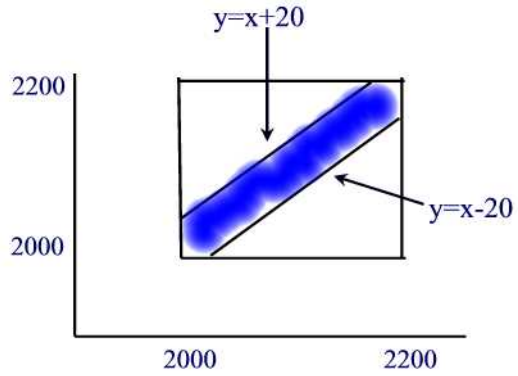
23.16



37.17



37.18



Section 38

38.1 $\int_0^1 \int_x^{x+1} f(x,y)dydx$
38.2

$$\int_0^1 \int_0^{\frac{x}{2}} f(x,y)dydx \text{ or } \int_0^{\frac{1}{2}} \int_{2y}^1 f(x,y)dxdy$$

38.3 $\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 f(x,y)dydx$
38.4 $\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(x,y)dydx + \int_{\frac{1}{2}}^1 \int_0^1 f(x,y)dydx$
38.5 $\int_{20}^{30} \int_{20}^{50-x} f(x,y)dydx$
38.6 $\int_0^1 \int_0^{x+1} f(x,y)dydx + \int_1^\infty \int_{x-1}^{x+1} f(x,y)dydx$
38.7 $1 - 2e^{-1}$
38.8 $\frac{1}{6}$
38.9 $\frac{L^4}{3}$
38.10 $\int_{0.5}^1 \int_{0.5}^1 f(x,y)dydx$
38.11 $\frac{7}{8}$
38.12 $\frac{3}{8}$
38.13 0.0423
38.14 $(2 + \alpha - \alpha^2)^{-1}$
38.15 0.375
38.16 0.708
38.17 0.488
38.18 0.008
38.19 0.35

38.20 0.375
38.21 0.6035
38.22 0.5
38.23 0.3301
38.24 0.0625
38.25 0.875
38.26 0.7619
38.27 0.340
38.28 0.6667
38.29 0.4

Section 39

39.1 (a) From the table we see that the sum of all the entries is 1.
(b) 0.25 (c) 0.55 (d) $p_X(0) = 0.3, p_X(1) = 0.5, p_X(2) = 0.125, p_X(3) = 0.075$ and 0 otherwise
39.2 0.25
39.3 (a) 0.08 (b) 0.36 (c) 0.86 (d) 0.35 (e) 0.6 (f) 0.65 (g) 0.4
39.4 (a) 0.4 (b) 0.8
39.5 0.043
39.6 (a) The cdf od X and Y is

$$\begin{aligned} F_{XY}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)dudv \\ &= \left(\int_0^x ue^{-\frac{u^2}{2}}du\right)\left(\int_0^y ve^{-\frac{v^2}{2}}du\right) \\ &= (1-e^{-\frac{x^2}{2}})(1-e^{-\frac{y^2}{2}}),\ x>0, y>0 \end{aligned}$$

and 0 otherwise.
(b) The marginal pdf for X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^{\infty} xye^{-\frac{x^2+y^2}{2}}dy = xe^{-\frac{x^2}{2}},\ x>0$$

and 0 otherwise. The marginal pdf for Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{\infty} xye^{-\frac{x^2+y^2}{2}}dx = ye^{-\frac{y^2}{2}}$$

for $y > 0$ and 0 otherwise

39.7 We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^a y^{1-a}dxdy = \int_0^1\int_0^1 x^a y^{1-a}dxdy$$
$$=(2+a-a^2)^{-1} \neq 1$$

so $f_{XY}(x,y)$ is not a density function. However one can easily turn it into a density function by multiplying $f(x,y)$ by $(2+a-a^2)$ to obtain the density function

$$f_{XY}(x,y) = \begin{cases} (2+a-a^2)x^a y^{1-a} & 0 \leq x,y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

39.8 0.625

39.9 0.708

39.10 0.576

39.11 0.488

39.12 $f_Y(y) = \int_y^{\sqrt{y}} 15ydx = 15y^{\frac{3}{2}}(1-y^{\frac{1}{2}}), \quad 0 < y < 1$ and 0 otherwise

39.13 5.778

39.14 0.83

39.15 0.008

39.16 $\frac{7}{20}$

39.17 $1-2e^{-1}$

39.18 $\frac{12}{25}$

39.19 $\frac{3}{8}$

39.20 (a) We have

X\ Y	1	2	$p_X(x)$
1	0.2	0.5	0.7
2	0.2	0.1	0.3
$p_Y(y)$	0.4	0.6	1

(b) We have

$$F_{XY}(x,y) = \begin{cases} 0 & x < 1 \text{ or } y < 1 \\ 0.2 & 1 \leq x < 2 \text{ and } 1 \leq y < 2 \\ 0.7 & 1 \leq x < 2 \text{ and } y \geq 2 \\ 0.4 & x \geq 2 \text{ and } 1 \leq y < 2 \\ 1 & x \geq 2 \text{ and } y \geq 2 \end{cases}$$

39.21 $\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(s,t)dsdt + \int_0^1 \int_0^{\frac{1}{2}} f(s,t)dsdt$
39.22 $\frac{1}{800} \int_0^{60} \int_0^{60-x} e^{-\frac{x}{40}-\frac{y}{20}} dydx$
39.23 0.049
39.24 $\frac{23}{49}$
39.25 $\frac{5}{12}$
39.26 764.2811
39.27 2
39.28 2/5
39.29 2.41
39.30 0.0004
39.31 0.099

Section 40

40.1 (a) Yes (b) 0.5 (c) $1 - e^{-a}$
40.2 (a) The joint density over the region R must integrate to 1, so we have

$$1 = \int \int_{(x,y)\in R} cdxdy = cA(R).$$

(b) Note that $A(R) = 4$ so that $f_{XY}(x,y) = \frac{1}{4} = \frac{1}{2}\frac{1}{2}$. Hence, by Theorem 40.2, X and Y are independent with each distributed uniformly over $(-1,1)$.
(c) $P(X^2 + Y^2 \leq 1) = \int \int_{x^2+y^2\leq 1} \frac{1}{4} dxdy = \frac{\pi}{4}$
40.3 (a) 0.484375 (b) We have

$$f_X(x) = \int_x^1 6(1-y)dy = 6y - 3y^2\Big|_x^1 = 3x^2 - 6x + 3,\ 0 \leq x \leq 1$$

and 0 otherwise. Similarly,

$$f_Y(y) = \int_0^y 6(1-y)dy = 6\left[y - \frac{y^2}{2}\right]_0^y = 6y(1-y),\ 0 \leq y \leq 1$$

and 0 otherwise.
(c) X and Y are dependent
40.4 (a) $k = 4$ (b) We have

$$f_X(x) = \int_0^1 4xydy = 2x, \quad 0 \leq x \leq 1$$

and 0 otherwise. Similarly,

$$f_Y(y) = \int_0^1 4xydx = 2y, \quad 0 \le y \le 1$$

and 0 otherwise.
(c) Since $f_{XY}(x, y) = f_X(x)f_Y(y)$, X and Y are independent.
40.5 (a) $k = 6$ (b) We have

$$f_X(x) = \int_0^1 6xy^2dy = 2x, \quad 0 \le x \le 1, \quad 0 \text{ otherwise}$$

and

$$f_Y(y) = \int_0^1 6xy^2dy = 3y^2, \quad 0 \le y \le 1, \quad 0 \text{ otherwise}$$

(c) 0.15 (d) 0.875 (e) X and Y are independent
40.6 (a) $k = \frac{8}{7}$ (b) Yes (c) $\frac{16}{21}$
40.7 (a) We have

$$f_X(x) = \int_0^2 \frac{3x^2 + 2y}{24}dy = \frac{6x^2 + 4}{24}, 0 \le x \le 2, 0 \text{ otherwise}$$

and

$$f_Y(y) = \int_0^2 \frac{3x^2 + 2y}{24}dx = \frac{8 + 4y}{24}, 0 \le y \le 2, \quad 0 \text{ otherwise}$$

(b) X and Y are dependent. (c) 0.340
40.8 (a) We have

$$f_X(x) = \int_x^{3-x} \frac{4}{9}dy = \frac{4}{3} - \frac{8}{9}x, \quad 0 \le x \le \frac{3}{2}, \quad 0 \text{ otherwise}$$

and

$$f_Y(y) = \begin{cases} \frac{4}{9}y & 0 \le y \le \frac{3}{2} \\ \frac{4}{9}(3 - y) & \frac{3}{2} \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) $\frac{2}{3}$ (c) X and Y are dependent
40.9 0.469
40.10 0.191
40.11 0.4

40.12 0.19
40.13 0.295
40.14 0.414
40.15 $f(z) = e^{-\frac{1}{2}z} - e^{-z}, \;\; z > 0, \; 0$ otherwise
40.16 $f(x) = \frac{2}{(2x+1)^2}, \; x > 0, \;\; 0$ otherwise
40.17 $\frac{3}{5}$
40.18 $\theta_1 = \frac{1}{4}$ and $\theta_2 = 0$
40.19 $\frac{511}{512}$
40.20 0.2857
40.21 0.5
40.22 2025
40.23 0.0495
40.24 90

Section 41

41.1 We have

$$\begin{aligned}
P(Z=0) =& P(X=0)P(Y=0) = (0.1)(0.25) = 0.025 \\
P(Z=1) =& P(X=1)P(Y=0) + P(Y=1)P(X=0) \\
=& (0.2)(0.25) + (0.4)(0.1) = 0.09 \\
P(Z=2) =& P(X=1)P(Y=1) + P(X=2)P(Y=0) + P(Y=2)P(X=0) \\
=& (0.2)(0.4) + (0.3)(0.25) + (0.35)(0.1) = 0.19 \\
P(Z=3) =& P(X=2)P(Y=1) + P(Y=2)P(X=1) + P(X=3)P(Y=0) \\
=& (0.3)(0.4) + (0.35)(0.2) + (0.4)(0.25) = 0.29 \\
P(Z=4) =& P(X=2)P(Y=2) + P(X=3)P(Y=1) \\
=& (0.3)(0.35) + (0.4)(0.4) = 0.265 \\
P(Z=5) =& P(X=3)P(X=2) = (0.4)(0.35) = 0.14
\end{aligned}$$

and 0 otherwise
41.2 $p_{X+Y}(k) = \left(\begin{array}{c} 30 \\ k \end{array} \right) 0.2^k 0.8^{30-k}$ for $0 \leq k \leq 30$ and 0 otherwise.
41.3 $p_{X+Y}(n) = (n-1)p^2(1-p)^{n-2}, \;\; n = 2, \cdots$ and $p_{X+Y}(a) = 0$ otherwise.

41.4

$$\begin{aligned}
p_{X+Y}(3) =& p_X(0)p_Y(3) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}\\
p_{X+Y}(4) =& p_X(0)p_Y(4) + p_X(1)p_Y(3) = \frac{4}{12}\\
p_{X+Y}(5) =& p_X(1)p_Y(4) + p_X(2)p_Y(3) = \frac{4}{12}\\
p_{X+Y}(6) =& p_X(2)p_Y(4) = \frac{3}{12}
\end{aligned}$$

and 0 otherwise.

41.5 $\frac{1}{64}$

41.6 0.03368

41.7 $P(X+Y=2) = e^{-\lambda}p(1-p) + e^{-\lambda}\lambda p$ (b) $P(Y>X) = e^{-\lambda p}$

41.8

$$\begin{aligned}
p_{X+Y}(0) =& p_X(0)p_Y(0) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\\
p_{X+Y}(1) =& p_X(0)p_Y(1) + p_X(1)p_Y(0) = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4}\\
p_{X+Y}(2) =& p_X(0)p_Y(2) + p_X(2)p_Y(0) + p_X(1)p_Y(1) = \frac{5}{16}\\
p_{X+Y}(3) =& p_X(1)p_Y(2) + p_X(2)p_Y(1) = \frac{1}{8}\\
p_{X+Y}(4) =& p_X(2)p_Y(2) = \frac{1}{16}
\end{aligned}$$

and 0 otherwise.

41.9

$$\begin{aligned}
p_{X+Y}(1) =& p_X(0)p_Y(1) + p_X(1)p_Y(0) = \frac{1}{6}\\
p_{X+Y}(2) =& p_X(0)p_Y(2) + p_X(2)p_Y(0) + p_X(1)p_Y(1) = \frac{5}{18}\\
p_{X+Y}(3) =& p_X(0)p_Y(3) + p_X(1)p_Y(2) + p_X(2)p_Y(1) + p_X(3)p_Y(0) = \frac{6}{18}\\
p_{X+Y}(4) =& p_X(0)p_Y(4) + p_X(1)p_Y(3) + p_X(2)p_Y(2) + p_X(3)p_Y(1) + p_X(4)p_Y(0) = \frac{3}{18}\\
p_{X+Y}(5) =& p_X(0)p_Y(5) + p_X(1)p_Y(4) + p_X(2)p_Y(3) + p_X(3)p_Y(2)\\
&+ p_X(4)p_Y(1) + p_X(4)p_Y(1) = \frac{1}{18}
\end{aligned}$$

and 0 otherwise.

41.10 We have

$$p_{X+Y}(a) = \sum_{n=0}^{a} p(1-p)^n p(1-p)^{a-n} = (a+1)p^2(1-p)^a = {}_{a+1}C_a p^2(1-p)^a$$

Thus, $X+Y$ is a negative binomial with parameters $(2,p)$.

41.11 $9e^{-8}$

41.12 $e^{-10\lambda}\frac{(10\lambda)^{10}}{10!}$

41.13 0.185

41.14 0.1512

41.15 0.577

41.16 The range of Z is $0,1,2,\cdots,2n$. Write down the joint pmf of X and Y as a rectangular array or matrix with n rows (numbered $0,1,2,\cdots,n$ from left to right) and n columns (numbered $n, n-1,\cdots,1,0$ from top to bottom). This rectangular array will have $2n+1$ diagonals with the first diagonal at $X=0$ and $Y=0$ and the last diagonal at $X=n$ and $Y=n$. Then for $k=0,1,2,\cdots,2n,\ P(X+Y=k)$ is the sum of the entries on the k^{th} diagonal of this array. Note that the first diagonal corresponds to $k=0$ and the $2n+1$ diagonal corresponds to $k=2n$. In this case, we find

$$P(X+Y=k) = \begin{cases} \frac{k+1}{(n+1)^2} & 0 \le k \le n \\ \frac{2n-k+1}{(n+1)^2} & n+1 \le k \le 2n \end{cases}$$

41.17 $X+Y$ is a negative binomial distribution with parameters $r+s$ and p

41.18 $\frac{1}{z-1}$

41.19 (a) We have

$$\begin{aligned} E(XY) &= \sum_{x\in S_X}\sum_{y\in S_Y} xy p_{XY}(x,y) \\ &= \sum_{x\in S_X}\sum_{y\in S_Y} xy p_X(x) p_Y(y) \\ &= \left(\sum_{x\in S_X} x p_X(x)\right)\left(\sum_{y\in S_Y} y p_Y(y)\right) \\ &= E(X)E(Y). \end{aligned}$$

(b) We have

$$\begin{aligned}
E(X+Y) =& \sum_x \sum_y (x+y) p_{XY}(x,y) \\
=& \sum_x \sum_y x p_{XY}(x,y) + \sum_x \sum_y y p_{XY}(x,y) \\
=& \sum_x x \sum_y p_{XY}(x,y) + \sum_y y \sum_x p_{XY}(x,y) \\
=& \sum_x x p_X(x) + \sum_y y p_Y(y) \\
=& E(X) + E(Y).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\text{Var}(X+Y) =& E[(X+Y)^2] - [E(X+Y)]^2 \\
=& E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 \\
=& E(X^2) + 2E(XY) + E(Y^2) - [(E(X))^2 + 2E(X)E(Y) + (E(Y))^2] \\
=& E(X^2) + 2E(X)E(Y) + E(Y^2) - [(E(X))^2 + 2E(X)E(Y) + (E(Y))^2] \\
=& [E(X^2) - (E(X))^2] + [E(Y^2) - (E(Y))^2] \\
=& \text{Var}(X) + \text{Var}(Y)
\end{aligned}$$

41.20 We have

$$\begin{aligned}
E[e^{t(X+Y)}] &= \sum_{x \in S_X} \sum_{y \in S_Y} e^{t(x+y)} p_{XY}(x,y) \\
&= \sum_{x \in S_X} \sum_{y \in S_Y} e^{tx} e^{ty} p_X(x) p_Y(y) \\
&= \left(\sum_{x \in S_X} e^{tx} p_X(x) \right) \left(\sum_{y \in S_Y} e^{ty} p_Y(y) \right) \\
&= E(e^{tX}) E(e^{tY})
\end{aligned}$$

Section 42

42.1

$$f_{X+Y}(a) = \begin{cases} 2\lambda e^{-\lambda a}(1 - e^{-\lambda a}) & 0 \leq a \\ 0 & \text{otherwise} \end{cases}$$

42.2

$$f_{X+Y}(a) = \begin{cases} 1 - e^{-\lambda a} & 0 \leq a \leq 1 \\ e^{-\lambda a}(e^{\lambda} - 1) & a \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

42.3 $f_{X+2Y}(a) = \int_{-\infty}^{\infty} f_X(a-2y) f_Y(y) dy$
42.4 If $0 \leq a \leq 1$ then $f_{X+Y}(a) = 2a - \frac{3}{2}a^2 + \frac{a^3}{6}$. If $1 \leq a \leq 2$ then $f_{X+Y}(a) = \frac{7}{6} - \frac{a}{2}$. If $2 \leq a \leq 3$ then $f_{X+Y}(a) = \frac{9}{2} - \frac{9}{2}a + \frac{3}{2}a^2 - \frac{1}{6}a^3$. If $a > 3$ then $f_{X+Y}(a) = 0$.
42.5 If $0 \leq a \leq 1$ then $f_{X+Y}(a) = \frac{2}{3}a^3$. If $1 < a < 2$ then $f_{X+Y}(a) = -\frac{2}{3}a^3 + 4a - \frac{8}{3}$. If $a \geq 2$ then $f_{X+Y}(a) = 0$ and 0 otherwise.
42.6 $f_{X+Y}(a) = \frac{\alpha\beta}{\alpha-\beta}\left(e^{-\beta a} - e^{-\alpha a}\right)$ for $a > 0$ and 0 otherwise.
42.7 $f_W(a) = e^{-\frac{a}{2}} - e^{-a}$, $a > 0$ and 0 otherwise.
42.8 If $2 \leq a \leq 4$ then $f_{X+Y}(a) = \frac{a}{4} - \frac{1}{2}$. If $4 \leq a \leq 6$, then $f_{X+Y}(a) = \frac{3}{2} - \frac{a}{4}$ and $f_{X+Y}(a) = 0$ otherwise.
42.9 If $0 < a \leq 2$ then $f_{X+Y}(a) = \frac{a^2}{8}$. If $2 < a < 4$ then $f_{X+Y}(a) = -\frac{a^2}{8} + \frac{a}{2}$ and 0 otherwise.
42.10 $\frac{1}{8}$
42.11 $f_Z(z) = \int_0^z e^{-z} ds = ze^{-z}$ for $z > 0$ and 0 otherwise.
42.12 $1 - 2e^{-1}$
42.13 We have

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \left(\int_0^{\infty} e^{-u} u^{x-1} du\right)\left(\int_0^{\infty} e^{-v} v^{y-1} dv\right) \\
&= \int_0^{\infty}\int_0^{\infty} e^{-u-v} u^{x-1} v^{y-1} du dv \\
&= \int_0^{\infty}\int_0^{1} e^{-z}(zt)^{x-1}[z(1-t)]^{y-1}\left|\frac{\partial(u,v)}{\partial(z,t)}\right| dt dz \\
&= \int_0^{\infty}\int_0^{1} e^{-z}(zt)^{x-1}[z(1-t)]^{y-1} z dt dz \\
&= \left(\int_0^{\infty} e^{-z} z^{x+y-1} dz\right)\left(\int_0^{1} t^{x-1}(1-t)^{y-1} dt\right) \\
&= \Gamma(x+y)B(x,y)
\end{aligned}$$

where we used the substitution $u = zt$ and $v = z(1-t)$

42.14 We have

$$\begin{aligned} E(e^{t(X+Y)}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x+y)} f_{XY}(x,y)dxdy \\ &= \left(\int_{-\infty}^{\infty} e^{tx} f_X(x)dx\right)\left(\int_{-\infty}^{\infty} e^{ty} f_Y(y)dy\right) \\ &= E(e^{tX})E(e^{tY}) \end{aligned}$$

42.15 10,560

42.16 (a) We have

$$\begin{aligned} E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} f_{XY}(x,y)dy\right)dx + \int_{-\infty}^{\infty} y\left(\int_{-\infty}^{\infty} f_{XY}(x,y)dx\right)dy \\ &= \int_{-\infty}^{\infty} x f_X(x)dx + \int_{-\infty}^{\infty} y f_Y(y)dy \\ &= E(X) + E(Y). \end{aligned}$$

(b) We have

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} y f_Y(y)dy\right) f_X(x)dx \\ &= \left(\int_{-\infty}^{\infty} x f_X(x)dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)dy\right) \\ &= E(X)E(Y). \end{aligned}$$

(c) We have

$$\begin{aligned}\text{Var}(X+Y) =& E[(X+Y)^2] - [E(X+Y)]^2 \\ =& E(X^2+2XY+Y^2) - [E(X)+E(Y)]^2 \\ =& E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - E(Y)^2 - 2E(X)E(Y) \\ =& E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2E(X)E(Y) - 2E(X)E(Y) \\ =& \text{Var}(X) + \text{Var}(Y)\end{aligned}$$

42.17 200
42.18 te^{-t}
42.19 0.892

Section 43

43.1 $p_{X|Y}(0|1) = 0.25$ and $p_{X|Y}(1|1) = 0.75$ and 0 otherwise.
43.2 (a) For $1 \leq x \leq 5$ and $y = 1, \cdots, x$ we have $p_{XY}(x,y) = \left(\frac{1}{5}\right)\left(\frac{1}{x}\right)$ and 0 otherwise.
(b) $p_{X|Y}(x|y) = \frac{\frac{1}{5x}}{\sum_{k=y}^{5}\left(\frac{1}{5k}\right)}$ and 0 otherwise.
(c) X and Y are dependent
43.3

$$\begin{aligned}P(X=3|Y=4) =& \frac{P(X=3,Y=4)}{P(Y=4)} = \frac{0.10}{0.35} = \frac{2}{7} \\ P(X=4|Y=4) =& \frac{P(X=4,Y=4)}{P(Y=4)} = \frac{0.15}{0.35} = \frac{3}{7} \\ P(X=5|Y=4) =& \frac{P(X=5,Y=4)}{P(Y=4)} = \frac{0.10}{0.35} = \frac{2}{7}\end{aligned}$$

43.4

$$\begin{aligned}P(X=0|Y=1) =& \frac{P(X=0,Y=1)}{P(Y=1)} = \frac{1/16}{6/16} = \frac{1}{6} \\ P(X=1|Y=1) =& \frac{P(X=1,Y=1)}{P(Y=1)} = \frac{3/16}{6/16} = \frac{1}{2} \\ P(X=2|Y=1) =& \frac{P(X=2,Y=1)}{P(Y=1)} = \frac{2/16}{6/16} = \frac{1}{3} \\ P(X=3|Y=1) =& \frac{P(X=3,Y=1)}{P(Y=1)} = \frac{0/16}{6/16} = 0\end{aligned}$$

and 0 otherwise.

43.5

x	1	2	3	4	5	6
$p_{Y\|X}(1\|x))$	1	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$p_{Y\|X}(2\|x)$	0	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$p_{Y\|X}(3\|x)$	0	0	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$p_{Y\|X}(4\|x)$	0	0	0	$\frac{1}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$p_{Y\|X}(5\|x)$	0	0	0	0	$\frac{1}{9}$	$\frac{2}{11}$
$p_{Y\|X}(6\|x)$	0	0	0	0	0	$\frac{1}{11}$

and 0 otherwise. X and Y are dependent since $p_{Y|X}(1|1) = 1 \neq \frac{11}{36} = p_Y(1)$.

43.6 (a) $p_Y(y) = {}_nC_y p^y (1-p)^{n-y}$ and 0 otherwise. Thus, Y is a binomial distribution with parameters n and p.

(b) For $0 \leq y \leq n$, we have

$$\begin{aligned} p_{X|Y}(x|y) =& \frac{p_{XY}(x,y)}{p_Y(y)} \\ =& \frac{\frac{n! y^x (pe^{-1})^y (1-p)^{n-y}}{y!(n-y)!x!}}{{}_nC_y p^y (1-p)^{n-y}} \\ =& \frac{y^x e^{-y}}{x!} \end{aligned}$$

for $x = 0, 1, 2, \cdots$ and 0 otherwise. Thus, $X|Y = y$ is a Poisson distribution with parameter y.

X and Y are dependent.

43.7

$$p_{X|Y}(x|0) = \frac{p_{XY}(x,0)}{p_Y(0)} = \begin{cases} 1/11 & x = 0 \\ 4/11 & x = 1 \\ 6/11 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{X|Y}(x|1) = \frac{p_{XY}(x,1)}{p_Y(1)} = \begin{cases} 3/7 & x = 0 \\ 3/7 & x = 1 \\ 1/7 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

The conditional probability distribution for Y given $X = x$ is

$$p_{Y|X}(y|0) = \frac{p_{XY}(0,y)}{p_X(0)} = \begin{cases} 1/4 & y = 0 \\ 3/4 & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{Y|X}(y|1) = \frac{p_{XY}(1,y)}{p_X(1)} = \begin{cases} 4/7 & y=0 \\ 3/7 & y=1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{Y|X}(y|2) = \frac{p_{XY}(2,y)}{p_X(2)} = \begin{cases} 6/7 & y=0 \\ 1/7 & y=1 \\ 0 & \text{otherwise} \end{cases}$$

43.8 (a) $\frac{1}{2^N-1}$ (b) $p_X(x) = \frac{2^x}{2^N-1}$ for $x = 0, 1, \cdots, N-1$ and 0 otherwise. (c) $p_{Y|X}(y|x) = 2^{-x}(1-2^{-x})^y$ for $x = 0, 1, \cdots, N-1$, $y = 0, 1, 2, \cdots$ and 0 otherwise
43.9 $P(X = k|X+Y = n) = C(n,k)\left(\frac{1}{2}\right)^n$ for $k = 0, 1, \cdots, n$ and 0 otherwise.
43.10 (a)

$$\begin{aligned} P(X=0, Y=0) &= \frac{48}{52}\frac{47}{51} = \frac{188}{221} \\ P(X=1, Y=0) &= \frac{48}{52}\frac{4}{51} = \frac{16}{221} \\ P(X=1, Y=1) &= \frac{4}{52}\frac{48}{51} = \frac{16}{221} \\ P(X=2, Y=1) &= \frac{4}{52}\frac{3}{51} = \frac{1}{221} \end{aligned}$$

and 0 otherwise.
(b) $P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0) = \frac{204}{221} = \frac{12}{13}$ and $P(Y=1) = P(X=1, Y=1) + P(X=2, Y=2) = \frac{1}{13}$
(c) $p_{X|Y}(1|1) = 13 \times \frac{16}{221} = \frac{16}{17}$ and $p_{X|Y}(2|1) = 13 \times \frac{1}{221} = \frac{1}{17}$
43.11 e^2
43.12 Suppose that X and Y are independent. Then $P(X=0|Y=1) = P(X=0) = 0.6$ and $P(X=1|Y=0) = P(X=1) = 0.7$. Since $P(X=0) + P(X=1) = 0.6 + 0.7 \neq 1$, it follows that X and Y can not be independent.
43.13 $P_{Y|X}(0|1) \approx 0.286$ and $P_{Y|X}(1|1) \approx 0.714$

43.14

$$\begin{aligned}
P(N+S=2) =& P(N=0,S=2)+P(N=1,S=1)+P(N=2,S=0)\\
=& 0.10+0.18+0.12=0.40\\
P(N=0|N+S=2) =& \frac{P(N=0,S=2)}{P(N+S=2)}=\frac{0.10}{0.40}=0.25\\
P(N=1|N+S=2) =& \frac{P(N=1,S=1)}{P(N+S=2)}=\frac{0.18}{0.40}=0.45\\
P(N=2|N+S=2) =& \frac{P(N=2,S=0)}{P(N+S=2)}=\frac{0.12}{0.40}=0.30
\end{aligned}$$

43.15

$$\begin{aligned}
P(X+Y=3) =& P(0,3)+P(1,2)+P(2,1)+P(3,0)\\
=& e^{-1.7}e^{-2.3}\left[\frac{2.3^3}{3!}+(1.7)\frac{2.3^2}{2!}+\frac{1.70^2}{2!}\frac{2.30}{1!}+\frac{1.70^3}{3!}\right]\\
\approx& 0.1954\\
P(X-Y=-3|X+Y=3) =& \frac{P(X=0,Y=3)}{P(X+Y=3)}\approx 0.1901\\
P(X-Y=-1|X+Y=3) =& \frac{P(X=1,Y=2)}{P(X+Y=3)}\approx 0.4215\\
P(X-Y=1|X+Y=3) =& \frac{P(X=2,Y=1)}{P(X+Y=3)}\approx 0.3116\\
P(X-Y=3|X+Y=3) =& \frac{P(X=0,Y=3)}{P(X+Y=3)}\approx 0.0768
\end{aligned}$$

43.16 0.0625
43.17 0.5581
43.18 0.3
43.19 1
43.20 Recall that

$$p_{X+Y}(n)=(n-1)p^2(1-p)^{n-2},\quad n\geq 2$$

and 0 otherwise. Thus,

$$\begin{aligned} p_{X|X+Y}(x|n) =& \frac{p_{X,X+Y}(x,n)}{p_{X+Y}(n)} = \frac{P(X=x, Y=n-x)}{p_{X+Y}(n)} \\ =& \frac{P(X=x)P(Y=n-x)}{p_{X+Y}(n)} = \frac{p(1-p)^{x-1}p(1-p)^{n-x-1}}{(n-1)p^2(1-p)^{n-2}} \\ =& \frac{1}{n-1}, \quad 1 \le x \le n-1 \ \blacksquare \end{aligned}$$

43.21 0.13125
43.22 $\frac{1}{54}$

Section 44

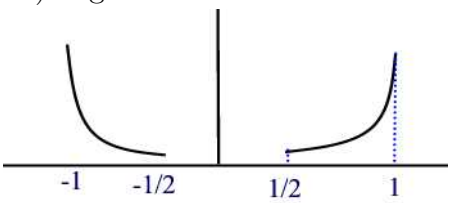
44.1 For $y \le |x| \le 1, 0 \le y \le 1$ we have

$$f_{X|Y}(x|y) = \frac{3}{2}\left[\frac{x^2}{1-y^3}\right]$$

If $y = 0.5$ then

$$f_{X|Y}(x|0.5) = \frac{12}{7}x^2, \quad 0.5 \le |x| \le 1.$$

The graph of $f_{X|Y}(x|0.5)$ is given below



44.2 $f_{X|Y}(x|y) = \frac{2x}{y^2}, \quad 0 \le x < y \le 1$
44.3 $f_{Y|X}(y|x) = \frac{3y^2}{x^3}, \quad 0 \le y < x \le 1$
44.4 $f_{X|Y}(x|y) = (y+1)^2 x e^{-x(y+1)}, \ x \ge 0$ and $f_{Y|X}(y|x) = xe^{-xy}, \ y \ge 0$
44.5 (a) For $0 < y < x$ we have $f_{XY}(x,y) = \frac{y^2}{2}e^{-x}$
(b) $f_{X|Y}(x|y) = e^{-(x-y)}, \quad 0 < y < x$
44.6 (a) $f_{X|Y}(x|y) = 6x(1-x), \quad 0 < x < 1$. X and Y are independent.
(b) 0.25
44.7 $f_{X|Y}(x|y) = \frac{\frac{1}{3}x-y+1}{\frac{3}{2}-y}$ (b) $\frac{11}{24}$

44.8 0.25
44.9 $\frac{8}{9}$
44.10 0.4167
44.11 $\frac{7}{8}$
44.12 0.1222
44.13 $f_X(x) = \int_{x^2}^{1} \frac{1}{\sqrt{y}} dy = 2(1-x), \quad 0 < x < 1$ and 0 otherwise
44.14 $\frac{1}{1-y}$ for $0 < y < x < 1$
44.15 mean$=\frac{1}{3}$ and $\text{Var}(Y) = \frac{1}{18}$
44.16 0.172
44.17 $f_{Y|X}(y|2) = \frac{1}{6} e^{-\frac{(y-2)}{6}}$ for $y > 2$ and 0 otherwise
44.18 $f_{Y|X}\left(y|\frac{1}{3}\right) = \frac{9}{2}(1-y)$ for $\frac{1}{3} < y < 1$ and 0 otherwise
44.19 $f_{Y|X}(y|x) = \frac{1}{1-x}$ for $0 < x < 1$ and 0 otherwise. Likewise, $f_{X|Y}(x|y) = \frac{1}{y}$ for $0 < y < 1$ and 0 otherwise
44.20 From

$$f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

we have

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) f_Y(y)}{f_X(x)}.$$

But

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y).$$

Hence,

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) dy}$$

44.21 0.0046

Section 45

45.1 $f_{ZW}(z,w) = \frac{f_{XY}(\frac{zd-bw}{ad-bc}, \frac{aw-cz}{ad-bc})}{|ad-bc|}$
45.2 $f_{Y_1Y_2}(y_1,y_2) = \frac{\lambda^2}{y_2} e^{-\lambda y_1}, \ y_2 > 1, \ y_1 \geq \ln y_2$ and 0 otherwise
45.3 $f_{R\Phi}(r,\phi) = r f_{XY}(r\cos\phi, r\sin\phi), \quad r > 0, \quad -\pi < \phi \leq \pi$ and 0 otherwise
45.4

$$\begin{aligned} f_{ZW}(z,w) = & \frac{z}{1+w^2}[f_{XY}(z(\sqrt{1+w^2})^{-1}, wz(\sqrt{1+w^2})^{-1}) \\ & + f_{XY}(-z(\sqrt{1+w^2})^{-1}, -wz(\sqrt{1+w^2})^{-1})] \end{aligned}$$

45.5 $f_{UV}(u,v) = \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$.
45.6 We have

$$f_{Y_1Y_2}(y_1,y_2) = \begin{cases} e^{-y_1}y_1 & y_1 \geq 0,\ 0 < y_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

45.7 $f_{Y_1Y_2}(y_1,y_2) = \frac{1}{\sqrt{2\pi}} e^{-(\frac{3}{7}y_1+\frac{1}{7}y_2)^2/2} \frac{1}{\sqrt{8\pi}} e^{-(\frac{1}{7}y_1-\frac{2}{7}y_2)^2/8} \cdot \frac{1}{7}$
45.8 We have $u = g_1(x,y) = \sqrt{2y}\cos x$ and $v = g_2(x,y) = \sqrt{2y}\sin x$. The Jacobian of the transformation is

$$J = \begin{vmatrix} -\sqrt{2y}\sin x & \frac{\cos x}{\sqrt{2y}} \\ \sqrt{2y}\cos x & \frac{\sin x}{\sqrt{2y}} \end{vmatrix} = -1$$

Also,

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-x}.$$

Thus,

$$f_{UV}(u,v) = f_{XY}(x(u,v),y(u,v))|J|^{-1} = \frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}}.$$

This is the joint density of two independent standard normal random variables.
45.9 $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_{XY}(a-y,y)dy$ If X and Y are independent then $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$
45.10 $f_{Y-X}(a) = \int_{-\infty}^{\infty} f_{XY}(y-a,y)dy$. If X and Y are independent then $f_{Y-X}(a) = \int_{-\infty}^{\infty} f_X(y-a)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(y)f_Y(a+y)dy$
45.11 $f_U(u) = \int_{-\infty}^{\infty} \frac{1}{|v|} f_{XY}(v,\frac{u}{v})dv$. If X and Y are independent then $f_U(u) = \int_{-\infty}^{\infty} \frac{1}{|v|} f_X(v)f_Y(\frac{u}{v})dv$
45.12 $f_U(u) = \frac{1}{(u+1)^2}$ for $u > 0$ and 0 elsewhere.
45.13 $g(x) = e^{-\frac{x}{2}} - e^{-x},\ x > 0$
45.14 $f(x) = \frac{2}{(2x+1)^2}$

Section 46

46.1 $\frac{(m+1)(m-1)}{3m}$ **46.2** $E(XY) = \frac{7}{12}$
46.3 $E(|X-Y|) = \frac{1}{3}$
46.4 $E(X^2Y) = \frac{7}{36}$ and $E(X^2+Y^2) = \frac{5}{6}$.
46.5 0
46.6 33

46.7 $\frac{L}{3}$
46.8 $\frac{30}{19}$
46.9 (a) 0.9 (b) 4.9 (c) 4.2
46.10 (a) 14 (b) 45
46.11 5.725
46.12 $\frac{2}{3}L^2$
46.13 27
46.14 5
46.15 1.25
46.16 Since X and Y are independent, $E(XY) = E(X)E(Y)$. We have

$$\begin{aligned}\mathrm{Var}(\alpha X+\beta Y+\gamma) =& E[(\alpha X+\beta Y+\gamma)^2]-(E[(\alpha X+\beta Y+\gamma)])^2\\ =& E(\alpha^2X^2+\beta^2Y^2+\gamma^2+2\alpha\beta XY+2\alpha\gamma X+2\beta\gamma Y)-[\alpha E(X)+\beta E(Y)+\\ =& \alpha^2E(X^2)+\beta^2E(Y^2)+\gamma^2+2\alpha\beta E(XY)+2\alpha\gamma E(X)+2\beta\gamma E(Y)\\ & -\alpha^2(E(X))^2-\beta^2(E(Y))^2-\gamma^2-2\alpha\beta E(X)E(Y)-2\alpha\gamma E(X)-2\beta\gamma E(Y)\\ =& \alpha^2[E(X^2)-(E(X))^2]+\beta^2[E(Y^2)-(E(Y))^2]\\ =& \alpha^2\mathrm{Var}(X)+\beta^2\mathrm{Var}(Y)\end{aligned}$$

46.17 11
46.18 $E(XY) = E(X)E(Y)$
46.19 $E(XY) \neq E(X)E(Y)$
46.20 3
46.21 $\frac{1}{6}$
46.22 0.125
46.23 102.50
46.24 44.64
46.25 1.1

Section 47

47.1 $2\sigma^2$
47.2 $-\frac{n}{36}$
47.3 24

47.4 We have

$$\begin{aligned}\text{Var}(X+Y) =& E[(X+Y)^2] - (E(X+Y))^2 = E(X^2+2XY+Y^2) - (E(X)+E(Y))^2\\ =& E(X^2+2XY+Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2\\ =& [E(X^2) - E(X)^2] + [E(Y^2) - E(Y)^2] + 2[E(XY) - E(X)E(Y)]\\ =& \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)\end{aligned}$$

47.5 19,300
47.6 $\frac{1}{12}$
47.7 (a) $f_{XY}(x,y) = 5, \ \ -1 < x < 1, \ x^2 < y < x^2 + 0.1$ and 0 otherwise (b) 0
47.8 We have

$$E(X) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta d\theta = 0$$

$$E(Y) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta d\theta = 0$$

$$E(XY) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\sin\theta d\theta = 0$$

Thus, $\text{Cov}(X,Y) = 0$. However, X and Y are clearly dependent since $X^2 + Y^2 = 1$
47.9 $\frac{3}{160}$
47.10 19,300
47.11 0
47.12 0.04
47.13 6
47.14 8.8
47.15 (a) We have

X \ Y	0	1	2	$p_X(x)$
0	0.25	0.08	0.05	0.38
1	0.12	0.20	0.10	0.42
2	0.03	0.07	0.10	0.2
$P_Y(y)$	0.4	0.35	0.25	1

(b) $E(X) = 0.82$ and $E(Y) = 0.85$. (c) $\text{Cov}(X,Y) = 0.243$. (d) $E(C) = E(100X + 75Y) = \$145.75$
47.16 $E(W) = 4$ and $\text{Var}(W) = 67$

47.17 $\frac{5}{12}$
47.18 -0.15
47.19 -0.123
47.20 $\frac{10}{72}$
47.21 43
47.22 2.5
47.23 (C)

Section 48

48.1 -0.33
48.2 (a) $f_{XY}(x,y) = 5, \quad -1 < x < 1,\ x^2 < y < x^2 + 0.1$ and 0 otherwise (b) 0
48.3 (a) We have

$$\rho(X_1 + X_2, X_2 + X_3) = \frac{\text{Cov}(X_1 + X_2, X_2 + X_3)}{\sqrt{\text{Var}(X_1 + X_2)}\sqrt{\text{Var}(X_2 + X_3)}} = \frac{\text{Var}(X_2)}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

(b)Replacing the term $X_2 + X_3$ in the above expression with $X_3 + X_4$, we can see that the numerator becomes 0 (because the variables are all pairwise uncorrelated); hence the correlation between $X_1 + X_2$ and $X_3 + X_4$ is 0
48.4 $\rho(X,Y) = \text{Cov}(X,Y) = 0$
48.5 $\text{Cov}(X,Y) = \frac{3}{160}$ and $\rho(X,Y) \approx 0.397$
48.6 0
48.7 -0.2
48.8 2
48.9 $\frac{n-2}{n+2}$
48.10 0.71
48.11 0.5
48.12 -0.61823
48.13 4036

Section 49

49.1 $E(X|Y=y) = \frac{2}{3}y$ and $E(Y|X=x) = \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right)$
49.2 $\frac{3}{4}x$
49.3 2.3
49.4 $\frac{2}{3}\left(\frac{1-x^6}{1-x^4}\right)$

49.5 We have

$$E(X|Y=1)=\sum_x \frac{p_{XY}(x,1)}{p_Y(1)}=\frac{9}{5}[\frac{1}{\times}\frac{1}{9}+2\times\frac{1}{3}+3\times\frac{1}{9}]=2$$

$$E(X|Y=2)=\sum_x \frac{p_{XY}(x,2)}{p_Y(2)}=6[1\times\frac{1}{9}+2\times 0+3\times\frac{1}{18}]=\frac{5}{3}$$

$$E(X|Y=3)=\sum_x \frac{p_{XY}(x,1)}{p_Y(3)}=\frac{18}{5}[1\times 0+2\times\frac{1}{6}+3\times\frac{1}{9}]=\frac{12}{5}.$$

Since $E(X|Y=y)$ changes with Y, X and Y are dependent
49.6 0.714
49.7 $x+\frac{1}{2}$
49.8 0.88
49.9 $\frac{1+x}{2}$
49.10 0.4167
49.11 3.89764
49.12 $\frac{4}{7}$
49.13 $E(X|Z=z)=\frac{z}{2}$
49.14 (a) $c=8$ (b) Dependent (c) $E(Y|X=x)=\frac{2}{3}x$ for $0\le x\le 1$
49.15 0.25
49.16 $\frac{y}{2}$
49.17 $\frac{1-3e^{-2}}{1-e^{-2}}$
49.18 (a) 1 (b) dependent (c) $f_X(x)=\int_x^\infty e^{-y}dy=e^{-x}$ for $x\ge 0$, and 0 otherwise $f_Y(y)=\int_0^y e^{-y}dx=ye^{-y}$ for $y\ge 0$, and 0 otherwise (d) $E(Y|X=x)=x+1$ for $x\ge 0$ (e) $E(X|Y=y)=\int_0^y xe^{x-y}dx=y-1+e^{-y}$ for $y\ge 0$
49.19 (a) $\frac{1}{5}$ (b) $f_X(x)=\int_x^{x+1}\frac{x+y}{5}dy=\frac{4x+1}{10}$ for $0<x<2$ and 0 otherwise. Also, $f_{Y|X}(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}=\frac{2(x+y)}{4x+1}$ for $0<x<2, x<y<x+1$ and 0 otherwise (c) $E(Y|X=x)=\frac{12x^2+9x+2}{12x+3}$
49.20 6.6
49.21 0.534
49.22 2.392
49.23 3.435
49.24 1.1
49.25 $-\frac{25}{9}$

Section 50

50.1 We prove the result in the continuous case. We have

$$\begin{aligned} E(a(X)+b(X)Y|X=x) =& \int_{-\infty}^{\infty}(a(x)+b(x)y)f_{Y|X}(y|x)dy \\ =& a(x)\int_{-\infty}^{\infty}f_{Y|X}(y|x)dy + b(x)\int_{-\infty}^{\infty}yf_{Y|X}(y|x)dy \\ =& a(x)+b(x)E(Y|X=x) \ \blacksquare \end{aligned}$$

50.2 We prove the result for continuous random variables, the proof for discrete random variables is almost identical. First, we define

$$h(x) = E[g(X,Y)|X=x] = \int_{-\infty}^{\infty} g(x,y)f_{Y|X}(y|x)dy.$$

We have

$$\begin{aligned} E[E(g(X,Y)|X)] =& E(h(X)) = \int_{-\infty}^{\infty} h(x)f_X(x)dx \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_X(x)f_{Y|X}(y|x)dydx \\ =& \int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)dydx = E[g(X,Y)] \end{aligned}$$

50.3 15
50.4 $\frac{1}{n+1}$
50.5 $\frac{7}{9}$
50.6 $E(X) = 15$ and $E(Y) = 5$
50.7 -1.4
50.8 15
50.9 $E(Y|X=1) = 0.714$ and $E(Y^2|X=1) = 0.714$
50.10 $E(Y|X) = x + \frac{1}{2}$ and $E(Y^2|X) = x^2 + x + \frac{1}{3}$
50.11 $E(X|Y=0) = 0.88$ and $E(X^2|Y=0) = 1.76$
50.12 $E(Y|X=x) = \frac{1+x}{2}$ and $E(Y^2|X=x) = \frac{1+x+x^2}{3}$
50.13 $\alpha\lambda$
50.14 $\frac{2}{9}[1+(0.5)^3]$
50.15 13.5

50.16 We have

$$\begin{aligned}
E(\sum_{i=1}^{N} X_i) =& E[E[\sum_{i=1}^{N} X_i|N]] = \sum_{n=1}^{\infty} E[\sum_{i=1}^{N} X_i|N=n]]P(N=n) \\
=& \sum_{n=1}^{\infty} E[\sum_{i=1}^{n} X_i|N=n]]P(N=n) \\
=& \sum_{n=1}^{\infty}\sum_{i=1}^{n} E(X_i|N=n)P(N=n) \\
=& \sum_{n=1}^{\infty}\sum_{i=1}^{n} E(X_i)P(N=n) \\
=& E(X_1)\sum_{n=1}^{\infty} nP(N=n) = E(N)E(X_1) \blacksquare
\end{aligned}$$

50.17 400
50.18 35
50.19 $\frac{\lambda}{\lambda+\mu}$
50.20 First, note that

$$F_U(a-x) = \begin{cases} 1-e^{-\lambda(a-x)} & 0 \le x \le a \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
F_{X+Y}(a) =& P(X+Y<a) = \int_0^a F_Y(a-x)f_X(x)dx = \int_0^a (1-e^{-\lambda(a-x)})\lambda e^{-\lambda x}dx \\
=& 1-\lambda e^{-\lambda a} - \lambda a e^{-\lambda a}.
\end{aligned}$$

Hence,

$$f_{X+Y}(a) = \frac{d}{da}(1-\lambda e^{-\lambda a} - \lambda a e^{-\lambda a}) = \lambda^2 a e^{-\lambda a}.$$

That is, $X+Y$ is a Gamma random variable with parameters $(2,\lambda)$
50.21 1%
50.22 0.547
50.23 882
50.24 21.19
50.25 2.6608

50.26 23.75

Section 51

51.1 0.20
51.2 $\frac{1}{12}$
51.3 0.9856
51.4 13
51.5 $\frac{(1-x)^2}{12}$
51.6 $\beta^2(\lambda+\lambda^2)+\alpha^2\lambda$
51.7 2.25
51.8 0.076
51.9 0.0756
51.10 8 millions
51.11 3.75
51.12 1743.75
51.13 1
51.14 $\frac{13x^2+8x+1}{18(3x+1)^2}$
51.15 $\frac{1-x^2}{12}$
51.16 1
51.17 0.25
51.18 24
51.19 36
51.20 0.9953
51.21 1/3
51.22 2.933

Section 52

52.1 $E(X)=\frac{n+1}{2}$ and $\text{Var}(X)=\frac{n^2-1}{12}$
52.2 $E(X)=\frac{1}{p}$ and $\text{Var}(X)=\frac{1-p}{p^2}$
52.3 The moment generating function is

$$M_X(t)=\sum_{n=1}^{\infty}e^{tn}\frac{6}{\pi^2n^2}.$$

By the ratio test we have

$$\lim_{n\to\infty} \frac{e^{t(n+1)}\frac{6}{\pi^2(n+1)^2}}{e^{tn}\frac{6}{\pi^2 n^2}} = \lim_{n\to\infty} e^t \frac{n^2}{(n+1)^2} = e^t > 1$$

and so the summation diverges whenever $t > 0$. Hence there does not exist a neighborhood about 0 in which the mgf is finite

52.4 $E(X) = \frac{\alpha}{\lambda}$ and $\text{Var}(X) = \frac{\alpha}{\lambda^2}$

52.5 $M_X(t) = \begin{cases} 1 & t = 0 \\ \infty & \text{otherwise} \end{cases}$

52.6 $M(t) = E(e^{ty}) = \frac{19}{27} + \frac{8}{27}e^t$

52.7 2

52.8 $M_X(t) = \frac{e^t(6-6t+3t^2)-6}{t^3}$

52.9 -38

52.10 $\frac{1}{2e}$

52.11 $e^{\frac{t^2}{2}}$

52.12 $e^{\frac{k^2}{2}}$

52.13 4

52.14 41.9

52.15 $M_X(t) = \frac{\lambda^2}{\lambda^2 - t^2},\ |t| < \lambda$

52.16 $M_X(t) = \frac{12}{(t-4)(t-3)},\ t < 3$

52.17 We have

$$\begin{aligned} M_{aX+b}(t) =& E[e^{t(aX+b)}] = E[e^{bt}e^{(at)X}] \\ =& e^{bt}E[e^{(at)X}] = e^{bt}M_X(at) \end{aligned}$$

52.18 First we find the moment of a standard normal random variable with parameters 0 and 1. We can write

$$\begin{aligned} M_Z(t) =& E(e^{tZ}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tz}e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left\{-\frac{(z^2-2tz)}{2}\right\}dz \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left\{-\frac{(z-t)^2}{2} + \frac{t^2}{2}\right\}dz = e^{\frac{t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}}dz = e^{\frac{t^2}{2}}. \end{aligned}$$

Now, since $X = \mu + \sigma Z$, we have

$$\begin{aligned} M_X(t) =& E(e^{tX}) = E(e^{t\mu+t\sigma Z}) = E(e^{t\mu}e^{t\sigma Z}) = e^{t\mu}E(e^{t\sigma Z}) \\ =& e^{t\mu}M_Z(t\sigma) = e^{t\mu}e^{\frac{\sigma^2 t^2}{2}} = exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}. \end{aligned}$$

By differentiation, we obtain

$$M'_X(t) = (\mu + t\sigma^2)exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}$$

and

$$M''_X(t) = (\mu + t\sigma^2)^2 exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\} + \sigma^2 exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}$$

and thus

$$E(X) = M'_X(0) = \mu \text{ and } E(X^2) = M''_X(0) = \mu^2 + \sigma^2$$

The variance of X is

$$Var(X) = E(X^2) - (E(X))^2 = \sigma^2$$

52.19 5000
52.20 0.87
52.21 $\frac{3}{10} + \frac{e^{7t}-1}{10t}$

Section 53

53.1 $e^{-2t}\frac{\lambda}{\lambda-3t}, \quad t < \frac{\lambda}{3}$
53.2 Y has the same distribution as $3X - 2$ where X is a binomial distribution with $n = 15$ and $p = \frac{3}{4}$
53.3 15
53.4 $f_{X+Y}(a) = \frac{1}{\sqrt{2\pi(a^2\sigma_1^2+b^2\sigma_2^2)}} e^{-(a-(a\mu_1+b\mu_2))^2/[2(a^2\sigma_1^2+b^2\sigma_2^2)]}$
53.5 0.84
53.6 Let $Y = X_1 + X_2 + \cdots + X_n$ where each X_i is an exponential random variable with parameter λ. Then

$$M_Y(t) = \prod_{k=1}^{n} M_{X_k}(t) = \prod_{k=1}^{n} \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n, \quad t < \lambda.$$

Since this is the mgf of a gamma random variable with parameters n and λ we can conclude that Y, is a gamma random variable with parameters n and λ
53.7 $e^{t_1^2+t_2^2}$
53.8 10560

53.9 (a) $M_X(t) = \frac{pe^t}{1-(1-p)e^t},\ t < -\ln(1-p)$ (b) $\left(\frac{pe^t}{1-(1-p)e^t}\right)^n,\ t < -\ln(1-p)$ (c) $M_Y(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^n$
53.10 $M(t_1, t_2) == \frac{(e^{t_1}-1)(e^{t_2}-1)}{t_1 t_2}$
53.11 $\frac{2}{9}$
53.12 $M_{X+2Y}(t) = e^{13t^2+4t}$
53.13 0.4
53.14 $-\frac{15}{16}$
53.15 $(0.7 + 0.3e^t)^9$
53.16 0.70
53.17 0.6293
53.18 $\left[M_{X_1}\left(\frac{t}{n}\right)\right]^n$
53.19 Y is a gamma distribution with $\alpha = 21$ and $\lambda = 5$
53.20 $\overline{X}$ is gamma distribution with $\alpha = 21$ and $\lambda = 15$ **53.21** 0.223
53.22 0.557
53.23 0.1003
53.24 (i) and (iii)
53.25 0.23
53.26 $(0.0003e^s + 0.0297e^t + 0.97)^{30}$
53.27 37.8

Section 54

54.1 0.2119
54.2 0.9876
54.3 0.0094
54.4 0.692
54.5 0.1367
54.6 0.383
54.7 0.0088 **54.8** 0
54.9 23
54.10 6,342,637.5
54.11 0.8185
54.12 16
54.13 0.8413
54.14 0.1587
54.15 0.9887

54.16 (a) $\overline{X}$ is approximated by a normal distribution with mean 100 and variance $\frac{400}{100} = 4$. (b) 0.9544.
54.17 0.77
54.18 0.2743
54.19 0.8202
54.20 0.5335
54.21 0.7486
54.22 1,150,000
54.23 0.9713

Bibliography

[1] Sheldon Ross, *A first course in Probability*, 8th Edition (2010), Prentice Hall.

[2] SOA/CAS, *Exam P Sample Questions*

Index

$E(aX^2 + bX + c)$, 155
$E(ax + b)$, 155
$E(g(X))$, 153
n^{th} moment about the origin, 156
n^{th} order statistics, 391
n^{th} raw moment, 156

Absolute complement, 16

Bayes' formula, 97
Bernoulli experiment, 177
Bernoulli random variable, 177
Bernoulli trial, 177
Beta function, 412
Bijection, 7
Binomial coefficient, 46
Binomial random variable, 177
Binomial Theorem, 46
Birthday problem, 60
Bivariate normal distribution, 492

Cardinality, 7
Cartesian product, 24
Chi-squared distribution, 343
Classical probability, 56
Coefficient of Correlation, 464
Coefficient of Variation, 462
Combination, 44
Complementary event, 58, 67
Conditional cumulative distribution, 416
Conditional cumulative distribution function, 426
Conditional density function, 425
Conditional expectation, 471
Conditional probability, 88
Conditional probability mass function, 415
Continuity correction, 324
Continuous random variable, 125, 267
Convergent improper integral, 256
Convolution, 398, 406
Corner points, 354
Countable additivity, 57
Countable sets, 7
Covariance, 456
Cumulative distribution function, 135, 179, 233, 268

De Moivre-Laplace theorem, 323
Decreasing sequence of sets, 233
Degrees of freedom, 343
Dependent events, 112
Discrete random variable, 125
Disjoint sets, 17
Distribution function, 135, 268
Divergent improper integral, 256

Empty set, 6
Equal sets, 7
Equally likely, 56
Euler integral, 412

Event, 55
Expected value of a continuous RV, 282
Expected value of a discrete random variable, 143
Experimental probability, 56
Exponential distribution, 328

Factorial, 38
Feasible region, 354
Finite sets, 7
First order statistics, 391
First quartile, 297
Floor function, 179

Gamma distribution, 339
Gamma function, 338
Geometric random variable, 167, 207

Hyper-geometric random variable, 224

Identically distributed, 390
Improper integrals, 255
Inclusion-Exclusion Principle, 23
Increasing sequence of sets, 233
Independent events, 110
Independent random variables, 385
Indicator function, 134
Infinite sets, 7
Interquartile range, 298
Intersection of events, 67
Intersection of two sets, 17
Iterated integrals, 359

Joint cumulative distribution function, 369
Joint probability mass function, 371

Kolmogorov axioms, 56

Law of large numbers, 56
Law of Total Probability, 99
Linear inequality, 353

Marginal distribution, 370
mathematical induction, 10
Mean, 144
Median, 296
Memoryless property, 330
Mixed random variable, 125
Mode, 297
Moment generating function, 499
Moments, 499
Multiplication rule of counting, 32
Mutually exclusive, 57, 67
Mutually independent, 113

Negative binomial distribution, 216
Negatively corrolated, 465
Non-equal sets, 7
Normal distribution, 312

Odds against, 121
Odds in favor, 121
One-to-one, 7
Onto function, 7
Order statistics, 389
Ordered pair, 24
Outcomes, 55

Pairwise disjoint, 22
Pairwise independent events, 114
Pascal's identity, 45
Pascal's triangle, 47
Percentile, 297
Permutation, 38
Poisson random variable, 167, 194
Positively corrolated, 465
Posterior probability, 91

Power set, 10
Prime number, 12
Prior probability, 91
Probability density function, 267
Probability histogram, 133
Probability mass function, 133
Probability measure, 57
Probability trees, 79
Proper subset, 9

Quantile, 297

Random experiment, 55
Random variable, 125
Rate parameter, 328
Relative complement, 16
Reliability function, 248

Same Cardinality, 7
Sample space, 55
Scale parameter, 339
Set, 6
Set-builder, 6
Shape parameter, 339
Standard Deviation, 162
Standard deviation, 289
Standard normal distribution, 313
Standard uniform distribution, 304
Subset, 8
Survival function, 248

test point, 353
Tree diagram, 31
Trinomial Distribution, 382

Uncorrelated, 465
Uncountable sets, 7
Uniform discrete R.V., 171
Uniform distribution, 304
Union of events, 67
Union of sets, 17
Universal Set, 16
Universal set, 8

Variance, 162, 286
Vendermonde's identity, 225
Venn Diagram, 9