PHYS-404
Lecture 6
Hermite Polynomials

How they come up
The quantum mechanical SHO

- The quantum mechanical Hamiltonian of a simple harmonic oscillator gets the form:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \equiv \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$x \to x, \quad p \to -i\hbar\frac{d}{dx}$$

$$H \to -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

How they come up
The quantum mechanical SHO

- We can show that the Schrödinger eq. takes the form:

$$\psi'' + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 x^2\right)\psi = 0$$

- By introducing the dimensionless parameters:

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x, \qquad K \equiv \frac{2E}{\hbar\omega}$$

- We get

$$\frac{d^2\psi}{d\xi^2} = \left(\xi^2 - K\right)\psi$$

The Hermite Differential Equation

- The fact that the wavefunction of a system must satisfy the condition $\lim_{\xi \to \infty} \psi = 0$ leads to the following value for the constant *K*:

$$K = 2n + 1, \quad n = 0,\ 1,\ 2,\ldots$$

- And thus we get the so called Hermite-Weber differential equation:

$$\frac{d^2\psi}{d\xi^2} + \left(2n + 1 - \xi^2\right)\psi = 0 \quad (6.1)$$

The generating function

- The solutions of this D.E. are the so called Hermite polynomials.
- The function which generates the Hermite polynomials is given by:

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \qquad (6.2)$$

The recurrence relations

- Hermite polynomials satisfy the following relations:

$$H_{n+1}(x) = 2xH_n\left(x\right) - 2nH_{n-1}(x)$$

$$H_n^{'}(x) = 2nH_{n-1}(x)$$

$$H_n(x) = \left(-1\right)^n H_n(-x) \qquad (6.3)$$

$$H_{2n}(0) = \left(-1\right)^n \frac{\left(2n\right)!}{n!}$$

$$H_{2n+}(0) = 0$$

Alternate Representations

- From the generating function we can get the expression

$$H_n(x) = \left(-1\right)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \qquad (6.4)$$

- We can also have a complex integral representation as

$$H_n(x) = \frac{n!}{2\pi i} \oint t^{-m-1^n} e^{-t^2+2tx} dt \qquad (6.5)$$

Orthogonality

- The Hermite polynomials satisfy the following orthogonality conditions:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 0, \quad m \neq n \qquad (6.6)$$

$$\int_{-\infty}^{\infty} \left[H_n(x)\right]^2 e^{-x^2} dx = 2^n \pi^{1/2} n! \qquad (6.7)$$

- As Hermite polynomials make up a complete set of ortonormal functions

Orthogonality

- The Hermite polynomials form a complete basis of orthonormal functions then we can develop a function $f(x)$ (continuous or continuous by parts) as a series of them as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n H_n(x) \qquad (6.8)$$

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x)\, dx \qquad (6.9)$$

Hermite polynomials' properties

$$H_0 = 1$$

$$H_1 = 2\xi$$

$$H_2 = 4\xi^2 - 2$$

$$H_3 = 8\xi^3 - 12\xi$$

$$H_4 = 16\xi^4 - 48\xi^2 + 12$$

$$H_5 = 32\xi^5 - 160\xi^3 + 120\xi$$

$$H_n(\xi) = (-1)^n e^{\xi^2} \left(\frac{d}{d\xi} \right)^n e^{-\xi^2}$$

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$$

$$\frac{dH_n}{d\xi} = 2nH_{n-1}(\xi)$$

$$H_n(x) = \sum_{s=0}^{[n/2]} (-1)^s (2x)^{n-2s} \frac{n!}{(n-2s)!s!}$$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2/2} dx = 0$$