Applications of The Definite Integral

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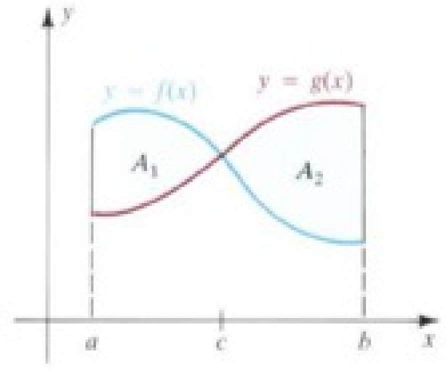
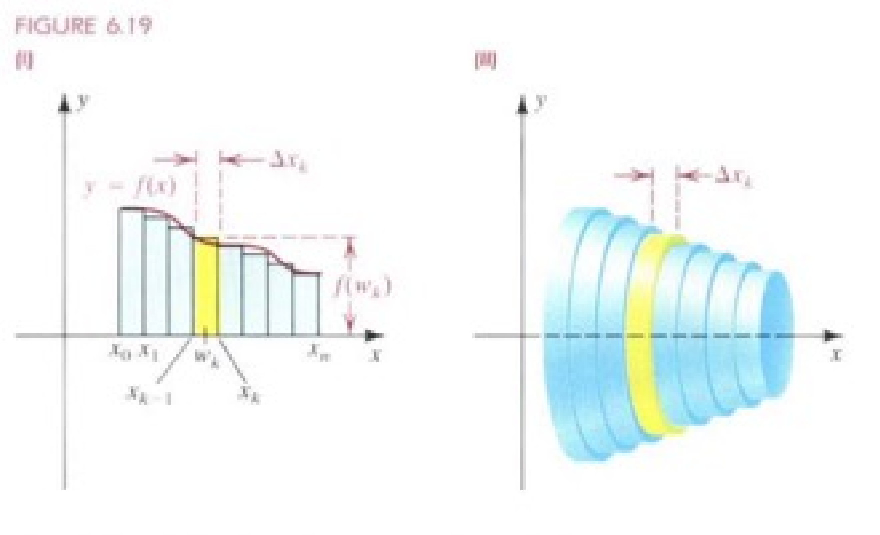
Area



$$A = A_1 + A_2 = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

Solids of revolution



k^{th} rectangle generates a circular disk of base radius $f(w_k)$ and altitude $\triangle x_k = x_k - x_{k-1}$.

The volume of the solid is the sum of the volumes of all such disks:

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$$\sum_k \pi \left[f\left(w_k\right)\right]^2 \Delta x_k$$

Definition

Let f be continuous on $[a, b]$, and let R be the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$. The volume V of the solid of revolution generated by revolving R about the x-axis is

$$V = \lim_{\|P\| \to 0} \sum_k \pi \left[f\left(w_k\right)\right]^2 \Delta x_k = \int_a^b \pi [f(x)]^2 dx$$

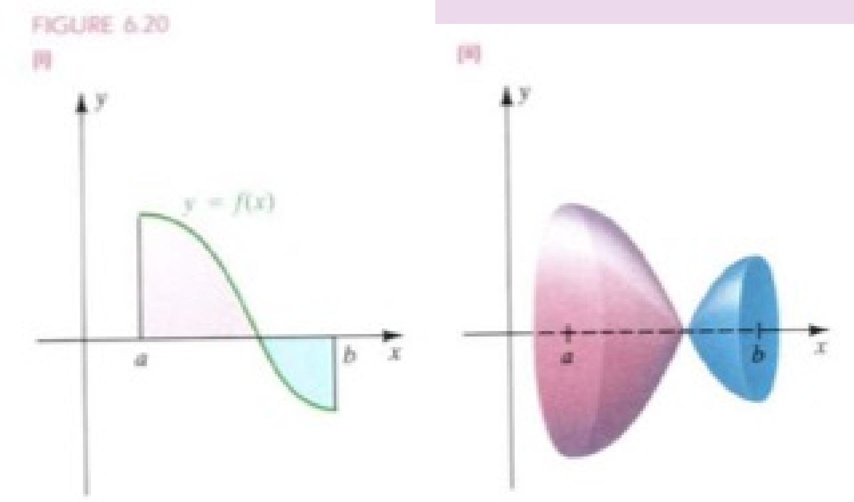
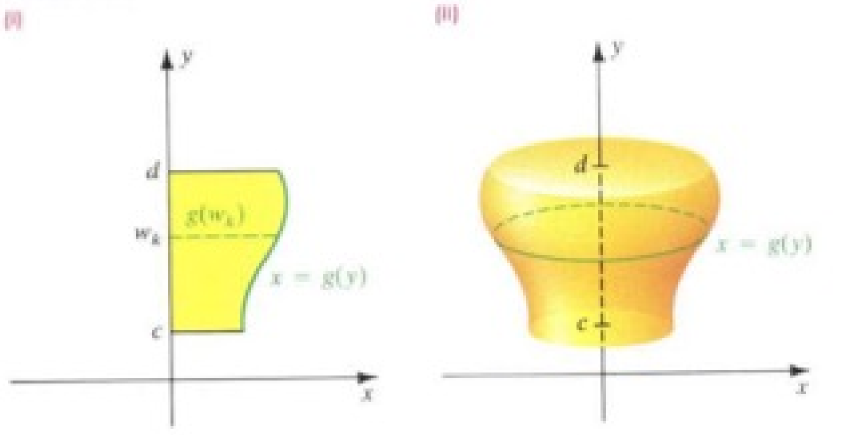


FIGURE 6.20
y
y = f(x)
a
b
x
y
a
b
x

FIGURE 6.21



Definition

$V = \lim_{||P|| \to 0} \sum_k \pi \left[g\left(w_k\right)\right]^2 \Delta y_k = \int_c^d \pi [g(y)]^2 dy$

Volume V of a circular disk

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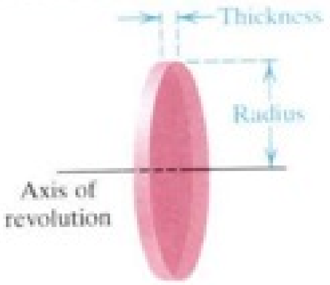
$$V = \pi(radius)^2.(thickness).$$

FIGURE 6.22



Examples

- Find the volume of the region bounded by $x - axis$, the graph $y = x^2 + 1$ and the line $x = -1$ and $x = 1$ which is revolving about the $x - axis$?

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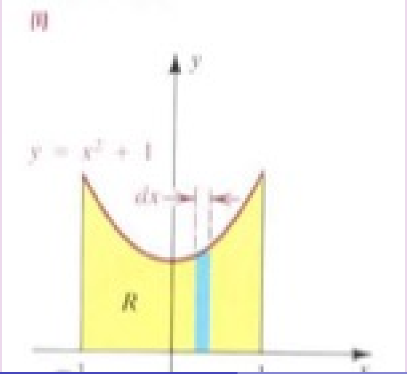

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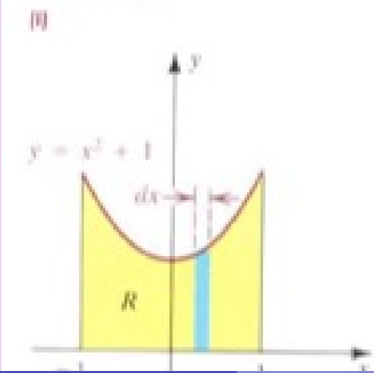

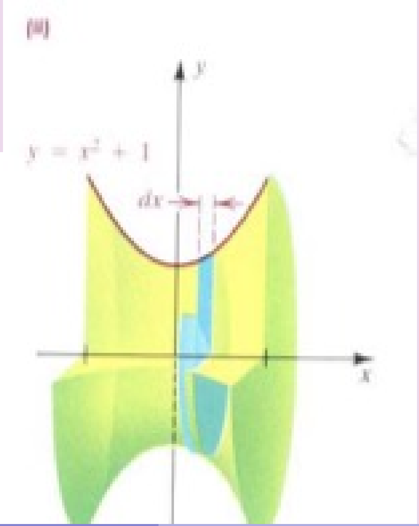

thickness: dx

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radius of disk: $x^2 + 1$

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volume of disk: $\pi(x^2 + 1)^2 dx$.

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$$V = \int_{-1}^{1} \pi \left(x^2 + 1\right)^2 dx, \tag{1}$$

$$= 2 \int_{0}^{1} \pi \left(x^4 + 2x^2 + 1\right) dx,$$

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$$= 2\pi \left[\frac{x^5}{5} + 2\left(\frac{x^3}{3}\right) + x\right]_0^1$$

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$$= 2\pi \left(\frac{1}{5} + \frac{2}{3} + 1\right) = \frac{56}{15}\pi \approx 11.7$$

Examples

- Find the volume of the region bounded by $y - axis$, the graph $y = x^3$ and the line $y = 1$ and $y = 8$ which is revolving about the $y - axis$?

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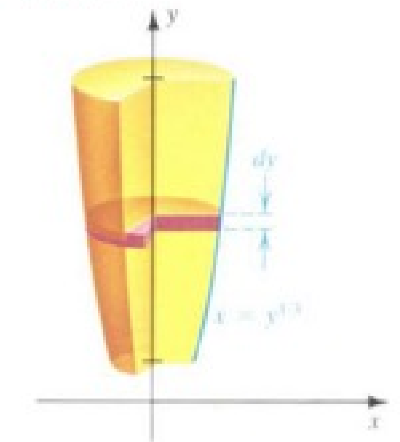
FIGURE 6.24



thickness: dy

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radius of disk: $\sqrt[3]{y}$

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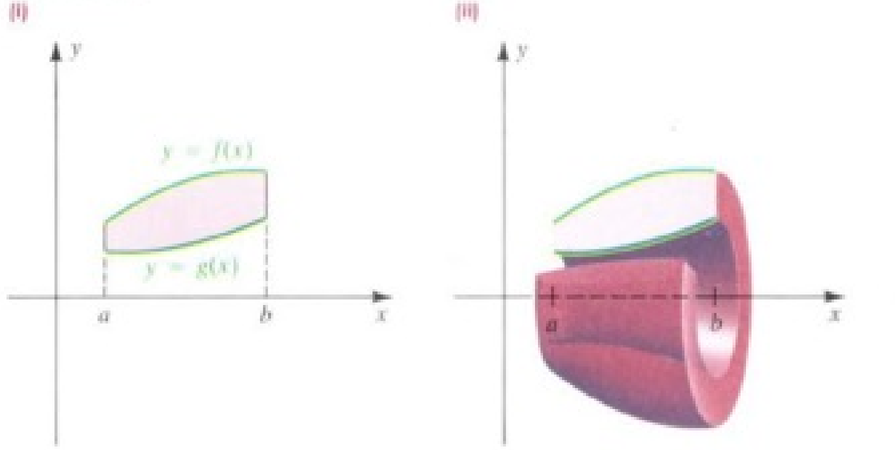
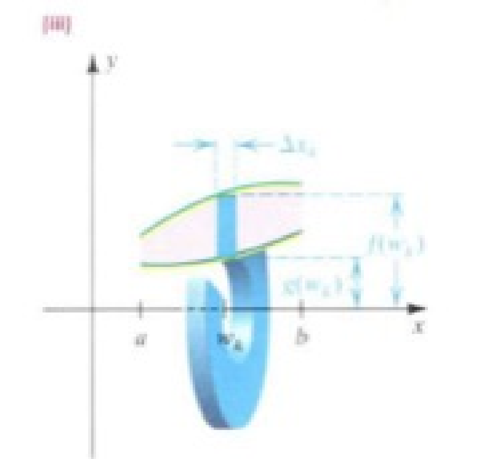
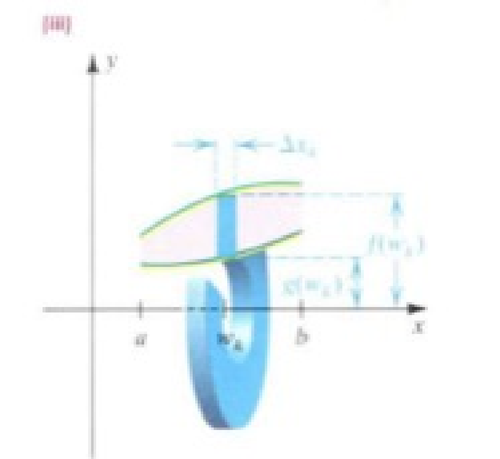
$$= \frac{3}{5}\pi[32 - 1] = \frac{93}{5}\pi \approx 11.7$$

FIGURE 6.25




y
a
b
x



$$V = \pi\left[f\left(w_k\right)\right]^2 \Delta x_k - \pi\left[g\left(w_k\right)\right]^2 \Delta x_k = \pi\left\{\left[f\left(w_k\right)\right]^2 - \left[g\left(w_k\right)\right]^2\right\} \Delta x_k$$

Volume V of a washer

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$$V = \pi[(outerradius)^2 - (innerradius)^2].(thickness)]$$

Examples

- Find the volume of the region bounded by the graph $x^2 = y - 2$ and $2y - x - 2 = 0$ and the vertical lines $x = 0$ and $x = 1$ which is revolving about the $x - axis$?

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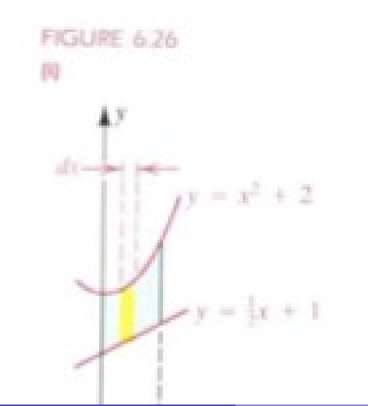

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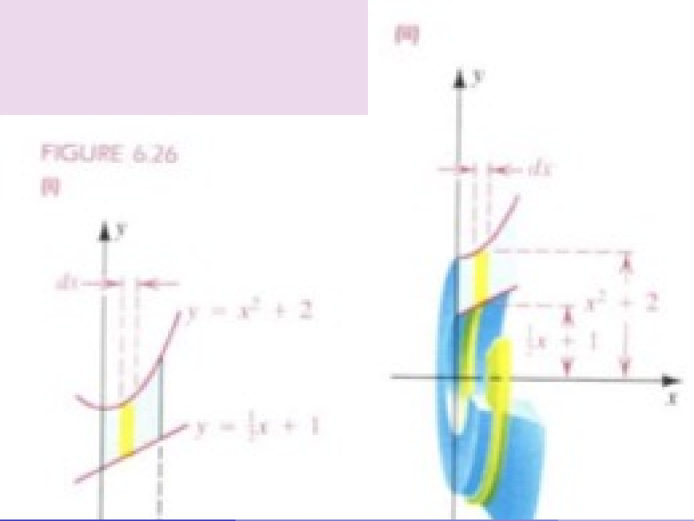

thickness: dx

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outer radius of disk: $x^2 + 2$

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$$= \pi \int_0^1 (x^4 + \frac{15}{4}x^2 - x + 3)dx$$

$$= \pi[\frac{1}{5}x^5 + \frac{15}{12}x^3 - \frac{1}{2}x^2 + 3x]_0^1$$

thickness: dx
outer radius of disk: $x^2 + 2$
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$$= \pi[\frac{1}{5}x^5 + \frac{15}{12}x^3 - \frac{1}{2}x^2 + 3x]_0^1$$

$$= \frac{79}{20}\pi = \approx 12.4$$

Examples

- Find the volume of the region bounded by the graph $y = \frac{1}{8}x^3$ and $y = 2x$ which is revolving about the $y - axis$?

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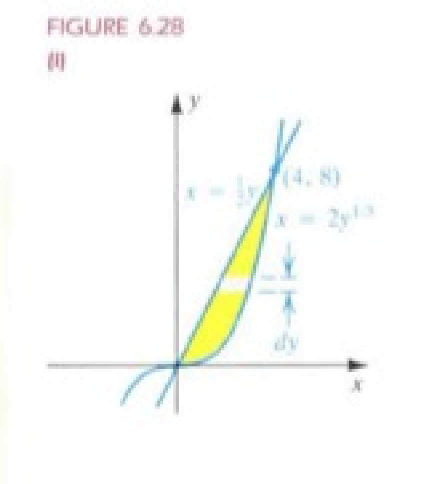


Examples

- Find the volume of the region bounded by the graph $y = \frac{1}{8}x^3$ and $y = 2x$ which is revolving about the $y - axis$?

FIGURE 6.28
(i)

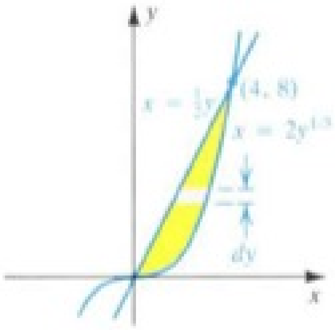
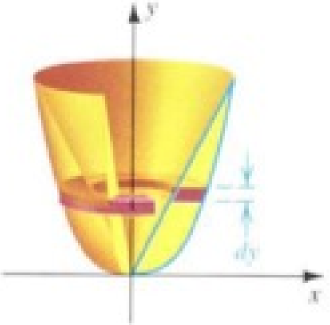


FIGURE 6.28
(ii)



$x = \frac{1}{2}y$ and $x = 2y^{\frac{1}{3}}$

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outer radius of disk: $2y^{\frac{1}{3}}$

$x = \frac{1}{2}y$ and $x = 2y^{\frac{1}{3}}$
thickness: dy
outer radius of disk: $2y^{\frac{1}{3}}$
inner radius of disk: $\frac{1}{2}y$

$x = \frac{1}{2}y$ and $x = 2y^{\frac{1}{3}}$
thickness: dy
outer radius of disk: $2y^{\frac{1}{3}}$
inner radius of disk: $\frac{1}{2}y$
volume of disk: $\pi[(2y^{\frac{1}{3}})^2 - (\frac{1}{2}y)^2]dy$.

$x = \frac{1}{2}y$ and $x = 2y^{\frac{1}{3}}$
thickness: dy
outer radius of disk: $2y^{\frac{1}{3}}$
inner radius of disk: $\frac{1}{2}y$
volume of disk: $\pi[(2y^{\frac{1}{3}})^2 - (\frac{1}{2}y)^2]dy$.

$$V = \int_0^8 \pi[4y^{\frac{2}{3}} - \frac{1}{4}y^2]dy \quad (4)$$

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thickness: dy
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$$= \pi[\frac{12}{5}y^{\frac{5}{3}} - \frac{1}{12}y^3]_0^8$$

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thickness: dy
outer radius of disk: $2y^{\frac{1}{3}}$
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$$V = \int_0^8 \pi[4y^{\frac{2}{3}} - \frac{1}{4}y^2]dy \tag{4}$$

$$= \pi[\frac{12}{5}y^{\frac{5}{3}} - \frac{1}{12}y^3]_0^8$$

$$= \frac{512}{15}\pi \approx 107.2$$