Representing Relations

Objectives:

The main purpose for this lesson is to introduce the following:

- ✓ In this section, we will discuss two alternative methods for representing relations.
- ✓ One method uses zero–one matrices.
- ✓ The other method uses pictorial representations called directed graphs.

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to

$B = \{b_1, b_2, \ldots, b_n\}$ (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when A = B we use the same ordering for A and B.) The relation R can be represented by the matrix MR = $[m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \\ 0 \; if \; (a_i, b_j) \notin R \end{cases}$$

EXAMPLE 1:

- Suppose that A = {1, 2, 3} and B = {1, 2}. Let R be the relation from A to B containing (a, b) , Where a ∈ A, b ∈ B, and $a > b$.

What is the matrix representing R ?

- Solution: R = {(2, 1), (3, 1), (3, 2)}

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Remark:

- The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties.

- Recall that a relation R on A is reflexive if (a, a) ∈ R whenever a ∈ A.
- R is reflexive if all the elements on the main diagonal of M_R are equal to 1,(R is reflexive if and only if m_{ii} = 1, for i = 1, 2,...,n)

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$ is reflexive

- The relation R is symmetric if and only if (a, b) ∈ R then (b, a) ∈ R.

(R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with i = 1, 2,...,n and j = 1, 2,...,n).

$$M_R = \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

The Zero–One Matrices for Symmetric Relation

we see that R is symmetric if and only if $M_R = (M_R)^t$,

- The relation R is antisymmetric if and only if (a, b) ∈ R and (b, a) ∈ R imply that a = b.

Consequently, the matrix of an antisymmetric relation has the property that if m_{ij} = 1 with i ≠ j , then m_{ji} = 0.

$$M_R = \begin{bmatrix} & 1 & & \\ 0 & & 0 & \\ & 1 & & 0 \\ & & 1 & \end{bmatrix}$$

The Zero–One Matrices for Antisymmetric Relations

Example 3: Suppose that the relation R on a set is represented by the matrix ?

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not Antisymmetric.

- Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R1} and M_{R2}, respectively. The matrix representing the union of these relations has a 1 in the positions where either M_{R1} or M_{R2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both M_{R1} and M_{R2} have a 1. Thus, the matrices representing the union and intersection of these relations are
- $\boldsymbol{M_{R1\cup R2}} = \boldsymbol{M_{R1}} \vee \boldsymbol{M_{R2}}$ and $\boldsymbol{M_{R1\cap R2}} = \boldsymbol{M_{R1}} \wedge \boldsymbol{M_{R2}}$.

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Representing Relations Using Digraphs

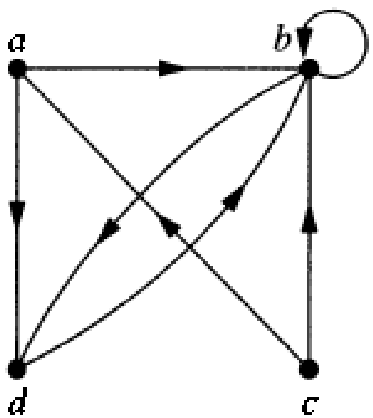

FIGURE 3
A Directed Graph.

DEFINITION 1

- A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex *a* is called the *initial vertex* of the edge (*a,b*), and the vertex *b* is called the *terminal vertex* of this edge.
- An edge of the form (a,a) is represented using an arc from the vertex a back to itself. Such an edge is called a *loop.*

EXAMPLE 7

The directed graph with vertices a, b, c, and d, and edges (a,b), (a,d), (b,b), (b,d), (c,a),(c,b), and (d,b) is displayed in Figure 3 .

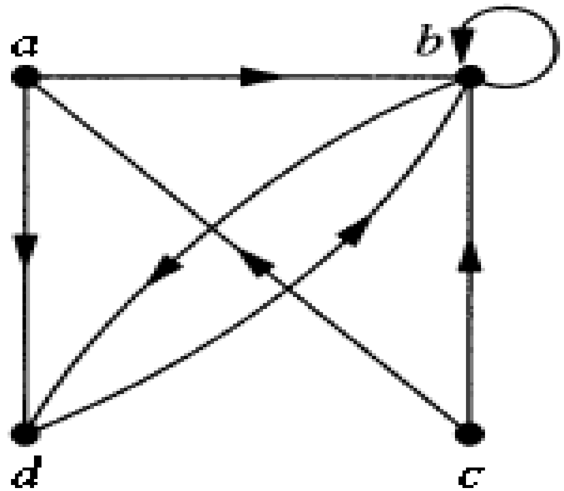

FIGURE 3
A Directed Graph.

EXAMPLE 8 The directed graph of the relation

R = { (1,1) , (1,3), (2,1), (2,3), (2,4) , (3,1) , (3,2), (4,1) }

on the set { 1 , 2 , 3 , 4 } is shown in Figure 4.

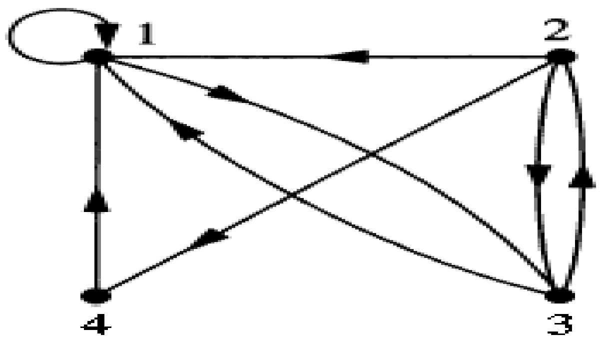

FIGURE 4 The Directed Graph of the Relation *R*.

Homework

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- 1 (a,b)
- 3 (a)
- 7
- 13 (c)
- 14 (a)
- 23
- 24