10.2 Graph Terminology and Special Types of Graphs

Objectives:

The main purpose for this lesson is to introduce the following:

- We define Basic Terminology.
- Define the neighborhood and degree of vertex.
- THE HANDSHAKING THEOREM.
- In-degree & Out-degree (directed graph).
- Some Special Simple Graphs.
- Bipartite Graphs.

Basic Terminology

DEFINITION 1

- Two ***vertices*** u and v in an undirected graph G are called ***adjacent*** (or ***neighbors***) in G if u and v are endpoints of an edge of G .
- If e is associated with {u,v}, the ***edge*** e is called ***incident*** with the vertices u and v.
- The ***edge*** e is also said to ***connect*** u and v.
- The ***vertices*** u and v are called ***endpoints*** of an edge associated with { u , v}.

DEFINITION 2

- The set of all neighbors of a vertex *v* of *G* = *(V ,E)*, denoted by *N(v)*, is called the *neighborhood* of *v*. If *A* is a subset of *V*, we denote by *N(A)* the set of all vertices in *G* that are adjacent to at least one vertex in *A*. So, *N(A)* = $\sum_{v \in A} N(v)$.

The degree of a vertex

DEFINITION 3

- The degree of a vertex in an **undirected** graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.
- The degree of the vertex v is denoted by **deg(v).**
- A vertex of ***degree zero*** is called ***isolated***.
- A vertex is ***pendant*** if and only if it has degree one.

Example 1:

What are the degrees of the vertices in the graphs G and H displayed in Figure 1?

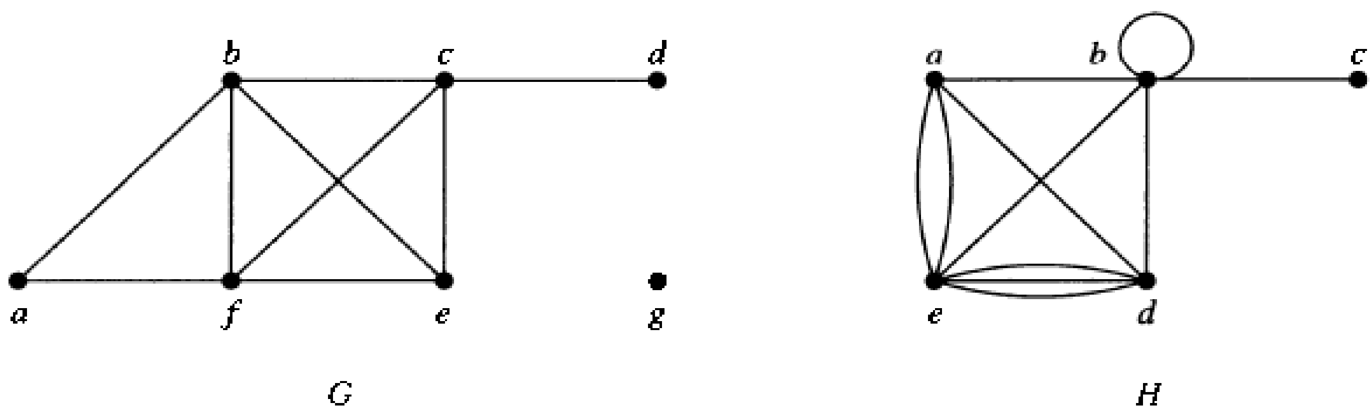

FIGURE 1 **The Undirected Graphs *G* and *H*.**

Solution:

In G:

deg(a) = 2 , deg(b) = deg(c) = deg(f) = 4 , deg(d) = 1 , deg(e) = 3 , and deg(g) = 0.

- The neighborhoods of these vertices are

N(a) = {b, f }, N(b) = {a, c, e, f }, N(c) = {b, d, e, f }, N(d) = {c}, N(e) = {b, c, f }, N(f) = {a, b, c, e}, and N(g) = ∅.

In H:

deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1 ,

and deg(d) = 5 .

The neighborhoods of these vertices are N(a) = {b, d, e}, N(b) = {a, b, c, d, e}, N(c) = {b}, N(d) = {a, b, e}, and N(e) = {a, b, d}.

- A vertex of degree zero is called ***isolated***.
- It follows that an isolated vertex is not adjacent to any vertex.
- Vertex g in graph G in Example 1 is isolated.
- A vertex is ***pendant*** if and only if it has degree one.
- Consequently, a pendant vertex is adjacent to exactly one other vertex.
- Vertex d in graph G in Example 1 is pendant.

THE HANDSHAKING THEOREM

THEOREM 1

Let G = (V, E) be an undirected graph with e edges. Then $2e = \sum_{v \in V} \deg(v)$.

(Note that this applies even if multiple edges and loops are present.)

EXAMPLE 3

How many edges are there in a graph with 1 0 vertices each of degree 6?

Solution:

Because the sum of the degrees of the vertices is 6 · 10x6 = 60,

it follows that 2e = 60.

Therefore, e = 30.

THEOREM 2

An undirected graph has an **even** number of **vertices** of **odd degree.**

DEFINITION 4

- When (u ,v) is an edge of the graph G with **directed edges**, u is said to be ***adjacent to*** v and v is said to be adjacent from u.
- The vertex u is called ***the initial vertex*** of (u,v), and v is called ***the terminal or end vertex*** of (u,v).
- The initial vertex and terminal vertex of a loop are the same.

In-degree & Out-degree (directed graph)

DEFINITION 4

- In a graph with **directed edges** the ***in-degree*** of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their **terminal vertex**.
- The ***out-degree*** of v, denoted by $\deg^{+}(v)$, is the number of edges with v as **their initial vertex.**
- (Note that a loop at a vertex contributes 1 to **both the in-degree and the out-degree** of this vertex.)

EXAMPLE 4

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2 .

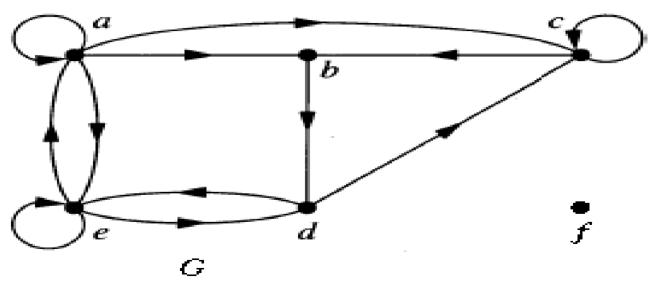

FIGURE 2 The Directed Graph G.

Solution:

The in-degrees in G are $\deg^-(a)=2$, $\deg^-(b)=2$, $\deg^-(c)=3$, $\deg^-(d)=2$, $\deg^-(e)=3$, and $\deg^-(f)=0$.

The out-degrees are $\deg^+(a)=4$, $\deg^+(b)=1$, $\deg^+(c)=2$, $\deg^+(d)=2$, $\deg^+(e)=3$, and $\deg^+(f)=0$.

THEOREM 3

Let G = (V,E) b e a graph with **directed edges**.

Then:

$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$.

Some Special Simple Graphs

EXAMPLE 5

Complete Graphs The complete graph on n vertices, denoted by K_n , is the **simple graph** that contains exactly one edge between each pair of distinct vertices.

The graphs K_n , for n=1,2,3,4,5,6, are displayed in Figure 3.

A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called noncomplete

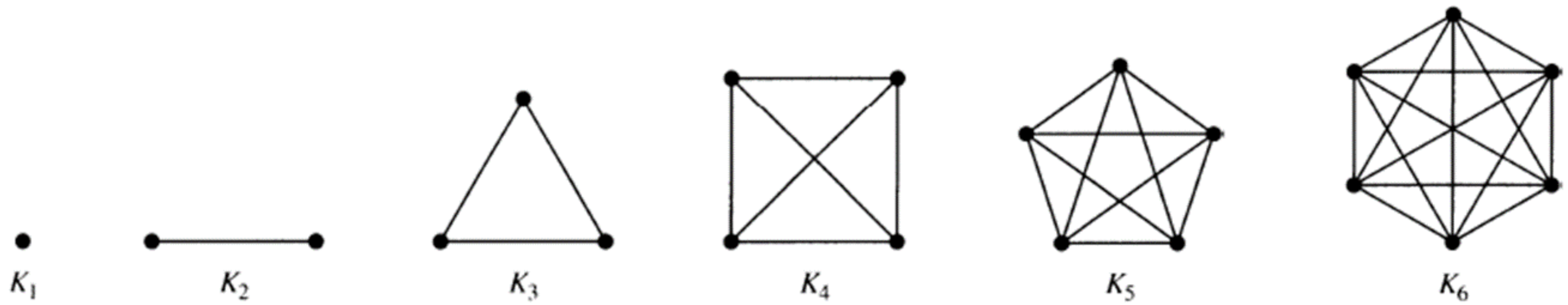

FIGURE 3 The Graphs K_n for $1 \leq n \leq 6$.

EXAMPLE 6

Cycles The cycle $C_n \geq 3$, consists of n vertices V_1, V_2, . . . , V_n and edges $\{V_1, V_2\}$, $\{V_2, V_3\}$, . . . , $\{V_{n-1}, V_n\}$, and $\{V_n, V_1\}$.

The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.

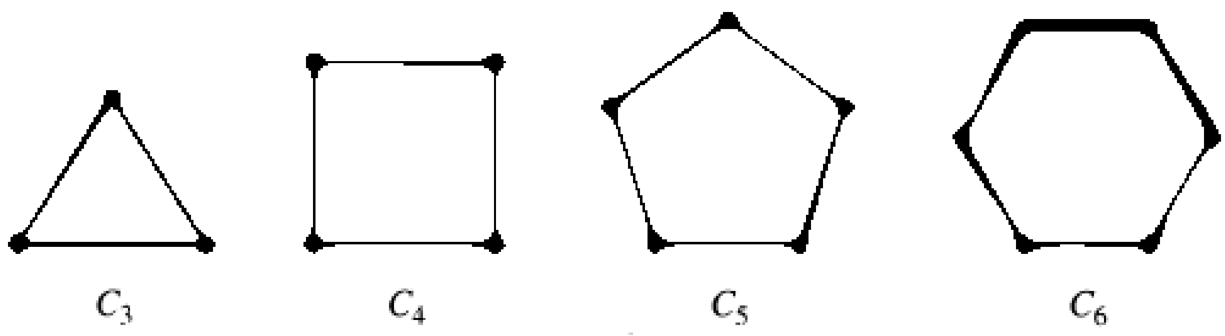

FIGURE 4 The Cycles C_3, C_4, C_5, and C_6.

EXAMPLE 7

Wheels We obtain the wheel W_n when we add an additional vertex to the cycle C_n , for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n , by new edges.

The wheels W_3 , W_4 ,W_5 , and W_6 are displayed in Figure 5 .

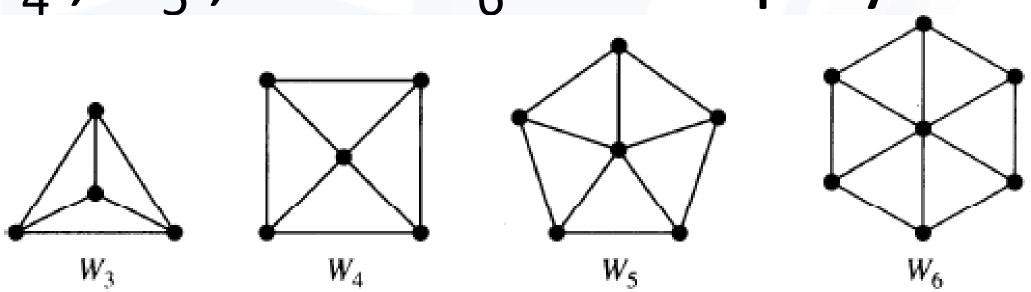

FIGURE 5 The Wheels W_3, W_4, W_5, and W_6.

How many vertices & edges in each type?

Type of the Simple Graph	Number of Vertices	Number of Edges
K_n	n	$\frac{n(n-1)}{2}$
C_n	n	n
W_n	n+1	2n

Bipartite Graphs

DEFINITION 6

A **simple graph** G is called ***bipartite*** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

(so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1,V_2) a ***bipartition*** of the vertex set V of G .

EXAMPLE 11

Are the graphs G and H displayed in Figure 8 bipartite?

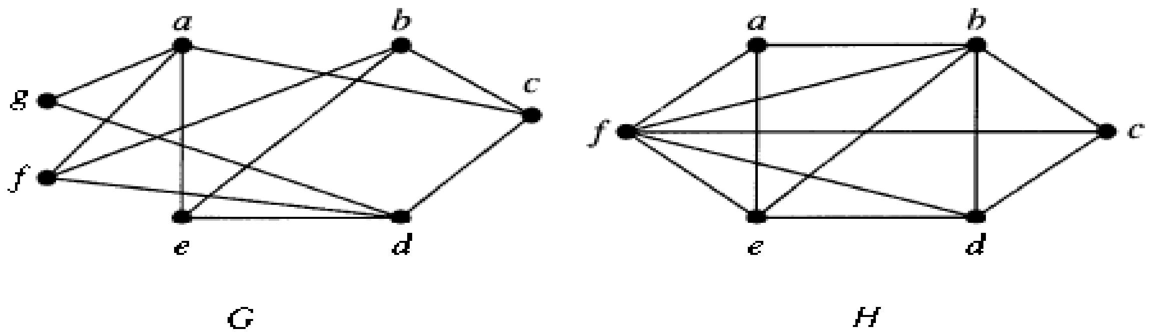

FIGURE 8 **The Undirected Graphs *G* and *H*.**

Solution:

- Graph G is bipartite because its vertex set is the union of two disjoint sets, {a ,b,d} and {c, e,f, g}, and each edge connects a vertex in one of these subsets to a vertex in the other subset.

(Note that for G to be bipartite it is not necessary that every vertex in {a ,b,d} be adjacent to every vertex in {c,e,f, g}. For instance, b and g are not adjacent.)

- Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. verify this by (consider the vertices a , b, and f.)

Homework

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- 1
- 2
- 3
- 7
- 8
- 9
- 10
- 29(a,b,c)
- 37(a,b,d,e,f).