Equivalence Relations

Objectives:

The main purpose for this lesson is to introduce the following:

- ❑ Define Equivalence Relations.
- ❑ Define equivalence class.
- ❑ Define Partial Ordering.

Equivalence Relations

DEFINITION 1

A relation on a set A is called an ***equivalence relation*** if it is **reflexive**, **symmetric**, and **transitive**.

DEFINITION 2

Two elements a and b that are related by an equivalence relation are called ***equivalent***. The

notation ***a ~ b*** is often used to denote that *a* and *b* are ***equivalent elements*** with respect to a particular equivalence relation.

EXAMPLE 1

- Let R be the relation on the set of integers such that aRb if and only if $a = b \ or \ a = -b$.

Is R an equivalence relation?

Solution:

- **Reflexive:** for every integer a, $a = a$ so, aRa.
- **Symmetric:** for if $a = b \ or \ a = -b, then$

$$b = a \ or \ b = -a.$$

- **Transitive:** because $a = \pm b \ and \ b = \pm c$ imply that $a = \pm c$.

Hence R is equivalence relation.

EXAMPLE 2

Let R be the relation on the set of real numbers such that a R b if and only if a - b is an integer.

Is R an equivalence relation?

Solution:

- Because a - a = 0 is an integer for all real numbers a, aRa for all real numbers a . Hence, **R is reflexive.**

Now suppose that aRb. Then a - b is an integer, so b - a is also an integer. Hence, bRa. It follows **that R is symmetric.**

- If a R b and b R c, then a - b and b - c

are integers. Therefore, a - c = (a - b) + (b - c) is also an integer. Hence, aRc. Thus, **R is transitive.**

Consequently, **R is an equivalence relation.**

Equivalence Classes

- DEFINITION 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$ When only one relation is under consideration, we can delete the subscript R and write [a] for this equivalence class.

- In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s \mid (a, s) \in R\}$.
- If $b \in [a]_R$, then b is called a representative of this equivalence class.

- Example 3: What is the equivalence class of an integer for the equivalence relation of Example 1?
- Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$.
- For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0]=\{0\}$

Partial Ordering.

DEFINITION 4:

A relation on a set A is called a **Partial order** if it is reflexive, antisymmetric, and transitive.

Example:

- Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.

Homework

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- 1 (a,b,c,d,e)
- 15
- 37
- 39(a)
- 47 (a)
- 55