Engineering Mechanics
AGE 2330

Lect 15: Moment of Inertia

Dr. Feras Fraige

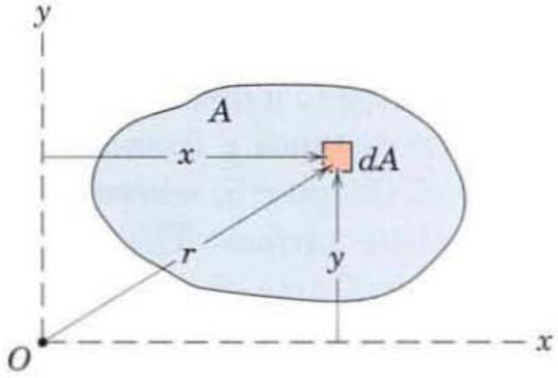
Moment of Inertia Def.

$$I_x = \int y^2 \, dA$$

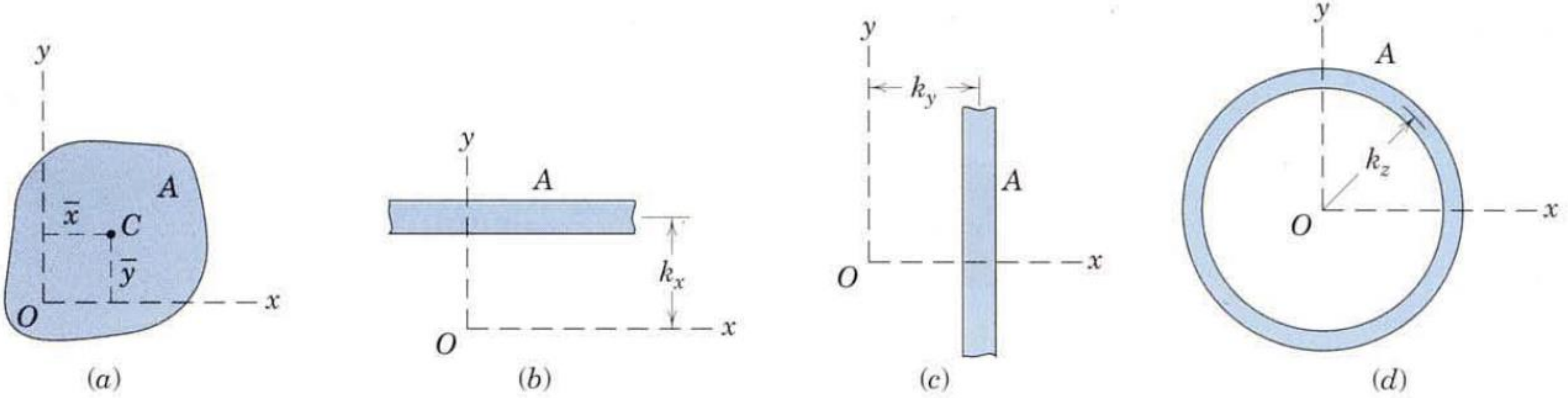
$$I_y = \int x^2 \, dA$$

$$I_z = \int r^2 \, dA$$

$$I_z = I_x + I_y$$



Radius of gyration



$$I_x = k_x^2 A$$
$$I_y = k_y^2 A$$
$$I_z = k_z^2 A$$

$$k_x = \sqrt{I_x/A}$$
$$k_y = \sqrt{I_y/A}$$
$$k_z = \sqrt{I_z/A}$$

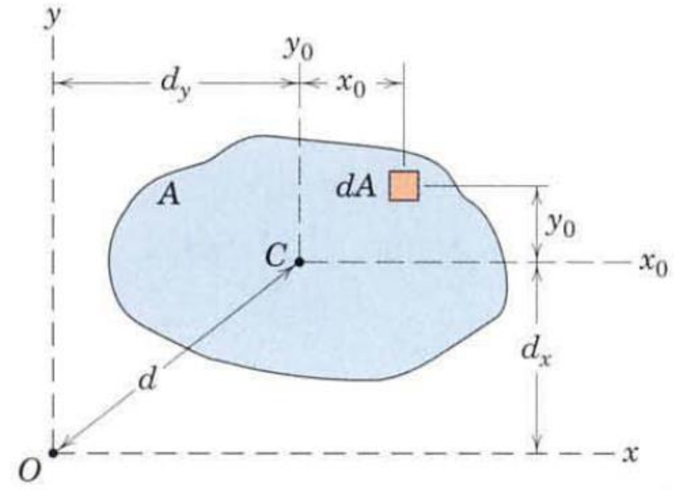
$$k_z^2 = k_x^2 + k_y^2$$

Transfer of Axes

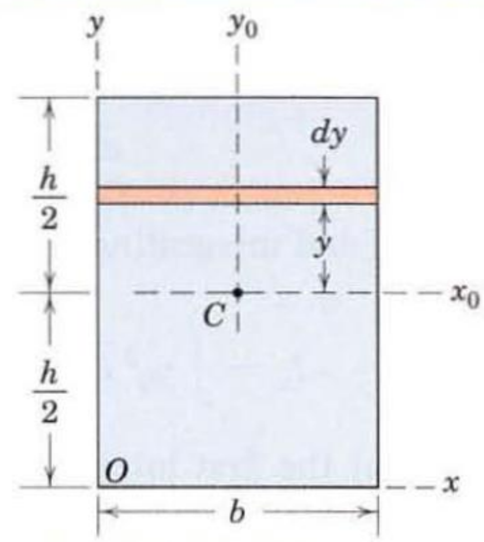
$$I_x = \bar{I}_x + Ad_x^{\,2}$$

$$I_y = \bar{I}_y + Ad_y^{\,2}$$

$$I_z = \bar{I}_z + Ad^2$$



Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



$$[I_x = \int y^2\, dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \tfrac{1}{12}bh^3$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2)$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

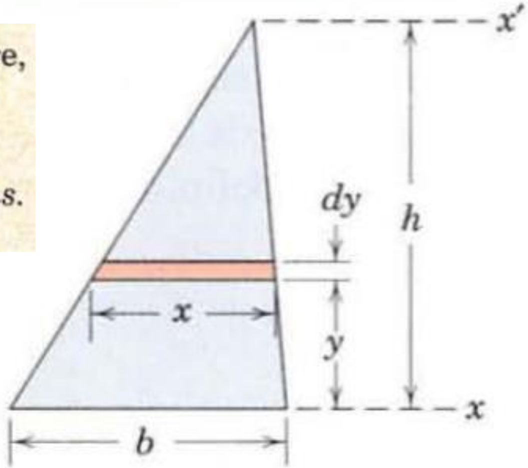
$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.

Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$



By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

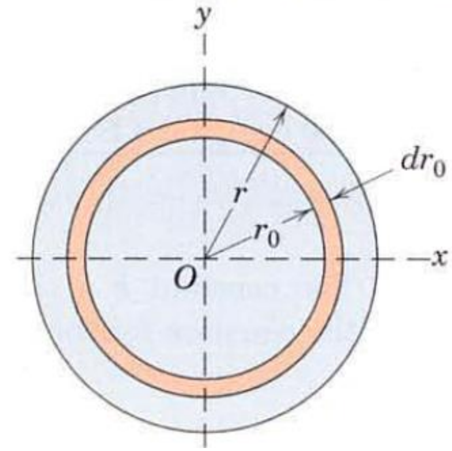
A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.

Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus,

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \tfrac{1}{2}Ar^2 \qquad \textit{Ans.}$$



The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \tfrac{1}{2}I_z = \frac{\pi r^4}{4} = \tfrac{1}{4}Ar^2$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_0\, dr_0\, d\theta$ shown in the lower figure. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^r (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

$$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \qquad \textit{Ans.}$$

Composite Areas

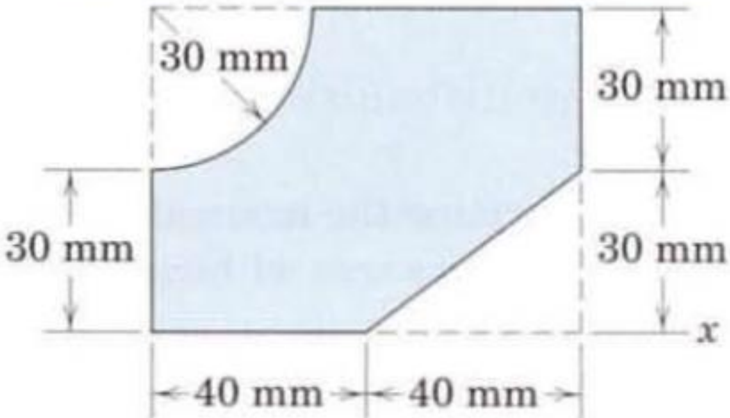
- In order to evaluate the moment of inertia of planar areas, evaluate the following table

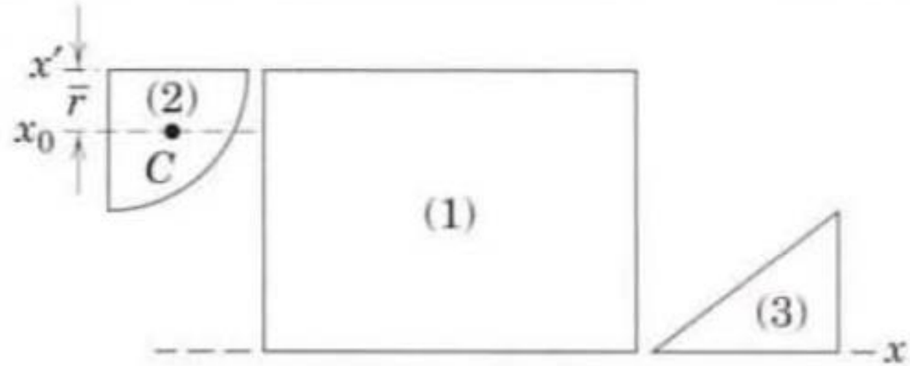
Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

- And calculate the I_x and I_y as follows:

$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^2$$
$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^2$$

Calculate the moment of inertia and radius of gyration about the x-axis for the shaded area shown.





Solution. The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \tfrac{1}{3}Ah^2 = \tfrac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\text{ mm}^4$$

From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\text{ mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$$

$$= -0.0445(10^6) \text{ mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$$

$$= -1.624(10^6) \text{ mm}^4$$

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\tfrac{1}{12}bh^3 = -\tfrac{1}{12}(40)(30)^3 = -0.09(10^6) \text{ mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6) \text{ mm}^4 \qquad \textit{Ans.}$$

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490 \text{ mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0 \text{ mm} \qquad \textit{Ans.}$$

TABLE D/3 PROPERTIES OF PLANE FIGURES

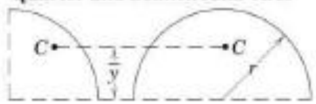
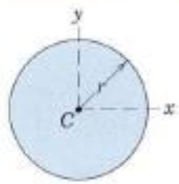
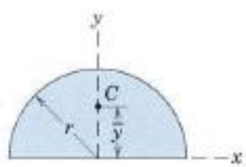
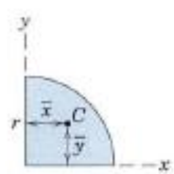
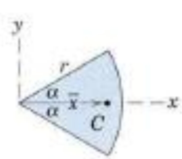
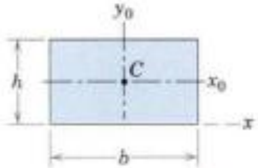
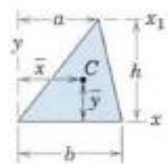
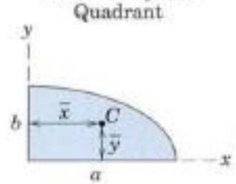
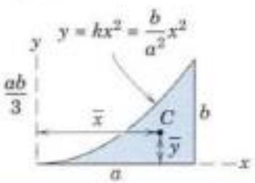
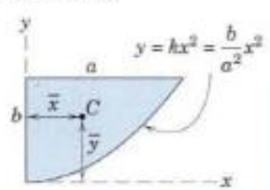
FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \frac{r \sin\alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \frac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha}$	$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2}\sin 2\alpha)$ $I_y = \frac{r^4}{4}(\alpha + \frac{1}{2}\sin 2\alpha)$ $I_z = \frac{1}{2}r^4\alpha$

TABLE D/3 PROPERTIES OF PLANE FIGURES *Continued*

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$