Faculty of Engineering
Mechanical Engineering Department

CALCULUS FOR ENGINEERS
MATH 1110

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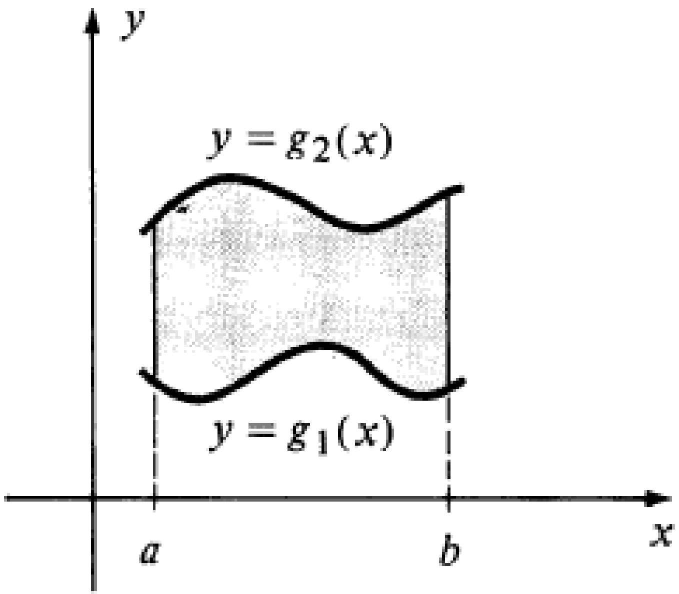
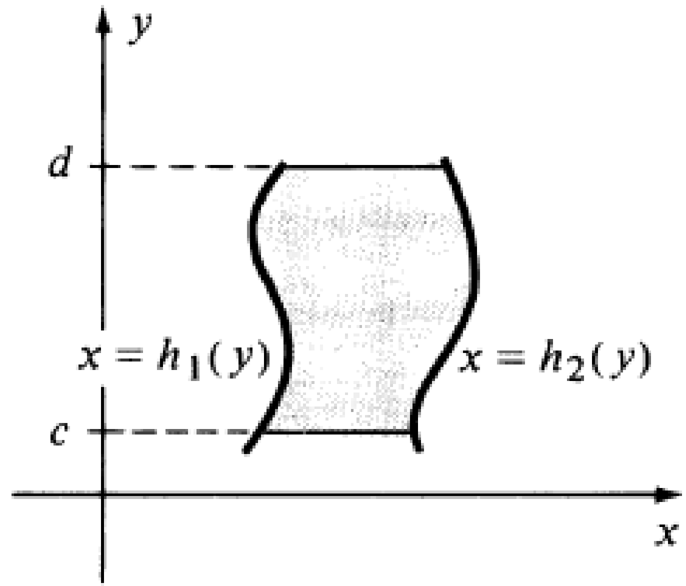
DOUBLE INTEGRALS



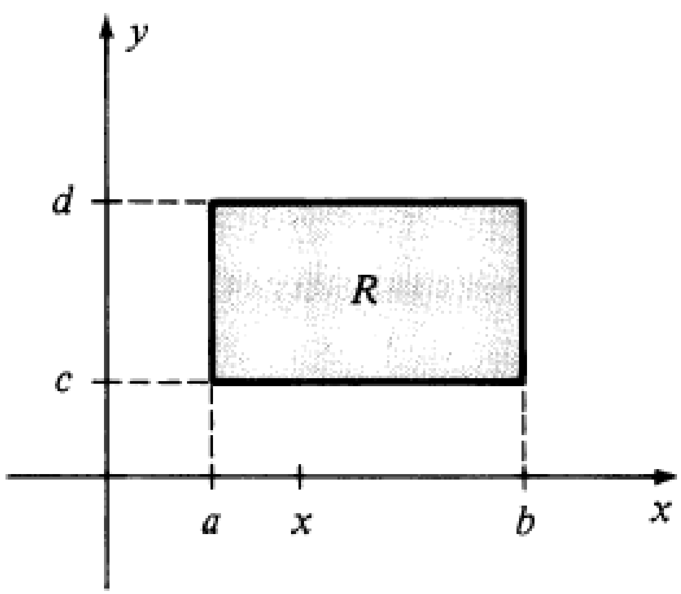
(i) Region of Type I



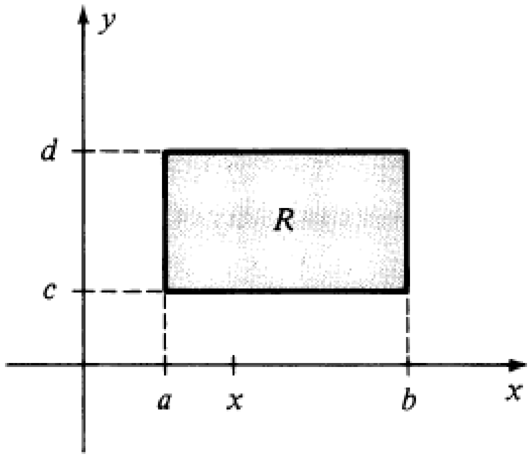
(ii) Region of Type II

EVALUATION OF DOUBLE INTEGRALS

- Suppose f is a function of two variables that is continuous on a closed rectangular region R of the type illustrated in the figure shown below.



Integration Method



$$A(x) = \int_c^d f(x, y)\, dy.$$

As an illustration, if $f(x, y) = x^3 + 4y$, $c = 1$, and $d = 2$, then

$$\begin{aligned} A(x) &= \int_1^2 (x^3 + 4y)\, dy = x^3 y + 2y^2 \Big]_1^2 \\ &= (2x^3 + 8) - (x^3 + 2) \\ &= x^3 + 6. \end{aligned}$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

***Definition* (17.6)**

$$\boxed{\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.}$$

Example 1 Evaluate $\int_{1}^{4} \int_{-1}^{2} (2x + 6x^2 y)\, dy\, dx.$

Solution As in Definition (17.6), the integral equals

$$\begin{aligned}
\int_{1}^{4} \left[\int_{-1}^{2} (2x + 6x^2 y)\, dy \right] dx &= \int_{1}^{4} \Big[2xy + 3x^2 y^2 \Big]_{-1}^{2} dx \\
&= \int_{1}^{4} [(4x + 12x^2) - (-2x + 3x^2)]\, dx \\
&= \int_{1}^{4} (6x + 9x^2)\, dx \\
&= 3x^2 + 3x^3 \Big]_{1}^{4} = 234.
\end{aligned}$$

Example 2 Evaluate $\int_{-1}^{2} \int_{1}^{4} (2x + 6x^2 y)\, dx\, dy$.

Solution Applying Definition (17.7),

$$\begin{aligned}
\int_{-1}^{2} \left[\int_{1}^{4} (2x + 6x^2 y)\, dx \right] dy &= \int_{-1}^{2} \Big[x^2 + 2x^3 y \Big]_{1}^{4} dy \\
&= \int_{-1}^{2} [(16 + 128y) - (1 + 2y)]\, dy \\
&= \int_{-1}^{2} (126y + 15)\, dy \\
&= 63y^2 + 15y \Big]_{-1}^{2} = 234.
\end{aligned}$$

Example 3 Evaluate $\int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\,dy\,dx.$

Solution By (i) of Definition (17.8) the integral equals

$$\int_0^2 \left[\int_{x^2}^{2x} (x^3 + 4y)\,dy \right] dx = \int_0^2 \left[x^3 y + 2y^2 \right]_{x^2}^{2x} dx$$

$$= \int_0^2 [(2x^4 + 8x^2) - (x^5 + 2x^4)]\,dx$$

$$= \frac{8}{3}x^3 - \frac{1}{6}x^6 \bigg]_0^2 = \frac{32}{3}.$$

Example 4 Evaluate $\int_1^3 \int_{\pi/6}^{y^2} 2y \cos x \, dx \, dy$.

Solution By (ii) of Definition (17.8) the integral equals

$$\begin{aligned}
\int_1^3 \left[\int_{\pi/6}^{y^2} 2y \cos x \, dx \right] dy &= \int_1^3 \Big[2y \sin x \Big]_{\pi/6}^{y^2} dy \\
&= \int_1^3 (2y \sin y^2 - y) \, dy \\
&= -\cos y^2 - \frac{1}{2} y^2 \Big]_1^3 \\
&= (-\cos 9 - \tfrac{9}{2}) - (-\cos 1 - \tfrac{1}{2}) \\
&= \cos 1 - \cos 9 - 4 \approx -2.55.
\end{aligned}$$